Distribution of relativistic particles behind magnetized mixed shocks. Application to synchrotron radio sources

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Summary. Introduction of moderate magnetic corrections to the theory of mixed shocks gives rise to new constraints on the structure of the shock and interesting corrections to the radio index of the synchrotron spectrum. Whereas we found a wide range of possible mixed shocks in our previous work (Pelletier and Roland, 1986) in the limit $M_s \to \infty$, our present analysis shows that the finite Alfvenic Mach number effects involve an unavoidable heating significantly greater than the unmagnetized case. When the magnetic field is quasi parallel, the streaming instability in the precursor is responsible for the increase of the heating ratio and consequently the efficiency of the shock in converting kinetic power into luminosity is lowered. When the magnetic field is quasi perpendicular, one cannot avoid that a heating occurs with the growth of the magnetic pressure. One of the most interesting result of this work is that we extend the validity of the spectral index formula given by the linear theory to the most common astrophysical situations and we show in such case the result does not depend on the diffusion coefficient and thus its energy dependence does not affect the radio spectral index. It turns out that the corrections in the quasi parallel shock are more sensitive than in the quasi perpendicular shock. In the latter case, the radio spectral index remains smaller than 0.6, whereas in the former it can easily reach the value 0.8. Regarding the astrophysical objects our theory of the spectral radio index is a suitable key to get a diagnosis of the medium responsible for the synchrotron emission. We show how it is possible to use the radio spectral index to make a diagnosis of young supernova remnants with a flat radio spectral index, radio active galactic nuclei, hot spots similar to those of Cygnus A and finally hot spots with a very steep radio spectral index.

Key words: synchrotron radio sources—shock waves—particle acceleration

1. Introduction

The acceleration of suprathermal particles by repeated first order Fermi process through a shock in a diffusive medium proposed ten years ago by several authors (Axford et al., 1977, Krimsky, 1977, Bell, 1978 and Blandford and Ostriker, 1978) to explain the formation of cosmic rays distributions, has been widely accepted and studied by many authors.

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The pioneering works developed a linear theory to predict the power law spectra, neglecting the back reaction of the accelerated particles on the flow. However the efficiency of the acceleration process renders the pressure of the accelerated particles too easily dominant compared to the thermal pressure, requiring a nonlinear treatment. The first nonlinear theory of the mixed shock structure has been performed by Drury and Völk (1981) and a spectral theory has been derived in the extreme case where the thermal pressure is negligible by Drury et al. (1982) assuming an energy independent diffusion coefficient.

The understanding of the nonlinear structure of the two fluid shocks is an important task which has recently progressed. However, several points emphasized by Achterberg et al. (1984) remain open. Numerical works will surely bring important insights as those soon available do (Ellison and Eichler, 1984, 1985; Drury and Falle, 1986; Achterberg, 1987 and Falle and Giddings, 1987). But the existence of three very different scales in this problem, namely: the subshock depth, the diffusion length of the relativistic particles and the typical loss length, makes a fair computation, especially that of the prediction of the spectrum, difficult. However, one can take advantage of these scale contrasts to simplify the theory and to progress toward simple reliable formulae and improve the remarkable result of the linear theory or at least look at its validity in a situation where the back reaction of the accelerated particles on the shock structure is important.

We are motivated to use these theories to describe the physics of hot spots of extragalactic radio sources similar to Cygnus A. Indeed, these intense synchrotron emission regions are considered as the downstream flow of a strong shock suffered by the jet coming from the nucleus of the galaxy which interacts with the intergalactic medium and the synchrotron emission results from the conversion of a part of the kinetic power into luminosity in the shock through the acceleration of relativistic electrons.

In a previous paper (Pelletier and Roland, 1986), we developed a spectral theory of mixed shocks parametrized by the ratio $\theta$ between the classical pressure $p_c$ and the relativistic one $p_r$, namely $\theta \equiv p_c/p_r$.

We showed the shock structure can be simply described as long as the ratio $\delta$ between the subshock width $\delta_s$ and the diffusion length $\delta$, of the relativistic particles, namely $\delta = \delta_s/\delta$, is small. The mixed shock structure exists for $\theta_{\min} < \theta < \theta_{\max}$ with $\theta_{\min} \sim \delta$ and $\theta_{\max} \sim \delta^{-1}$. When $\theta$ becomes smaller than $\theta_{\min}$, we
get the purely "radiative" smooth transition considered by Drury and Völk (1981). We derived the spectral index for arbitrary value of \( \theta \), owing to a method that we will briefly recall in Sect. 2, and found again the results soon obtained in the two extreme limits, \( \theta \rightarrow 0 \) (Drury et al., 1982) and \( \theta \rightarrow \infty \) (the linear theory). It turns out that the index of the post shock energy distribution remains close to 2 like in the two extreme cases mentioned above and even becomes smaller than 2, as in Blandford's perturbation theory (Blandford, 1980), reaching a minimum equal to 1.5 for \( \theta \approx 10^{-1} \). Thus losses are necessarily invoked to cut the spectrum at high energies. In the case of hot spots, the relevant dissipation is the synchrotron radiation. However, synchrotron losses have a negligible effect on the shock structure which can be considered as adiabatic and the synchrotron luminosity of the hot spot is only a small fraction of the kinetic power of the jet. This analysis was performed under the assumption that the magnetic field can be ignored in first approximation. But we were able to construct a closed system providing us with the hot spot plasma parameters and it turned out that the magnetic pressure, in spite that it is not in equipartition with the particle pressure, has a significant contribution.

In this paper, we will assume that the shock wave is perpendicular to the jet and we will calculate the magnetic corrections to the spectral theory for two opposite situations, first when the magnetic field is quasi parallel to the flow (Sect. 3), second when it is quasi perpendicular (Sect. 4). We disregard the case of perpendicular shocks where the de Hoffman–Teller transformation is impossible.

