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Chapter 3

The Brauer group of ample log K3 surfaces

This chapter is based on the author's article [10] written jointly with Martin Bright.

In the previous chapter we have looked at the arithmetic of del Pezzo surfaces. Let us briefly discuss the more complicated arithmetic of K3 surfaces. These surfaces will not play an important role in this chapter or even this thesis, but we will use them to put this chapter in context.

Let X be a K3 surface over a number field k. Here $\operatorname{Br} \bar{X}$ is infinite, but it was proved by Skorobogatov and Zarhin [49] that the quotient $\operatorname{Br} X / \operatorname{Br}_1 X$ is finite. The question then arises of trying to bound this finite group; there has been quite a body of work on this in recent years. Ieronymou, Skorobogatov and Zarhin proved in [34] that, when X is a diagonal quartic surface over the field $\mathbb{Q}$ of rational numbers, the order of $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$ divides $2^{25} \times 3^2 \times 5^2$. When X is the Kummer surface associated to $E \times E$, with $E/\mathbb{Q}$ an elliptic curve with full complex multiplication, Newton [44] described the odd-order part of $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$. When X is the Kummer surface associated to a curve of genus 2 over a number field k, Cantoral Farfán, Tang, Tanimoto and Visse [11] described an algorithm for computing a bound for $\operatorname{Br} X / \operatorname{Br} k$. More generally, Várilly-Alvarado [56, Conjectures 4.5, 4.6] has conjectured that there should be a uniform bound on $\operatorname{Br} X / \operatorname{Br} k$ for any K3 surface X, depending only on the geometric Picard lattice of the surface. Recent progress towards this conjecture has been made by Várilly-Alvarado and Viray for certain Kummer surfaces associated to non-CM elliptic curves [57, Theorem 1.8] and by Orr and Skorobogatov for K3 surfaces of CM type [45, Corollary C.1].

So far we have been discussing proper varieties. However, non-proper varieties are also of arithmetic interest. A particular case is that of log K3 surfaces; the arithmetic of integral points on log K3 surfaces shows several features analogous to those of rational points on proper K3 surfaces. See [32] for an introduction to the arithmetic of log K3 surfaces. One example of a log K3 surface is the complement of an anticanonical divisor on an ordinary del Pezzo surface, and it is that case with which we concern ourselves in this chapter.

Some calculations of the Brauer groups of such varieties have already appeared in the literature. In [16], Colliot-Thélène and Wittenberg computed explicitly the Brauer group of the complement of a plane section in certain cubic surfaces. In [35], Jahnel and Schindler carried out extensive calculations in the case of an ordinary del Pezzo surface of degree 4. In this chapter, we compute the possible algebraic Brauer groups of these surfaces, and use uniform boundedness of torsion of elliptic curves to bound the possible transcendental Brauer groups, resulting in Theorem 3.3.4, which is the main result of this chapter.

3.1 Ample log K3 surfaces

We defined a surface in Definition 2.1.1 as a geometrically integral variety over a field k of dimension two. In this chapter all surfaces will be smooth over k. By [33, Proposition 6.11] we see that this implies that we can identify Weil and Cartier divisors and we will use the notions interchangeably.

The goal of this chapter is to describe the Brauer groups of certain surfaces over a number field. With this in mind we may restrict to surfaces over a field of characteristic 0 in the general definitions and statements leading up to the main results. Note for example that this is a standing convention in [32].

DEFINITION 3.1.1. Let C be a divisor on a smooth surface X over a field k. Let C_i be the geometrically irreducible components of $\bar{C}$. We say that C has *simple normal crossings* if the following three conditions are satisfied:

» each component C_i is smooth;

» every geometric point x of C lies on at most two components C_i; and

» any two distinct components C_i and C_j meet transversally.

Most of the divisors we will be interested in will only have one geometrically irreducible smooth component. Such divisors obviously have simple normal crossings. We have included this notion to give the general definition of log K3 surfaces.

