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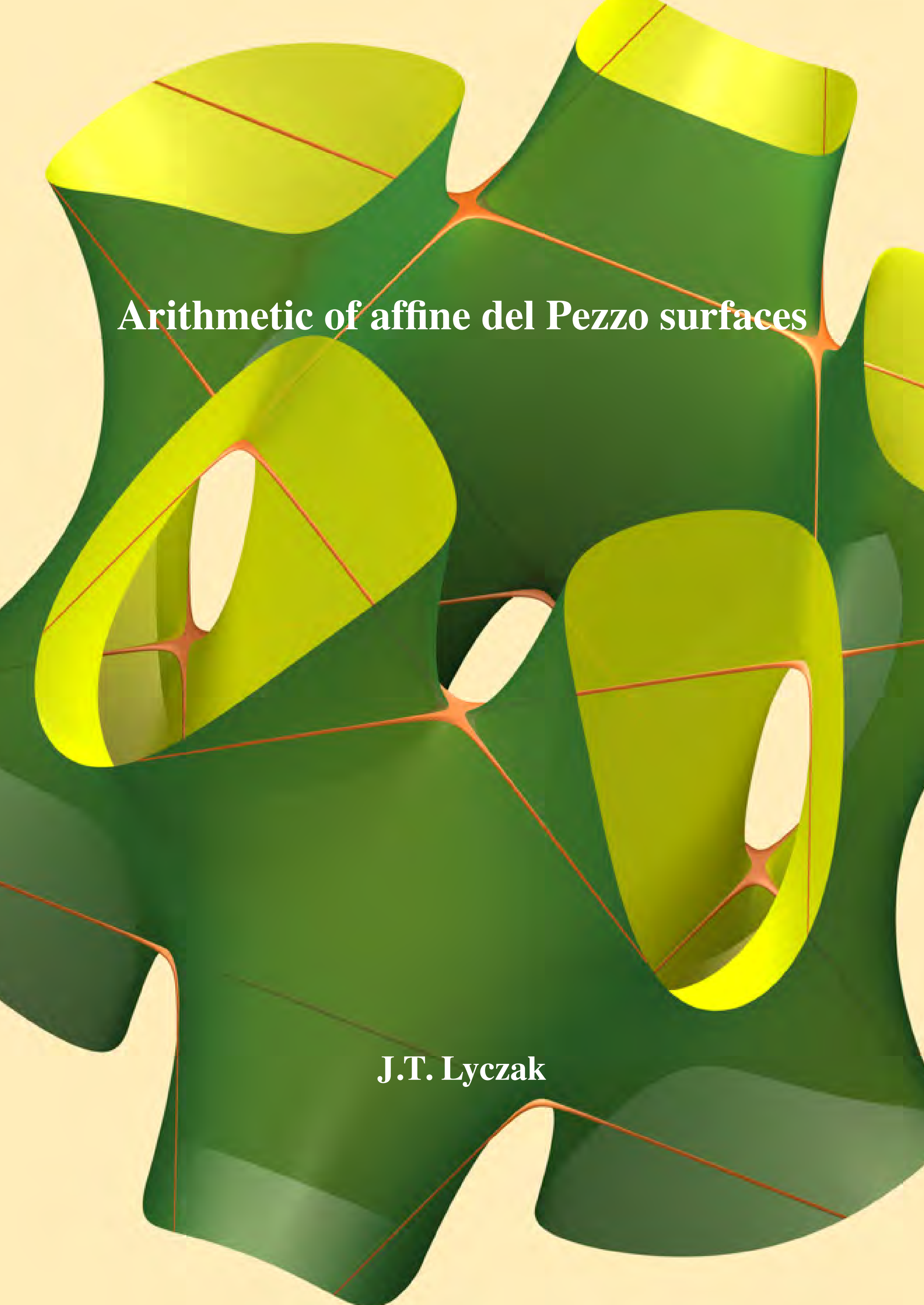

# Arithmetic of affine del Pezzo surfaces

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# Contents

# Introduction

## Background

The study of Diophantine equations is concerned with polynomial equations and their solutions in number fields and number rings. This field of research is thousands of years old. For example, triplets of integers $x$, $y$ and $z$ satisfying the equation $x^2 + y^2 = z^2$ have been studied in several ancient civilizations. A first question can be whether a solution to such an equation exists at all. The existence of solutions is usually proved by providing explicit values satisfying the equation. Next, one can wonder about how many solutions there are and if this number is finite or infinite. If on the other hand no solutions exist, one would want a reason to explain their absence. It is this question with which we concern ourselves in this thesis.