In the case of a quasi parallel magnetic field, the amplification of the magnetic field pressure can be neglected, but the streaming instability develops in the precursor which heats the thermal plasma and thus weakens the acceleration efficiency (Völk and McKenzie, 1982; Lagage and Cesarsky, 1983a, b; and Völk et al., 1984). This will provide anomalous corrections proportional to \( 1/M_s \) (\( M_s \) being the Alfvénic Mach number) instead of usual corrections in \( 1/M^2 \) to the compression ratios and the spectral index.

In the case of quasi perpendicular magnetic field, the streaming instability is negligible, but the amplification of the magnetic pressure is the most important effect. We limit our theory to situations where the growth of the magnetic pressure does not produce particle reflection in the precursor. It turns out that this limitation is not severe.

Regarding the astrophysical objects our theory of the spectral radio index is a suitable key to get a diagnosis of the medium responsible for the synchrotron emission. In Sect. 5, we show how it is possible from the radio spectral index to make a diagnosis in the case of young supernova remnants with a flat radio spectral index, in radioactive galactic nuclei, in hot spots similar to those of Cygnus A and finally in hot spots with a very steep radio spectral index.

2. The method to calculate the spectral index including the magnetic effects

First let us set the basic assumptions and steps of the general theory (Pelletier and Roland, 1986).

(i) The shock is adiabatic, i.e. losses do not perturb its structure. The losses do not affect the formation of the spectrum of the relativistic particles on a scale length of order of the diffusion one (\( l_{\text{diff}} \ll l_{\text{rms}} \)). These assumptions are valid for shocks responsible for the formation of hot spots.

(ii) The shock is a mixed one. So we have two pressures, namely \( p_e \) and \( p_s \) which describe respectively the classical and the relativistic components. The ratio \( p_e/p_s = \theta \), is arbitrary. We will distinguish the velocity \( v \) of the thermal flow carrying most of the mass and the velocity \( u \) of the scattering centers, responsible for the acceleration of the relativistic particles. Throughout the paper, we deal with "classical" hydrodynamics, in the sense of non relativistic hydrodynamics, i.e. \( v < c/\sqrt{3} \). However part of the pressure is due to relativistic particles, namely \( p_s \), and the non relativistic particles are responsible for a classical pressure \( p_e \).

(iii) The parameter \( \delta \equiv \delta_e/\delta_s \), will be supposed very small, i.e. \( \delta \ll 1 \), where \( \delta_s \) is the diffusion length of the classical particles and \( \delta_e \) the diffusion length of the relativistic ones.

(iv) The momentum distribution \( f(p,x) \) of the relativistic electrons related to the energy distribution \( \rho(E,x) \) by \( p^2 f(p,x) \propto \rho(E,x) \) is governed by the following transport equation (Skilling, 1975; Ginzburg and Ptuskin, 1985), when loss effects are disregarded and the flow is unidirectional

\[
\frac{\partial}{\partial t} f + u \frac{\partial}{\partial x} f = \frac{1}{3} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial p} f + \frac{\partial}{\partial x} \left( D_e \frac{\partial}{\partial x} f \right) \right)
\]

In the following we will disregard the energy dependence of the diffusion coefficient \( D_e \), and so the diffusivity \( \kappa_e \), is

\[
\kappa_e \equiv \left( \int \rho(E) D_e E dE \right) / \int \rho(E) E dE \approx D_e
\]

In fact this assumption will not change the final result as long as \( \delta \) is very small and \( \theta \approx 10^{-1} \) as we will see in Sect. 3.4 and 4.7.

Moreover in the Galaxy the diffusion coefficient is constant for electrons with an energy \( E \leq 1 \) GeV (Ginzburg and Ptuskin, 1985), and the approximation \( \kappa_e \approx D_e \) is reasonably good for the study of shocks accelerating the electrons responsible for the radio emission.

As every workers in this field, we make use of the Skilling equation in spite of the fact that we do not know if it is still valid for \( \delta \ll 1 \), since it has been established in the case of the opposite assumption.

(v) We will assume that the transition is a monotonic one, i.e. the shock does not oscillate, or that oscillations are small enough for considering the quantities as average ones.

In fact the subshock oscillates but its detailed structure is unimportant in the limit \( \delta \ll 1 \). In such case we can define a function \( \phi \) of the velocity field of the scattering centers, such that

\[
\frac{\partial \phi}{\partial x} = \phi(u)
\]

The function \( \phi \) is derived from the fluid equations and describes the structure of the mixed shock. We will call it the structure function. Thus the stationary Skilling equation, using the variable \( u \) instead of the space variable \( x \), can be written

\[
-1 \frac{1}{3} \frac{\partial}{\partial p} f = -u \frac{\partial}{\partial u} f + \frac{\partial}{\partial u} \left( D_e \frac{\partial}{\partial u} f + \frac{\partial}{\partial u} \left( D_e \frac{\partial}{\partial u} f \right) \right)
\]

As indicated in (iv) when the diffusion coefficient is independent of the energy, we have \( D_e = \kappa_e \).

(vi) The structure function \( \phi \) vanishes at \( u_s \) and \( u_i \), where indices 1 and 2 designate upstream and downstream quantities.
and can be written

$$\phi(u) \equiv g(u)(u-u_1)(u-u_2)$$  \hspace{1cm} (4)

with \( g(u) > 0 \) on \([u_2, u_1]\).