DEFINITION 3.1.2. Let U be a smooth surface over a field k. A *log K3 structure* on U is a triple (X, C, i) consisting of a proper smooth surface X over k, an effective anticanonical divisor C on X with simple normal crossings and an open embedding $i\colon U \to X$, such that i induces an isomorphism between U and $X\backslash C$. A log K3 structure is called *ample* if C is ample.
A log K3 surface is a simply connected, smooth surface U over k together with a choice of log K3 structure (X, C, i) on U. An *ample log K3 surface* is a surface together with an ample log K3 structure.

We will often use the notation U for $X\backslash C$ and assume the log K3 structure is understood.

PROPOSITION 3.1.3. *Let U be a log K3 surface over a field k with log K3 structure (X, C, i) such that C is not the trivial divisor on X. Then $\bar{X}$ is a rational surface over $\bar{k}$.*

Proof. See Proposition 2.0.18 in [32]. The standing convention in this paper is that k should be of characteristic 0. The proof works for log K3 surface over general fields. □

If C is the trivial divisor then $U \xrightarrow{\cong} X$ is a proper smooth surface. These types of surfaces are precisely the *K3 surfaces* touched upon in the introduction of this chapter. We will however be interested in log K3 structures for which the divisor is not trivial. We will usually even assume that C is an ample divisor. Proposition 3.1.3 shows that in this case the compactification X is an ordinary del Pezzo surface.

DEFINITION 3.1.4. Let $1 \leq d \leq 9$ be an integer. A *log K3 surface of $dP(d)$ type* or an *ample log K3 surface of degree d* is a log K3 surface U with log K3 structure (X, C, i) such that X is an ordinary del Pezzo surface of degree d. If the locally principal subscheme of X associated to the effective divisor C is geometrically irreducible, we say that U is a *log K3 surface of geometrically irreducible type*.

We will be concerned with ample log K3 surfaces, and in particular those of geometrically irreducible type.

Note that if (X, C, i) is a log K3 structure such that C is geometrically irreducible, then $\bar{C}$ has a unique irreducible component which must be smooth by the definition of normal crossing divisor. Equivalently we could assume that C is geometrically integral or geometrically connected.

3.2 The algebraic Brauer group

In the next sections we will compute Brauer groups of certain log K3 surfaces or bound the order of these groups. If an ample log K3 surface is of geometrically irreducible type we can compute the algebraic Brauer group modulo constants using Proposition 1.6.5.

PROPOSITION 3.2.1. *Let k be a number field and $U = X\backslash C$ an ample log K3 surface of geometrically irreducible type of degree d at most 7 over k, i.e. X is an ordinary del Pezzo surface of degree d and $U \subseteq X$ is the complement of a geometrically irreducible curve $C \in |-K_X|$. Then $\mathrm{Br}_1 U/\mathrm{Br}\, k$ depends only on $\mathrm{Pic}\, \bar{X}$ as a Galois module and its order is at most 256. For $d = 1$, the natural map $\mathrm{Br}_1 X \to \mathrm{Br}_1 U$ is an isomorphism. For $2 \leq d \leq 7$, the possible combinations of $\mathrm{Br}\, X/\mathrm{Br}\, k = \mathrm{Br}_1 X/\mathrm{Br}\, k$ and $\mathrm{Br}_1 U/\mathrm{Br}\, k$ are as shown in Table C.*

We will see that the proof does not use the fact that the geometrically irreducible C is smooth. The result is also true for any geometrically irreducible