Let $X$ be a scheme over a number field $k$ for which we want to determine whether $X(k)$ is empty or not. A usually first step is to consider a completion $k_v$ of $k$ and note that if $X(k_v)$ is empty then so is $X(k)$. For some schemes this is all one needs to consider; the Hasse–Minkowski theorem states that for a hypersurface $X \subseteq \mathbb{P}^n_k$ defined by a homogeneous quadric equation $F \in k[x_0, \ldots, x_n]$ the set $X(k)$ is non-empty precisely if $X(k_v)$ is non-empty for all completions $k_v$ of $v$. For other schemes over $k$ this need not be the case and several counterexamples to this principle were published. Then in 1970 Manin [40] introduced a technique which unified all these results. The technique he put forward is now known as the *Brauer–Manin obstruction*.

Up to then all known examples of schemes $X$ over the number field $k$ for which $X(k)$ was empty were either explained by the absence of local points or by the Brauer–Manin obstruction. We now however know that the Brauer–Manin obstruction is not sufficient to explain to the absence of integral points on all schemes. When we restrict to certain subclasses of schemes, such as the quadric hypersurfaces, there do exist positive results. For other types of schemes the study of integral points is still ongoing. This thesis aims to add to the case of *surfaces*, i.e. schemes of dimension two. Although there are still a lot of unanswered questions rational points on surfaces are relatively well understood. We have for example the conjecture by Colliot-Thélène and Sansuc [14, page 174] which states that the Brauer–Manin obstruction is the only one to the Hasse principle on rational surfaces. So despite the many open problems there is an understanding of what the final results should be or at least in what type of language they should be phrased.

Now let us turn our attention to integral points. There is no standard framework yet for working with integral points on surfaces. Even on log rationally connected surfaces, which are not unlike the rationally connected surface mentioned above, Manin's technique does not have to exclude the existence of integral points, as was proved in [16]. In [32] a refined conjecture was put forward, but for the slightly more complex class of log K3 surfaces no conjectures have been formulated yet. In [35] new concepts were introduced to explain the absence of integral solutions. But again, these techniques do not explain all known counter-examples. In the hope to find other techniques and terminology which allow us to formulate reasonable conjectures we still need more examples of the study of integral points on surfaces.

This thesis contributes in that sense to this field of research because one of the main result is the existence of Brauer–Manin obstructions to the integral Hasse principle. What makes these examples stand out is that these are of order 5, in contrast to the previously known examples which are all of order 2 and 3.

Another important result is the uniform bound on the Brauer group of ample log K3 surfaces. This result will help to understand the relevance of the Brauer–Manin obstruction to integral points on such surfaces.

## Overview

In the first chapter we recall general notions in the theory of the arithmetic of schemes. We will for example define the *ring of adeles* $\mathbb{A}_k$ associated to a number field $k$. Now consider a scheme $X$ over $k$. We will describe the elements of the set of adelic points $X(\mathbb{A}_k)$ as tuples of points $p_v \in X(k_v)$ indexed by all completions $k_v$ of $k$. This induces an inclusion $X(k) \subseteq X(\mathbb{A}_k)$ which could be used to prove that $X(k)$ is empty. The celebrated Hasse–Minkowski theorem can now be rephrased by saying that this technique is all one needs for quadric hypersurface; consider a projective scheme $X \subseteq \mathbb{P}^d_k$ over a number field defined by a homogeneous quadratic equation. The set $X(k)$ is empty precisely if the set $X(\mathbb{A}_k)$ is empty. One says that quadratic forms satisfy the *Hasse principle*.

For other schemes it is however possible that the set of adelic points is non-empty, but that there are no $k$-points on $X$. In this case, one could consider the Brauer group of $X$. This invariant of schemes can be useful in the following way. Any element $\mathcal{A} \in \operatorname{Br} X$ defines an intermediate subset

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathcal{A}} \subseteq X(\mathbb{A}_k).$$

If this explains the absence of rational points one says that there is a *Brauer–Manin obstruction* to the Hasse principle.

It was recognized in [17] that these techniques can be adapted to explain the absence of $\mathcal{O}_k$-points for a scheme $\mathcal{X}$ defined over $\mathcal{O}_k$. In this case one defines the *ring of integral adeles* $\mathbb{A}_{k,\infty}$ of $k$. For classes of schemes over $\mathcal{O}_k$ we have the *integral Hasse principle* which says that $\mathcal{X}(\mathcal{O}_k)$ is empty precisely if $\mathcal{X}(\mathbb{A}_{k,\infty})$ is empty. Again, if $\mathcal{X}(\mathcal{O}_k)$ is non-empty then so is $\mathcal{X}(\mathbb{A}_{k,\infty})$ since we have an inclusion $\mathcal{X}(\mathcal{O}_k) \subseteq \mathcal{X}(\mathbb{A}_{k,\infty})$. Also, for any element $\mathcal{A} \in \operatorname{Br} X$ of the Brauer group of the generic fibre $X = \mathcal{X}_k$ we have an intermediate subset

$$\mathcal{X}(\mathcal{O}_k) \subseteq \mathcal{X}(\mathbb{A}_{k,\infty})^{\mathcal{A}} \subseteq X(\mathbb{A}_{k,\infty}).$$

We will use this chain of inclusions to define the *Brauer–Manin obstruction* to the integral Hasse principle.