The distribution of the relativistic electrons of energy sufficiently greater than the threshold energy, namely \( E_{\text{thres}} \equiv m_e c V_{A2} \epsilon \), is of the form

$$\rho(E) \equiv KE^{-\eta_0}$$  \hspace{1cm} (5)

where \( \eta_0 \) is related to the eigenvalue \( \lambda_0 \) of the fundamental state of the operator

$$D = -\frac{\partial}{\partial u} + \frac{\partial}{\partial u} \phi \frac{\partial}{\partial u}$$  \hspace{1cm} (6a)

by the relation \( \eta_0 = 3\lambda_0 - 2 \). The operator \( D \) is self-adjoint for a unique density \( \omega_0 \) defined within a normalizing arbitrary constant factor, such that

$$u \omega_0 + \phi \frac{\partial \omega_0}{\partial u} = 0.$$  \hspace{1cm} (6b)

The energy spectral index \( \eta_0 \) is therefore

$$\eta_0 = \frac{3\gamma_2}{r_1 - 1} \left( 1 + \frac{1}{a(r_2 - 1)} \right) - 2,$$  \hspace{1cm} (7)

where \( r_1 = u_1/u_2 \) and \( a \) defined by \( g(u) = a(1 + \epsilon h(u)) \) with \( \epsilon < 1 \), is the mean value of the function \( g(u) \) such that the first order correction vanishes in the perturbation theory using the small parameter \( \epsilon \). Therefore \( a \) is the proper average of \( g(u) \)

$$a = \frac{\int_{u_2}^{u_1} g(u)(u-u_2)^\epsilon_\epsilon u(u-u_2)^\epsilon u \, du}{\int_{u_2}^{u_1} (u-u_2)^\epsilon u(u-u_2)^\epsilon u \, du}$$  \hspace{1cm} (9)

for the unperturbed density \( \omega_0 \)

$$\omega_0(u) \propto (u-u_2)^\epsilon u(u-u_1)^\epsilon u$$  \hspace{1cm} (10)

with

$$\alpha = -\frac{1}{r_1 - 1} a$$  \hspace{1cm} (11a)

$$\beta = r_1 - 1 \frac{1}{a}$$  \hspace{1cm} (11b)

see respectively Eqs (68), (69), (67), (58) and (57) of Pelletier and Roland, 1986.

When we have no magnetic corrections we have

$$a = \frac{\gamma_2 + 1}{2} \frac{7}{6} \leq a \leq \frac{\gamma_2 + 1}{2} = \frac{4}{3} \delta.$$

(vii) The magnetic field of the jet is supposed to be such that the Alfvénic Mach number \( M_A \) is greater than unity in the upstream flow and \( M_A^{-1} \) will be used as a small expansion parameter.

In this paper, we will calculate the magnetic corrections to the spectral index \( \eta_0 \), when the magnetic field is either quasi parallel to the flow, or quasi perpendicular to it. In both cases we assume the front shock to be perpendicular to the flow and do not study in this article oblique shocks.

When the magnetic field is quasi parallel to the jet, the streaming instability introduces a difference between the velocities of the hydrodynamical flow and of the scattering centers in the precursor i.e. \( u \neq v \) and then anomalous magnetic corrections, proportional to \( 1/M_A \), occur. The difference between \( u \) and \( v \) introduces a difference between the compression ratio of the scattering centers and the compression ratio of the hydrodynamical flow, i.e. \( r_s \neq r \). Then it introduces a modification of functions \( g \) and \( a \) and we want to calculate the function \( \eta(\theta, 1/M_A) \).

When the magnetic field is quasi perpendicular to the jet the velocities of the hydrodynamical flow and of the scattering centers are the same, i.e. \( u = v \), but the jump of the magnetic pressure introduces corrections proportional to \( (r/M_A)^2 \) in the compression ratio. It follows that functions \( g \) and \( a \) are also modified and we want to calculate the function \( \eta(\theta, 1/M_A) \).

In summary, we can say that magnetic corrections are given by the hydrodynamic modified on one hand by the streaming instability when the magnetic field is quasi parallel to the jet, and on the other hand by the magnetic pressure when the magnetic field is quasi perpendicular to the jet.

3. Magnetic field quasi-parallel to the jet: the streaming instability effect

3.1. The basic equations

3.1.1. The system of equations

When the magnetic field is quasi-parallel to the flow, there is a difference between the scattering center velocity \( u \) and the hydrodynamical velocity \( v \). This difference in the precursor is due to the streaming instability which excites backward Alfvén waves when \( v > V_A \) where \( V_A \) is the Alfvén velocity. When such are the conditions, the scattering frequency \( v_+ \) of the backward Alfvén waves is greater than the scattering frequency on the forward Alfvén waves \( v_- \), i.e. \( v_+ > v_- \) and then the difference between \( u \) and \( v \) (Skilling, 1975) is given by

$$u = v + \frac{3}{2} \frac{1 - \mu^2}{v_+ - v_-} V_A,$$  \hspace{1cm} (12)

where \( \mu \) is the pitch angle cosine. Thus, we have

$$u_1 \simeq v_1 - V_A.$$

This difference disappears in the downstream flow, where we have

$$u_2 = v_2.$$  \hspace{1cm} (13)

The cosmic ray stream looses momentum, which is given to the backward Alfvén waves which in turn heat the thermal plasma. The mediation of the Alfvén waves being short, the transfer from the cosmic rays to the thermal plasma can be written directly (McKenzie and Völk, 1982) and the basic system of equations is the following.

The mass conservation

$$J = \rho v$$  \hspace{1cm} (15)

The momentum conservation

$$F = Jv + p_x + p_y,$$  \hspace{1cm} (16)
because the magnetic pressure is constant through the shock and the Alfvén wave pressure negligible.

The thermal component experiences a compression and a heating described by the following equation

\[ \gamma_e p_e \frac{dv}{dx} + u \frac{dp_e}{dx} = -\kappa_e \frac{d}{dx} p_e + (\gamma_e - 1) V_0 \frac{dp_e}{dx}, \]  

(17)

where the last term \((\gamma_e - 1) V_0 \frac{dp_e}{dx}\) represents the anomalous heating by the cosmic rays via the streaming instability (McKenzie and Völk, 1982).

The compression of the relativistic component is governed by the following equation

\[ \gamma_e p_v \frac{dv}{dx} + u \frac{dp_v}{dx} = \frac{d}{dx} \frac{d}{dx} \kappa_r \frac{dp_v}{dx}, \]  

(18)

derived from the Skilling equation.

So we set

\[ u = v - V \]  

(19)

with \(V = V_{A1}\) in the upstream flow and \(V = 0\) downstream.