Degree	$\mathrm{Br}_1 X/\,\mathrm{Br}\,k$	Possibilities for $\mathrm{Br}_1 U/\,\mathrm{Br}\,k$
$\mathbf{d = 7}$	1	1
$\mathbf{d = 6}$	1	$1 \quad 2 \quad 3 \quad 6$
$\mathbf{d = 5}$	1	$1 \quad 5$
$\mathbf{d = 4}$	1	$1 \quad 2 \quad 2^2 \quad 2^3 \quad 2^4 \quad 4 \quad 2 \cdot 4$
	2	$2 \quad 2^2 \quad 2^3 \quad 2^4 \quad 2 \cdot 4$
	2^2	$2^3 \quad 2^4 \quad 2^2 \cdot 4$
$\mathbf{d = 3}$	1	$1 \quad 3 \quad 3^2$
	2	$2 \quad 6$
	2^2	$2^2 \quad 2 \cdot 6$
	3	$3 \quad 3^2$
	3^2	3^3
$\mathbf{d = 2}$	1	$1 \quad 2$
	2	$2 \quad 2^2 \quad 2 \cdot 4 \quad 4$
	2^2	$2^2 \quad 2^3 \quad 2 \cdot 4 \quad 2^2 \cdot 4 \quad 4^2$
	2^3	$2^3 \quad 2^4 \quad 2^2 \cdot 4$
	2^4	$2^4 \quad 2^5 \quad 2^3 \cdot 4$
	2^5	2^6
	2^6	2^7
	3	$3 \quad 6$
	3^2	$3^2 \quad 3 \cdot 6$
	$2 \cdot 4$	$2 \cdot 4 \quad 2^2 \cdot 4 \quad 4^2$
	$2^2 \cdot 4$	$2^2 \cdot 4 \quad 2^3 \cdot 4$
	4	$2 \cdot 4 \quad 4$
	4^2	$2 \cdot 4^2$

Table C: Possible group structures of $\mathrm{Br}_1 U/\,\mathrm{Br}\,k$. The notation is the same as in Table B on page 59. For example, $2 \cdot 4^2$ means $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^2$.

effective anticanonical divisor C on a del Pezzo surface X over a number field k.

Note that our computations agree with those of Jahnel and Schindler [35, Remark 4.7i)] on ordinary del Pezzo surfaces of degree 4.

To prove Proposition 3.2.1 we will use the following lemma.

LEMMA 3.2.2. *Let $U = X \setminus C$ be a log K3 surface of geometrically irreducible type over a number field k of degree $d \leq 7$. Let W be the minimal subgroup of the Weyl group W_{9-d} such that the action of G_k on $\operatorname{Pic} \bar{X}$ factors through the induced action of W on $\operatorname{Pic} \bar{X}$. The group $\operatorname{Br}_1 U / \operatorname{Br} k$ only depends on the conjugacy class of W as a subgroup of W_{9-d}.*

Proof. As C is geometrically irreducible, a section of $\mathbb{G}_m$ on $\bar{U}$ corresponds to a rational function on $\bar{X}$ whose divisor is a multiple of C. The intersection of this principal divisor with C must be zero and we see that $\mathrm{H}^0(\bar{U}, \mathbb{G}_m) = \bar{k}^\times$. This means that we can apply Proposition 1.6.5 to compute the algebraic Brauer group modulo constants.

By definition of W we have a surjective group homomorphism $G_k \twoheadrightarrow W$, such that G_k acts on $\operatorname{Pic} \bar{X}$ through $W \subseteq W_{9-d}$. We also find that the action of the kernel of $G_k \to W$ on $\operatorname{Pic} \bar{X}$ is trivial. Now we will determine the induced action of these groups on $\operatorname{Pic} \bar{U}$. By [33, Proposition II.6.5] we have an exact sequence of Galois modules

$$0 \to \mathbb{Z} \to \operatorname{Pic} \bar{X} \to \operatorname{Pic} \bar{U} \to 0$$

where the first map sends 1 to the anticanonical class in $\operatorname{Pic} \bar{X}$. This shows that the Galois module $\operatorname{Pic} \bar{U}$ only depends on the Galois module $\operatorname{Pic} \bar{X}$, since the class of C is the anticanonical class.

Let G' be the kernel of the group homomorphism $G_k \to W$. The inflation-restriction sequence for the actions of G_k and its normal subgroup G' on $\operatorname{Pic} \bar{U}$ gives the exact sequence

$$0 \to \mathrm{H}^1(G_k/G', (\operatorname{Pic} \bar{U})^{G'}) \to \mathrm{H}^1(G_k, \operatorname{Pic} \bar{U}) \to \mathrm{H}^1(G', \operatorname{Pic} \bar{U})^{G_k/G'}.$$

The kernel G' of $G_k \to W$ acts trivially on $\operatorname{Pic} \bar{X}$ and hence also on its quotient $\operatorname{Pic} \bar{U}$. Since G' is a torsion group and $\operatorname{Pic} \bar{U}$ is a free $\mathbb{Z}$-module we see that $\mathrm{H}^1(G', \operatorname{Pic} \bar{U})$ is trivial. From the minimality of W we derive that $G_k \to W$ is surjective and hence G_k/G' is isomorphic to W. We conclude that

$$\mathrm{H}^1(W, \operatorname{Pic} \bar{U}) \to \mathrm{H}^1(G_k, \operatorname{Pic} \bar{U})$$

is an isomorphism.