There are many sources which offer a more complete treatise on the arithmetic of schemes over number rings and number fields. See for example [46] and the introduction in [32].

In the second chapter we review del Pezzo surfaces. We start by treating the surfaces studied by del Pezzo himself in [22] which we will call *ordinary del Pezzo surfaces*. The most common extension in the literature of these surfaces are the *generalized del Pezzo surfaces*. These surfaces are also smooth and are indistinguishable from the ordinary del Pezzo surfaces if one only looks at the geometric Picard group. These two types of surfaces share many geometric properties, one of which is the fact that the anticanonical map turns out to be a morphism into projective space of relatively small dimension for both types of surfaces. The difference is that this morphism is an isomorphism onto its image for precisely the ordinary del Pezzo surfaces; for the generalized del Pezzo surfaces it will merely be a birational morphism onto its image. This brings us to the last common type of del Pezzo surface to be studied in literature. A *singular del Pezzo surface* is the image of a generalized del Pezzo surface under this morphism to projective space.

This chapter also contains the new concept of *peculiar del Pezzo surfaces*. This novel type of surface fits in between the generalized and singular del Pezzo surfaces in the following manner. A generalized del Pezzo surface $X$ over an algebraically closed field $k$ admits a birational morphism $\pi\colon X \to \mathbb{P}^2$ to the projective plane. Let $Y \subseteq \mathbb{P}^d$ be the image of $X$ under the anticanonical morphism to $\mathbb{P}^d$, i.e. the singular del Pezzo surface associated to $X$. The composition of $\pi$ with the birational inverse of the anticanonical morphism $X \to Y$ produces a birational map $Y \dashrightarrow \mathbb{P}^2$. Let us consider a splitting of the morphism $X \to X' \to Y$ into two birational morphisms. For each such splitting we have a birational map $X' \dashrightarrow \mathbb{P}^2$. A *peculiar del Pezzo surface* is the minimal $X'$ such that $X' \dashrightarrow \mathbb{P}^2$ is a morphism, i.e. $\pi$ factors through $X'$.

We will prove that the generalized del Pezzo surface $X$, the peculiar del Pezzo surface $X'$, and the singular del Pezzo surface $Y$ all determine each other. The main advantage is that a peculiar del Pezzo surface has a rather direct geometric construction; any peculiar del Pezzo surface is the blowup of the projective plane in a, possibly non-reduced, zero-dimensional scheme. In this manner, many geometric properties of $X$, $X'$ and $Y$ could be described in terms of this zero-dimensional scheme on $\mathbb{P}^2$. This allows one to solve problems concerning del Pezzo surfaces using the language of points and curves on the projective plane.

In the third chapter we change our attention to log K3 surfaces over a number field $k$ and specially those of ample type. A particular straightforward way to construct such a surface $U$ is by considering a projectively embedded ordinary del Pezzo surface $X \subseteq \mathbb{P}^d$ and taking the complement of a hyperplane, i.e. $U = \mathbb{A}^d \cap X$. We present the results from the author's work joint with Martin Bright [10]. The main theorem in this chapter is that the Brauer group of such surfaces over $k$ are uniformly bounded in the sense that $\#\left(\operatorname{Br} U / \operatorname{Br} k\right) < C$. We will show that the constant $C$ only depends on the degree of the number field $k$. This result is particularly encouraging with a view towards Várilly-Alvarado's conjecture [56, Conjectures 4.5, 4.6] which states that such a bound should exist for proper K3 surfaces, another class of log K3 surfaces which is disjoint from the ample log K3 surfaces considered in this chapter.

The fourth chapter presents another novel result of the author. It exhibits Brauer–Manin obstructions of order 5 to the integral Hasse principle. These examples are particularly interesting since all other known examples of the Brauer–Manin obstruction, be it to weak approximation or to any form of the Hasse principle, are all of either order 2 or order 3.

These examples arise from an element $\mathcal{A}$ of order 5 in the Brauer group $\operatorname{Br} U$ of the ample log K3 surface $U$ over $\mathbb{Q}$. Note however that the set of integral points on the scheme $U$ over $\mathbb{Q}$ is not well-defined. It depends on the choice of model $\mathcal{U}$ of $U$, by which we mean a scheme $\mathcal{U}$ over $\mathbb{Z}$ whose fibre $\mathcal{U}_{\mathbb{Q}}$ is isomorphic to $U$.