In the jet we have

\[ M_\Lambda = \frac{v_1}{V_{A1}}. \]

Using the definition of the structure function \(\phi, (2)\), (19) and \(u\) as the new variable, the previous relations become

\[ F = J(u + V) + p_e + p_v \]  

(20)

\[ \gamma_e p_e \frac{d(u + V)}{du} + (u + V) \frac{dp_e}{du} = \delta \frac{d}{du} \phi \frac{dp_e}{du} + (\gamma_e - 1) V_0 \frac{dp_e}{du}, \]  

(21)

\[ \gamma_v p_v + u \frac{dp_v}{du} = \frac{d}{du} \phi \frac{dp_v}{du}, \]  

(22)

where

\[ \delta \equiv \delta_c/\delta_r = \kappa_e/\kappa_r. \]

3.1.2. Relation between \(\beta_2\) and \(M_\Lambda\) in the strong shock limit

A useful and important relation is the relation between the Alfvénic Mach number in the jet \(M_\Lambda\) and the parameter \(\beta_2\) in the downstream flow, defined by \(\beta_2 = (p_{e1} + p_{v1})/p_{e2}\). From definitions of \(M_\Lambda\) and \(\beta_2\) and from relation (16), in the strong shock limit, we have

\[ M_\Lambda^2 \approx \frac{r}{2(r - 1)}, \]  

(23a)

3.1.3. The compression ratios in the strong shock limit

In the strong shock limit, the compression ratio \(r = v_2/v_1\) of the hydrodynamical flow is (see Appendix A)

\[ r \approx \frac{r_0}{1 + \frac{r_0}{r_0 - 1} 3M_\Lambda^3} \left(1 - \frac{r_0}{r_0 - 1} \frac{3}{2}\right). \]  

(23b)

In (23b), \(r\) contains the first order term of the expansion in \(M_\Lambda^{-1}\). The indices 0 refer to the zeroth order values given by (23c) and (23d). Using respectively \(\gamma_e = 4/3\) and \(\gamma_c = 5/3\), for the relativistic and the classical gas, we obtain

\[ r_0 = \frac{\gamma_e + 1 + (\gamma_e + 1)/2}{\gamma_e - 1 + (\gamma_e - 1)/2} = \frac{7 + 4\theta}{1 + \theta} \]  

(23c)

and

\[ r_c = \frac{1 + \theta r_0}{1 + \theta} \]  

(23d)

where \(r_0\) and \(r_c\) would be respectively the global compression ratio and the compression ratio of the subshock if the magnetic effect would be neglected. It turns out that the magnetic correction in (23b) is small provided that \(M_\Lambda \geq 5\) (see Sect. 3.4).

But the compression ratio experienced by the scattering centers is

\[ r_s = \frac{u_1}{u_2} = \frac{v_1 - V_{A1}}{v_2} = r \left(1 - \frac{1}{M_\Lambda}\right) \]  

(24)

3.2. The structure of the precursor

3.2.1. Derivation of the structure function \(\phi\)

The relativistic pressure causes a slowing down of the flow from \(v_1\) to a value \(v_0\), before the sharp thermal transition where the velocity suddenly decrease from \(v_0\) to \(v_2\).

In the precursor \(v_1 \geq v > v_0\) or \(u_1 = v_1 - V_1 \geq u > u_0 = v_0 - V_0\), the adiabatic growth of the classical pressure is small compared to the growth of the relativistic one and we can keep a linear approximation for its dependence on \(u\). In the same manner we will assume a linear dependence on \(u\) for \(V\) and according to (20) the dependence of \(p_e\) is also a linear one on \(u\). So

\[ p_e = p_{e1} + \alpha_1 J(u_1 - u) \]  

(25)

\[ V = V_{1} - \alpha_1 (u_1 - u) \]  

(26)

\[ p_r = p_{r1} + (1 - \alpha_1 + \alpha_1^2) J(u_1 - u). \]  

(27)

Inserting (27) into (22), we find

\[ \frac{d}{du} \phi = \frac{\gamma_r + 1}{(1 - \alpha_1 + \alpha_1^2) J - \gamma_r u_1}. \]  

(28)

So, writing

\[ \phi = \phi_1 \equiv \alpha_1 (u - u_1)(u - u_1^2) \]  

(29)

which vanishes in \(u_1\) and \(u_1^2\) (Fig. 1a), we deduce

\[ a_r = \frac{\gamma_r + 1}{2} \]  

(30)

![Fig. 1a. The shape of the structure function \(\phi\), assuming \(\delta_c/\delta_r < 1\)](image-url)
3.3. The structure of the subshock

3.3.1. Derivation of the structure function $\phi_c$

In the subshock $v_0 > v \geq v_2$, the function $V$ vanishes rapidly as $v \rightarrow v_0$. Anyway we do not pretend to describe the detailed structure of the subshock and to simplify the calculation, we will take a linear variation of $V$ and we will take the limit $\delta \rightarrow 0$. In addition, the relativistic pressure varies slowly and we can write

$$V = -\alpha_1 (u_2 - u)$$  \hspace{1cm} (36)

$$p_r = p_{r2} + \alpha_2 J(u_2 - u)$$  \hspace{1cm} (37)

and from (20)

$$p_c = p_{c2} + (1 - \alpha_2 + \alpha_2') J(u_2 - u)$$  \hspace{1cm} (38)

writing

$$\phi = \phi_2 = a_c (u - u_r^*) (u - u_2)$$  \hspace{1cm} (39)

which vanishes in $u_r^*$ and $u_2$ (Fig. 1a), and inserting (38) into (21) we obtain

$$a_c = \frac{\gamma_c - 1}{2 \alpha} (1 + \alpha_2')$$  \hspace{1cm} (40)

The fictitious upstream velocity $u_r^*$ is the value that would have the upstream velocity for a thermal shock having a Mach number approximately determined by the thermal pressure before the subshock.