Similarly to the proof of Proposition 2.8.6 using [58, Example 6.7.7] we see that the cohomology group $\mathrm{H}^1(W, \operatorname{Pic} \bar{U})$ only depends on the conjugacy class of W in W_{9-d} and by Proposition 1.6.5 this determines $\operatorname{Br}_1 U / \operatorname{Br} k$. □

Proof of Proposition 3.2.1. By the discussion following Proposition 2.3.5 there are essentially finitely many Galois actions on $\operatorname{Pic} \bar{X}$ as each action factors through a unique minimal subgroup W of the finite Weyl group W_{9-d}. Enumerating all possible subgroups W of W_{9-r}, or even just the conjugacy classes of subgroups, and computing the induced actions of W on $\operatorname{Pic}\bar{U}$ allows us to calculate the possible cohomology groups. For MAGMA code to accomplish this calculation, see [39]. In the case $d = 1$, the following lemma, its corollary and the Table B on page 59 spares us from what would be a lengthy calculation. $\square$

LEMMA 3.2.3. *The natural map* $\mathrm{H}^1(k, \operatorname{Pic}\bar{X}) \to \mathrm{H}^1(k, \operatorname{Pic}\bar{U})$ *is injective and the cokernel has exponent dividing δ, where δ is the minimal non-zero value of $|D \cdot K_X|$ for D a divisor on X.*

Proof. Let D be a divisor with $D \cdot K_X = \delta$. As above, we have the exact sequence of Galois modules

$$0 \to \mathbb{Z} \xrightarrow{j} \operatorname{Pic}\bar{X} \to \operatorname{Pic}\bar{U} \to 0.$$

The map $E \mapsto E \cdot D$ gives a map $s\colon \operatorname{Pic}\bar{X} \to \mathbb{Z}$ with the property that $s \circ j$ is multiplication by $-\delta$. Consider the following part of the long exact sequence associated to this short exact sequence:

$$0 \to \mathrm{H}^1(k, \operatorname{Pic}\bar{X}) \to \mathrm{H}^1(k, \operatorname{Pic}\bar{U}) \to \mathrm{H}^2(k, \mathbb{Z}) \underset{s}{\overset{j}{\rightleftarrows}} \mathrm{H}^2(k, \operatorname{Pic}\bar{X})$$

We see that the cokernel of $\mathrm{H}^1(k, \operatorname{Pic}\bar{X}) \to \mathrm{H}^1(k, \operatorname{Pic}\bar{U})$ is isomorphic to the kernel of j, which is contained in the kernel of $s \circ j$; but this map is multiplication by $-\delta$. $\square$

COROLLARY 3.2.4. *If X is an ordinary del Pezzo surface of degree* 1 *or X contains a* -1*-curve defined over k, then the map* $\operatorname{Br} X / \operatorname{Br} k = \operatorname{Br}_1 X / \operatorname{Br} k \to \operatorname{Br}_1 U / \operatorname{Br} k$ *is an isomorphism.*

We draw special attention to the possible algebraic Brauer groups modulo constants for log K3 surfaces of dP_5 type. The following proposition gives a criterion for computing $\operatorname{Br}_1 U / \operatorname{Br} k$.

PROPOSITION 3.2.5. *Let X be a del Pezzo surface of degree* 5 *and let W be the minimal subgroup of W_4 through which the action of G_k on* $\operatorname{Pic}\bar{X}$ *factors. The group* $\operatorname{Br}_1 U / \operatorname{Br} k$ *is cyclic of order* 5 *precisely for W in one conjugacy class of subgroups of W_4. In all other cases* $\operatorname{Br}_1 U / \operatorname{Br} k$ *is trivial.*

One can check that W_4 has 19 conjugacy classes of subgroups and only in one of those cases we find a non-trivial algebraic Brauer group modulo constants. We will study this specific action more in Chapter 4 and use it to produce examples of Brauer–Manin obstructions of order 5 to the integral Hasse principle.