A model is often chosen by writing down explicit equations with integral coefficients. For example, homogeneous equations with integral coefficients define a projective scheme over $\mathbb{Z}$. The situation is made even more complicated by the fact that we are dealing with affine schemes. Again, one could simply construct an affine scheme by supplying equations which now should be inhomogeneous. This was done for example in [16], [32], [35], [38], [15]. In each of these papers the application of the Brauer–Manin obstruction to the existence of integral points on certain ample log K3 surfaces is studied. For the examples of Brauer–Manin obstructions to the integral Hasse principle in this thesis another approach is used; we will study the set of integral points on a scheme $\mathcal{U}$ over $\mathbb{Z}$ which is constructed using geometrical tools. This geometric construction will allow for a relatively effortless study of the fibres $\mathcal{U}_\ell$ of $\mathcal{U}$ over a prime $\ell \in \mathbb{Z}$. We will use this to describe the set of integral adelic points $\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})$ and compute the subset $\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})^{\mathcal{A}}$ corresponding to the class $\mathcal{A} \in \operatorname{Br} U$ of order 5. This subset contains the set $\mathcal{U}(\mathbb{Z})$ of integral points on $\mathcal{U}$. We will give several explicit examples for which $\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})^{\mathcal{A}}$ and hence $\mathcal{U}(\mathbb{Z})$ is empty.

# Notation and conventions

We list some basic notation and conventions which will reoccur throughout this thesis. Some concepts will also be introduced in the text, but we include them here as well for the reader's convenience.

A *ring* is always assumed to be unitary. All rings will be commutative unless stated otherwise.

A *division ring* is a, not necessarily commutative, ring where each non-zero element has a multiplicative inverse.

Let $R$ be a, not necessarily commutative, topological ring. A *(left) $R$-module* is an abelian topological group together with a continuous left $R$-action. A *(left) $R$-algebra* is a, not necessarily commutative, topological ring together with a continuous left $R$-action.

Let $G$ be a topological group. A *(left) $G$-module* is a topological group with a continuous left $G$-action. The notions of abelian $G$-modules and $\mathbb{Z}[G]$-modules are equivalent.

Whenever the topology on a ring, module or group is not explicitly stated, we assume the topology to be discrete.

For a field $k$ we will write $\bar{k}$ for a fixed algebraic closure and $k^{\mathrm{sep}}$ for the separable closure of $k$ in $\bar{k}$. The absolute Galois group of a field $k$ is endowed with the profinite topology and this topological group is denoted by $G_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$.

The absolute Galois group of a finite field $\mathbb{F}_q$ is canonically isomorphic to $\widehat{\mathbb{Z}}$ and it is topologically generated by the *Frobenius* $\mathrm{Frob}_q \colon \bar{\mathbb{F}}_q \to \bar{\mathbb{F}}_q$, $x \mapsto x^q$.

A *variety* over a field $k$ is a separated scheme of finite type over $\operatorname{Spec} k$.

A *curve* over a field $k$ is a variety over $k$ of pure dimension 1, it need not be irreducible, reduced or smooth.

A *surface* over a field $k$ is a geometrically integral variety of dimension 2 over $k$.

A *curve on a surface* over a field $k$ is a closed subscheme of the surface which is a curve over $k$.

For a scheme $X$ over a field $k$ we will write $X_K$ for the base change $X \times_k K$ for any field extension $K$ of $k$. The notations $X^{\mathrm{sep}}$ and $\bar{X}$ will be synonymous for $X_{k^{\mathrm{sep}}}$ and $X_{\bar{k}}$, respectively.

For a Cartier divisor $D$ on a scheme $X$ we will denote the associated line bundle by $\mathcal{L}(D)$. It comes with a designated rational section denoted by $1_D$. The rational section $1_D$ extends to a global section precisely if $D$ is effective.

The complete linear system of a line bundle $\mathcal{L}$ on a scheme $X$ is denoted by $|\mathcal{L}|_X$. We might suppress the scheme $X$ in the subscript if it is clear from the context. For a Cartier divisor $D$ we write $|D|$ for $|\mathcal{L}(D)|$.

An integral variety $X$ has a unique generic point $\eta$. The local ring $\mathcal{O}_{X,\eta}$ is a field which is called the *function field* of $X$ and is denoted by $\kappa(X)$.

The *codimension* of a point $x$ on a scheme $X$ is the dimension of the local ring $\mathcal{O}_{X,x}$. The set of all points of codimension $d$ is denoted by $X^{(d)}$.

The canonical bundle on a normal scheme $X$ over a field $k$ will be denoted by $\omega_X$. The associated divisor class is $K_X$, and by abuse of notation we also use this notation to denote a divisor representing this class.

The category of abelian groups and group homomorphisms is denoted by **Ab**. The category of schemes and morphisms is denoted by **Sch**.