3.3.2. Determination of the parameters $\alpha_2$ and $\alpha'_2$

In a first step, it is simpler to calculate the parameters $u'_2$ and $\alpha_2$. From (36) and (33), we can write

$$V_0 = -\alpha_2 (u_2 - u_0) = V_r \left( \frac{u_0 + V_0}{u_1 + V_1} \right)^{1/2}$$

using (36) once more and defining

$$r_c = u_0 / u_2$$  \hspace{1cm} (41)

we get

$$\alpha_2' = \frac{1}{M_2} \left( \frac{r_c \rho_0}{r_c \rho_0 - 1} \right)^{1/2}$$  \hspace{1cm} (42)

As we will see further (Sect. 3.3.5 and Table 1, Sect. 3.4) when the Alfvénic Mach number has a finite value, the parameter $\theta$ has to be greater than a minimum value $\theta_\text{min}$ given by (58). When $\theta \rightarrow \theta_\text{min}$, $r_\text{c} \rightarrow 1$ and $\alpha'_2 \rightarrow \infty$ but the jump of $p_c$ given by (38) remains finite and no singularity appears.

Now, we can calculate $\alpha_2$ inserting (37) into (22) and taking $u = u_2$. We find

$$\alpha_2 = \frac{\gamma_0 p_{r2}}{J u_2 (r_\ast - 1) + 1}$$  \hspace{1cm} (44)

where

$$r_\ast = u_r^*/u_2.$$  \hspace{1cm} (45)

In the strong shock limit, from (20) we have

$$J u_2 + p_{c2} + p_{r2} = J r_\ast u_2$$  \hspace{1cm} (46)

or

$$p_{r2} = \frac{(r_\ast - 1) J u_2}{1 + \theta}$$  \hspace{1cm} (47)
and finally we deduce for $\alpha_2$

$$\alpha_2 = \frac{\gamma_e (r_s - 1)}{(1 + \theta) \frac{\gamma_e + 1}{2} (1 + \alpha_2)(r_{**} - 1) + 1}. \quad (48)$$

As we will see Sect. 3.3.6, $r_{**} \approx r_e$ and when \( M_A \to \infty \), $\alpha_2 \to 0$ and moreover when $\theta \to 0$, $r_{**} \to 1$ and a finite minimum value of $\theta$ is found, such that $\alpha_2 = 1$. It is

$$\theta_{\min} \approx \delta. \quad (49)$$

If $M_A$ has a finite value, when $\theta \to 0$, $\alpha_2 \to \delta \theta$ and when $\theta \to \infty$, $\alpha_2 \to 1$. So when $\delta \ll 1$, we have always $\alpha_2 \ll \alpha'_2$.

Since we will take $\delta \approx 10^{-4}$ from the numerical application, when $\theta \geq 10^{-3}$ we will further neglect $\alpha_2$ compared to $\alpha'_2$.

3.3.3. Derivation of the compression ratio $r_{**}$

The function $\phi_e$ defined by (39) will be completely determined if we calculate $r_{**}$ defined by (45).

Using the simplification $\alpha_2 \ll \alpha'_2$, we can calculate $r_{**}$ inserting (38) into (21). We get

$$\frac{\gamma_e + 1}{2} r_{**} = \frac{\gamma_e - 1}{2} + \alpha'_2 \frac{\gamma_e p_{c2}}{\gamma_e p_{c2}} \frac{1}{1 + \alpha'_2}. \quad (50)$$

We can transform the last term of (50) using (38) and (25) and we obtain

$$r_{c2} = p_{c1} + \Delta p_{c} + (1 + \alpha'_2)J(u_1^* - u_2)$$

with

$$\Delta p_{c} = \alpha_1 J(u_1 - u_0) - (1 + \alpha'_2)J(u_1^* - u_0)$$

and then, (50) becomes

$$\frac{\gamma_e + 1}{2} r_{**} = \frac{\gamma_e - 1}{2} + \alpha'_2 \frac{\gamma_e p_{c2}}{\gamma_e p_{c2}} \frac{1}{1 + \alpha'_2} + \gamma_e (r_{**} - 1)$$

and then

$$r_{**} = \frac{\gamma_e + 1 - 2\alpha'_2}{\gamma_e - 1 + 2/M_{**}} \quad (51)$$

with

$$M_{**}^2 = \frac{J u_1^*(1 + \alpha'_2)}{\gamma_e p_{c1} + \gamma_e \Delta p_{c}}. \quad (52)$$

Let us remark that contrary to $r_e$ (32), which is always equal to 7 in the strong shock limit, the compression ratio $r_{**}$, is within the range $[1, 4]$ in the strong shock limit when $M_A \to \infty$.

3.3.4. Derivation of the compression ratio $r_e$

If we neglect $\alpha_2$ compare to $\alpha'_2$, the compression ratio $r_e$ can be calculated from the continuity of the pressures at $u_0$, using (25), (38), (27) and (37), namely

$$p_{c0} = p_{c1} + \alpha_1 J(u_1 - u_0) = p_{c2} + (1 + \alpha'_2)J(u_2 - u_0) \quad (53)$$

$$p_0 = p_{c1} + (1 - \alpha_1 + \alpha'_1)J(u_1 - u_0) \approx p_{c2}. \quad (54)$$

So in the strong shock limit we have

$$\theta = \frac{p_{c2}}{p_{c1}} \approx \frac{\alpha_1 (r_e - r_s) + (1 + \alpha'_2)(r_e - 1)}{(1 - \alpha_1 + \alpha'_1)(r_e - r_s)} \quad (55)$$

and thus

$$r_e \simeq \frac{1 + \alpha'_2 - \alpha_1 r_s + \theta(1 - \alpha_1 + \alpha'_1) r_e}{1 + \alpha'_2 - \theta(1 - \alpha_1 + \alpha'_1) r_e}. \quad (56)$$

Using (24), (34), (35) and (42) we finally obtain

$$r_e \simeq \frac{1 + \theta + \frac{1}{M_A} \left( \frac{r_0 r_{c0}}{r_{c0} - 1}^{1/2} - \gamma_e (1 - \gamma_e)^{-1} \right) \frac{r_0 - 2}{2} \right]}{1 + \theta + \frac{1}{M_A} \left( \frac{r_0 r_{c0}}{r_{c0} - 1}^{1/2} - \gamma_e (1 - \gamma_e)^{-1} \right) \frac{r_0 - 3}{2} \right]} \quad (57)$$

When $\theta \to \infty$, $r_e \to r(1 - 1/M_A)$ and then the preheating instability decreases the compression ratio of the classical subshock.