3.3 Uniform bound for the order of the Brauer group

Next we will study the whole Brauer group of ample log K3 surfaces. The Brauer group modulo constants was computed for some instances of these surfaces by

Colliot-Thélène and Wittenberg [16], Jahnel and Schindler [35] and Harpaz [32]. For ample log K3 surfaces U over a number field k we will use the techniques used by Colliot-Thélène and Wittenberg [16] to give a bound for the order of $\operatorname{Br} U / \operatorname{Br} k$ in terms of the degree of k.

We will use the following important result.

PROPOSITION 3.3.1 (Merel). *Let m be a positive number. There exists an effective computable number $N(m)$ such that for an elliptic curve E over a number field of degree m the order of a torsion point in $E(k)$ is bounded by $N(m)$.*

Proof. See [42]. □

This bound will be the basis for practically all bounds in this section. Now let us first look at the case where at least one of the -1-curves on $\bar{X}$ is defined over k. Note that in this situation by Corollary 3.2.4 the image of $\operatorname{Br} X$ in $\operatorname{Br} U$ coincides with the algebraic Brauer group $\operatorname{Br}_1 U$.

LEMMA 3.3.2. *Let $U = X \backslash C$ be an ample log K3 surface of geometrically irreducible type such that a -1-curve $L \subseteq X$ is defined over k. Let m denote the degree $[k : \mathbb{Q}]$. Then the restriction map $\operatorname{Br} X \to \operatorname{Br} U$ is injective, and the order of its cokernel is bounded by $N(m)^2$.*

Proof. Since U of geometrically irreducible type we see that C is geometrically integral. Because C is a strict normal crossings divisor it is smooth, and we have the exact sequence

$$0 \to \operatorname{Br} X \to \operatorname{Br} U \xrightarrow{\partial_C} \mathrm{H}^1(C, \mathbb{Q}/\mathbb{Z})$$

from Proposition 1.7.5. So it is enough to bound the image of the residue map ∂_C.

We have $C \cdot L = 1$ and this implies that $C \cap L$ is geometrically integral zero-dimensional subscheme P on X. This means that we can apply Proposition 1.7.6 and we find the commutative diagram shown in (3.1).

$$\begin{tikzcd} \operatorname{Br} X \arrow[r] \arrow[d] & \operatorname{Br} U \arrow[r, "\partial_C"] \arrow[d] & \mathrm{H}^1(C, \mathbb{Q}/\mathbb{Z}) \arrow[d, "\alpha"] \\ \operatorname{Br} L \arrow[r] & \operatorname{Br}(L \setminus P) \arrow[r, "\partial_C"] & \mathrm{H}^1(P, \mathbb{Q}/\mathbb{Z}) \end{tikzcd} \tag{3.1}$$

We have $L \cong \mathbb{P}^1_k$ and $L \setminus P \cong \mathbb{A}^1_k$, both of which have Brauer group isomorphic to $\operatorname{Br} k$ by Proposition 1.4.12; exactness of the bottom row shows that ∂_P is the zero map. This implies that the image of ∂_C is contained in the kernel of the homomorphism α. The first cohomology group $\mathrm{H}^1(C, \mathbb{Q}/\mathbb{Z})$ classifies cyclic Galois covers of C, and $\ker \alpha$ corresponds to those cyclic Galois covers $D \to C$ for which the fibre above P is a trivial torsor for the structure group $\mathbb{Z}/n\mathbb{Z}$. So we consider such covers whose kernel is a disjoint union of k-points.

We first bound the degree of such a cover. Let $\pi \colon D \to C$ be a cyclic cover of degree n, and suppose that the fibre $F = \pi^{-1}(P)$ is trivial, so that F consists of

n distinct k-points. The Riemann–Hurwitz formula shows that D has genus 1. Pick a point Q in the fibre F. If we regard D and C as elliptic curves with base points Q and P respectively, then π is an isogeny of elliptic curves, and $F(k) = \ker \pi$ is a cyclic subgroup of order n in $D(k)$. In particular, since D is an elliptic curve over k with a point of order n, we have $n \leq N(m)$.