3.3.5. The minimum value of the heating ratio $\theta$ due to the magnetic corrections

In astrophysical conditions $M_A$ has a rather limited value (e.g. $M_A \approx 6$ in the jets powering Cygnus A hot spots) and this has an important consequence for the downstream flow.

Let us take into account a finite value for $M_A$. From (55) we have a minimum value of $\theta_{\min}$ of $\theta$ such that $r_e = 1$:

$$\theta_{\min} \approx \frac{\alpha_1 (r_e - 1) + \alpha'_2 (r_e - 1)}{(1 - \alpha_1 + \alpha'_1) r_e - 1} \approx \frac{2}{3 M_A + 1} \frac{1}{M_A r_0 - 1} \quad (58)$$

Since we must have $r_e \geq 1$, we must also have $\theta \geq \theta_{\min}$. The structure function $\phi$ is shown in Fig. 1b when $\theta \geq \theta_{\min}$, i.e. $r_e \approx 1$.

As an example for $M_A = 5$ we have $\theta_{\min} \approx 1.2$, which is a rather high lower bound.

So we come to the following conclusion: when one takes into account the Alfvenic Mach number of the upstream flow, the ratio between the classical thermal pressure and the relativistic pressure is greater than some significant critical value in the downstream flow. This value is given by (58). For different $M_A$ the corresponding values of $\theta_{\min}$ are given in Table 1 (See Sect. 3.4).

3.3.6. Relation between $r_{**}$ and $r_e$

We have seen that $r_{**}$ is not equal to 4 and moreover not a constant. But relations (51) and (52) are not convenient for a numerical application. The continuity of the function $\phi$ at $u_0$, namely $\phi_e(u_0) = \phi_o(u_0)$ provides the relation between $r_{**}$ and $r_e$.

Indeed, we have

$$a_e(r_e - r_{**})(r_e - r_{**}) = a_e(r_e - r_{**})(r_e - 1). \quad (59)$$

So we get

$$r_{**} = r_e + \frac{r_e - r_{**}}{r_{**} - r_e} \quad (60)$$

When $\delta \to 0$, $r_e \to \infty$ and we have $r_{**} = r_e$.

For numerical application, when $\delta \approx 10^{-4}$, (60) shows that we can take $r_{**} = r_e$.

3.4. The spectral index

We can now calculate the spectral index $\theta_0$ of the energy distribution of the relativistic electrons as a function of $\theta$ and $1/M_A$. It
is given by \( \eta_0 = 3\lambda_0 - 2 \), where \( \lambda_0 \) is the eigenvalue of the fundamental state of the operator \( D \) which is self-adjoint for a unique density \( \omega \) (see Eqs. (5), (6a, b) and (10)).

The loss-free radio spectral index \( \alpha_0 \) is \( \alpha_0 = \frac{1}{2}(\eta_0 - 1) \). The functions \( r(\theta, 1/M_A), r_s(\theta, 1/M_A), r(\theta, 1/M_A), r_\ast(\theta, 1/M_A), r_\ast(\theta) \) and \( r_\ast(\theta) \) are respectively given by (23b), (24), (57), (60), (23c) and (23d). We will take \( \alpha_0 = 10^4 \) (i.e. \( \delta \approx 10^{-6} \)) and \( \theta \approx 10^{-3} \).

From (58), for each value of \( M_A \) we deduce a finite value \( \theta_{\text{min}} \) such that \( r_\ast > 1 \). Moreover, bearing in mind that \( r \) (23b) is an expansion in power of \( M_A^{-1} \), for consistency we will limit the computation to \( r_\ast \geq 1.5 \) and then we must have \( \theta \geq \theta_{\text{lim}}(r_\ast = 1.5) \). The values of \( \theta_{\text{min}} \) and \( \theta_{\text{lim}} \) are given in Table 1.

Using the dimensionless variable \( y = (u_1 + u_2 - 2u)/(u_1 - u_2) \), the function \( g(y) \), defined by (4), is for \( y < y_0 \):

\[
g(y) = \frac{a_y + 1 - 2r_\ast}{r_\ast}(r_\ast - 1) \frac{y(r_\ast - 1)}{(r_\ast - 1)(1 + y)}
\]

Table 1

<table>
<thead>
<tr>
<th>( M_A )</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>30</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_{\text{min}} )</td>
<td>3.0</td>
<td>1.2</td>
<td>0.4</td>
<td>0.2</td>
<td>0.05</td>
<td>0.001</td>
</tr>
<tr>
<td>( \theta_{\text{lim}} )</td>
<td>5.0</td>
<td>2.0</td>
<td>1.0</td>
<td>0.5</td>
<td>0.2</td>
<td>---</td>
</tr>
</tbody>
</table>

![Magnetic corrections - B parallel](image)

Fig. 2. The variation of the radio spectral index \( \alpha_0(\theta, 1/M_A); B \) quasi parallel. For a given value of \( M_A \), we have plotted the radio spectral index \( \alpha_\ast \) as a function of \( \theta = p_\parallel/p_\perp \). For each value of \( M_A \), corresponds a limiting value of \( \theta \) namely \( \theta_{\text{lim}} \) for which \( r_\ast = 1.5 \) and thus \( (\theta, 1/M_A) \) given by (23b) can be considered as an expansion in \( M_A^{-1} \). As long as \( \theta \geq 10^{-1} \), the radio spectral index is given by the linear theory, i.e. \( \alpha_\ast = 3 - 1 \). This result is valid even if the diffusion coefficient is a function of the energy

\[
g(y) = \frac{a_y + 1 - 2r_\ast}{r_\ast}(r_\ast - 1) \frac{y(r_\ast - 1)}{(r_\ast - 1)(1 + y)}
\]

where \( y_0 = (r_\ast + 1 - 2r_s)/(r_\ast - 1) \) and \( r_s = 7 \).