We now fix n to be the maximal order of an element of $\ker \alpha$. The exponent of a finite abelian group is equal to the maximal order of its elements, so every element of $\ker \alpha$ has order dividing n.

Looking at the long exact sequence in cohomology associated to the short exact sequence of sheaves

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \xrightarrow{\times n} \mathbb{Q}/\mathbb{Z} \to 0$$

shows that the natural map $\mathrm{H}^1(C, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^1(C, \mathbb{Q}/\mathbb{Z})$ is injective. This identifies $\mathrm{H}^1(C, \mathbb{Z}/n\mathbb{Z})$ with the n-torsion in $\mathrm{H}^1(C, \mathbb{Q}/\mathbb{Z})$, which contains $\ker \alpha$. The Hochschild–Serre spectral sequence gives a short exact sequence

$$0 \to \mathrm{H}^1(k, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\beta} \mathrm{H}^1(C, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^1(\bar{C}, \mathbb{Z}/n\mathbb{Z})$$

in which the map α induces a left inverse to β. Thus $\ker \alpha$ is identified with a subgroup of $\mathrm{H}^1(\bar{C}, \mathbb{Z}/n\mathbb{Z})$, which is isomorphic to the n-torsion in $\operatorname{Pic} \bar{C}$ and so has order n^2. Combining this with the above bound on n gives the claimed bound. □

COROLLARY 3.3.3. *Under the conditions of Lemma 3.3.2, the order of* $\operatorname{Br} U / \operatorname{Br} k$ *is bounded by* $2^6 N(m)^2$.

Proof. If we blow down the -1-curve on X we find a del Pezzo surface X' of degree $d+1$, such that $\operatorname{Br} X \cong \operatorname{Br} X'$. So we see in Table B on page 59 that $\#(\operatorname{Br} X / \operatorname{Br} k) = \#(\operatorname{Br} X' / \operatorname{Br} k) \leq 64$ since $\deg X' \geq 2$. Corollary 3.2.4 gives an isomorphism $\operatorname{Br} X / \operatorname{Br} k \cong \operatorname{Br}_1 U / \operatorname{Br} k$. Combining this with Lemma 3.3.2 gives the bound for $\operatorname{Br} U / \operatorname{Br} k$. □

Now we can prove a bound for a general ample log K3 surface of geometrically irreducible type over a number field.

THEOREM 3.3.4. *Let k be a number field of degree m. For an ample log K3 surface of geometrically irreducible type $U = X \backslash C$ of degree d at most 7 the order of* $\operatorname{Br} U / \operatorname{Br} k$ *is bounded by*

$$2^{14} N(240m)^2.$$

Proof. Let K be a finite extension of k such that at least one -1-curve L on $\bar{X}$ is defined over K. The orbit-stabilizer theorem shows that we can always take the degree $[K : k]$ no larger than the number of -1-curves on $\bar{X}$. Since the maximal number of -1-curves on a del Pezzo surface is 240, we find $[K : \mathbb{Q}] \leq 240m$.

By Corollary 3.3.3 we have a bound on $\operatorname{Br} U_K / \operatorname{Br} K$. On the other hand, the kernel of the morphism $\operatorname{Br} U / \operatorname{Br} k \to \operatorname{Br} U_K / \operatorname{Br} K$ is contained in $\operatorname{Br}_1 U / \operatorname{Br} k$ and hence bounded by 256 by Proposition 3.2.1.

Combining these two bounds, we find that

$$\#\left(\operatorname{Br} U / \operatorname{Br} k\right) < 2^{14} N(240m)^2. \qquad \square$$

Remark. There are, of course, many ways in which the constants appearing in this bound could be improved, especially if we were to separate the various different degrees. For example, the group $\operatorname{Br} U_K / \operatorname{Br} K$ and the kernel of the homomorphism $\operatorname{Br}_1 U / \operatorname{Br} k \to \operatorname{Br}_1 U_K / \operatorname{Br} K$ are far from independent. Our interest here has been in showing the existence of a uniform bound, rather than in making that bound as small as possible.