Thus using the parameter \( a \) given by (9), \( z \) and \( \beta \) by (11a, b), we deduce \( \eta_0 \) from (7) and \( \alpha_0 \) from \( \alpha_0 = \frac{1}{2}(\eta_0 - 1) \). The curves \( \alpha_0(\theta, 1/M_A) \) and \( \eta_0 = \eta_0 + 2 \) are plotted in Fig. 2.

One of the most important result is the possibility to calculate \( \eta_0 \) using the linear theory result if \( \theta \geq 10^{-1} \) for all the values of \( M_A \) with \( \theta \geq \theta_{\text{lim}} \) for a given \( M_A \).

So, if \( \theta \geq 10^{-1} \), we validate the approximate result

\[
\eta_0 = \frac{3r_s}{r_s - 1} - 2
\]

or

\[
\alpha_0 = \frac{3 - 1}{2r_\ast - 1}
\]

with \( r_s = r(1 - 1/M_A) \) and where \( r(\theta, 1/M_A) \) is given by (23b), \( r_s(\theta, 1/M_A) \) by (57) and with the limitation \( \theta \geq \theta_{\text{lim}} \) which corresponds to \( r_\ast = 1.5 \) for a given value of \( M_A \).

This important result does not depend on the diffusion coefficient and thus its energy dependence does not affect the spectral index as long as \( \theta \geq 10^{-1} \) which corresponds to the astrophysical situations involving classical hydrodynamics (i.e. flow velocities \( < c/\sqrt{3} \)).

Anyway note that if \( D_\perp \) is an increasing function of the energy, the effective value of the parameter \( a \) increases with the energy and therefore the validity of the results (63) and (64) is weakened.

4. Magnetic field quasi-perpendicular to the jet: the magnetic pressure effect

4.1. Basic equations

We consider a magnetic field quasi-perpendicular to the jet and neglect the parallel component.

In such a case we have

\[
u = v
\]

and the streaming instability effect does not occur.

Now the basic equations are

\[
J = \rho u
\]

\[
J + p_e + p_r + p_m = F
\]

\[
\gamma e p_e + u\frac{dp_e}{du} = \delta \frac{dp_e}{du}
\]

\[
\gamma r p_r + u\frac{dp_r}{du} = \frac{dp_r}{du}
\]

\[
u m B = \frac{\partial B}{\partial x} + Cst = \frac{v_m}{\kappa_r} \frac{\partial B}{\partial u} + Cst
\]

where \( p_m = B^2/(8\pi) \) is the magnetic pressure, and \( v_m \) the magnetic viscosity. Even if, the magnetic field is not frozen-in through the shock, we have always \( u_1 B_1 = u_2 B_2 \) or \( B_1 = rB_1 \) and the jump of the magnetic pressure is \( p_{m2} = r^2 p_{m1} \).
4.2. Relation between $\beta_2$ and $M_\lambda$ in the strong shock limit

A useful and important relation is the relation between the Alfvénic Mach number in the jet $M_\lambda = v_1/V_{A1}$ and the parameter $\beta_2$ in the downstream flow defined by $\beta_2 = (p_{m2} + p_{s2})/p_{m2}$.

From definitions of $M_\lambda$ and $V_{A1}$ we have

$$\frac{1}{M_\lambda^2} = \frac{2p_{m1}}{Ju_1}. \quad (71)$$

Now, in the strong shock limit, from (66) and (71), we deduce

$$M_\lambda \approx \frac{r^3}{2(r-1)} (1 + \beta_2) \quad (72)$$

4.3. The global compression ratio in the strong shock limit

Let us calculate the compression ratio in the hydrodynamical model with a pressure which is the sum of partial ones $p_j$ with adiabatic coefficients $\gamma_j$. We have $\rho = \sum p_j$ and $\rho = \sum \frac{\gamma_j}{\gamma_j - 1} p_j$, where $\rho$ is the total pressure per mass unit and $\gamma_\mathrm{r} = 4/3$, $\gamma_\mathrm{c} = 5/3$ and $\gamma_\mathrm{m} = 2$.

The basic relations for the jump are

$$J = \rho u \quad (73)$$
$$Ju_1 + p_1 = Ju_2 + p_2 \quad (74)$$
$$\frac{Ju_1^2}{2} + w_1 u_1 = \frac{Ju_2^2}{2} + w_2 u_2. \quad (75)$$

Writing $\mu = \rho/\rho$, from (75) and (74), we get in the strong shock limit

$$r = 2\beta_2 - 1 \quad (76)$$

or

$$r = \left( \frac{2\gamma_\mathrm{r}}{\gamma_\mathrm{r} - 1} + \frac{2\gamma_\mathrm{c}}{\gamma_\mathrm{c} - 1} \right) \beta_2 + \frac{4}{\gamma_\mathrm{r} - 1 + \beta_2} - 1, \quad (77)$$

which using (72) can be written

$$r = \frac{r_0 - 3}{1 + \frac{r_0 - 3}{2(r-1)M_\lambda^2}}. \quad (78)$$

Let us remark that the correction is sensitive because it is in $r^2/M_\lambda^2$ and not simply in $M_\lambda^2$.

4.4. On the occurrence of a singularity in the precursor

In the precursor, a fraction of incident particles of the hydrodynamical flow can be reflected, which leads to a singularity in the hydrodynamical equations. Indeed, (68) can be written

$$\frac{d\gamma}{d\epsilon} = \frac{d}{d\epsilon} \left[ \frac{\gamma}{\gamma^2} - (\gamma - 1) u \right] \frac{dp_s}{d\epsilon} = \frac{d}{d\epsilon} \left[ \phi \frac{d\epsilon}{d\epsilon} \right]$$

and then we have

$$\phi(u) = \left[ \gamma(u_{p1} - u_{p1}) - (\gamma - 1) \int_{u_{p1}}^u u dp_s \right]. \quad (79)$$

If we neglect $p_s$ in the precursor, from (66) we get

$$J + \frac{dp_r}{du} + \frac{dp_m}{du} = 0. \quad (80)$$

Since the inequality $v_m \ll \kappa$, is widely satisfied in astrophysical plasma, the frozen-in assumption can be applied in the precursor, which simplifies (69) to $uB = C_{\mathrm{s1}}$, and from (71) we have

$$p_m = \frac{B^2}{8\pi} = p_{m1} \frac{u_1^2}{u^2} + \frac{J}{2M_\lambda^2} \frac{u_1^2}{u^2} \quad (81)$$

and then, from (80)

$$\frac{d\epsilon}{du} = J \left( \frac{u_1^2}{(M_\lambda^2)^{3/2}} - 1 \right). \quad (82)$$

In the precursor, $dp_r/du \leq 0$, but a pole appears in (79) when the RHS in (82) vanishes for $u = u_c, M_\lambda^{\gamma_\mathrm{r}/2}$. Thus, the precursor is not critical if $u_0 > u_c$ or

$$M_\lambda > M_{\lambda,\text{cr}} = \left( \frac{r}{r_0} \right)^{3/2} \quad (83)$$

For a quasi classical shock, $r \approx r_c$ and we must have $M_\lambda > 1$, but for a radiative shock, $r \approx 7$ and $r_c \approx 1$ and then $M_\lambda > 18$.

4.5. Structure of the precursor

4.5.1. Derivation of the function $\phi_r$

When $M_\lambda > (r/r_c)^{3/2}$, the growth of the thermal pressure is small compared to the growth of the relativistic pressure and we can keep a linear approximation for its dependence on $u$. In the same manner we will assume a linear dependence on $u$ for $p_m$ and according to (66) we can write

$$p_c = p_{c1} + \sigma_1 J(u_1 - u) \quad (84)$$
$$p_m = p_{m1} + \sigma_1 J(u_1 - u) \quad (85)$$
$$p_r = p_{r1} + (1 - \sigma_1 - \sigma_2) J(u_1 - u). \quad (86)$$

So, writing

$$\phi = \phi_r \equiv a_r(u - u_1)(u - u_2) \quad (87)$$

(see Fig. 1a) and inserting (86) into (68) we deduce

$$a_r = \frac{\gamma_i + 1}{2}\quad (88)$$

and

$$r_s = u_t/u_2 = \frac{\gamma_i + 1}{\gamma_i - 1 + \frac{2}{M_\lambda^2}} \quad (89)$$

with

$$M_\lambda^2 = \frac{(1 - \sigma_i - \sigma_i') J u_{11}}{\gamma_i p_{r1}} \quad (90)$$

In the strong shock limit $M_\lambda^2 \gg 1$ and we have always $r_s \approx 7$.

4.5.2. Determination of the parameters $\sigma_1$ and $\sigma_i$

From (81), we have

$$p_m = \frac{B^2}{8\pi} \frac{u_1^2}{u^2} \quad (81)$$

and developing $p_m$ for $u$ close to $u_1$, we have

$$p_m \approx p_{m1} \left( 1 - 2 \frac{u - u_1}{u_1} \right) = p_{m1} + \sigma_i' J(u_1 - u)$$
and thus from (71) we deduce
\[ \alpha_1' \approx \frac{1}{M_A^2}. \]
\[ (91) \]

Now we can calculate \( \alpha_1 \), inserting (84) into (67) and taking \( u = u_1 \). Since \( \delta \ll 1 \), we obtain
\[ \alpha_1 = \frac{\gamma e p_{c1}}{M_A^2}, \]
\[ (92) \]
which is always much smaller than one in the strong shock limit.

### 4.6. Structure of the subshock

#### 4.6.1. Derivation of the structure function \( \phi_c \)

We do not claim to describe the subshock transition and we will take the limit \( \delta \ll 1 \). So to simplify the calculation, we will assume a linear variation of \( p_{m} \) which satisfies the jump condition. In addition, the relativistic pressure varies slowly and we can write
\[ p_i = p_{i2} + \alpha_2 J(u_2 - u), \]
\[ p_m = p_{m2} + \alpha_2' J(u_2 - u), \]
\[ p_c = p_{c2} + (1 - \alpha_2 - \alpha_2') J(u_2 - u). \]
\[ (93) \]
\[ (94) \]
\[ (95) \]

Writing
\[ \phi = \phi_c = a_c(u - u_1^*) (u - u_2), \]
(see Fig. 1a) and inserting (95) into (67) we find
\[ a_c = \frac{\gamma_e + 1}{2 \delta}. \]
\[ (96) \]
\[ (97) \]

#### 4.6.2. Determination of the parameters \( \alpha_2 \) and \( \alpha_2' \)

From \( u B = C \) and (94), we can write
\[ p_m - p_{m0} = \frac{u_2^2}{u_0^2} = p_{m2} + \alpha_2' J(u_2 - u_0) \]
and thus
\[ \alpha_2' = \frac{p_{m2} r_c + 1}{J u_2 - \frac{r_c^2}{2M_A^2}}. \]
\[ (98) \]

From (66), in the strong shock limit, we have
\[ Ju_2 + p_{m2} + p_{c2} + p_{i2} = J u_2, \]
then
\[ p_{m2} = (r - 1) Ju_2 (1 + \beta) \]
and from (72), we finally obtain
\[ \alpha_2' = \frac{r_c^2 + 1}{r_c^2 2 M_A^2}. \]
\[ (99) \]

Since we must have \( \alpha_2' < 1 \), we deduce the following constraint
\[ M_A > M_{Alum} = \left( \frac{r_c^2 + 1}{2 r_c^2} - r \right)^{1/2}. \]
\[ (100) \]

Let us remark that we have
\[ M_{Alum} = \left( \frac{r_c^2 + 1}{2 r_c^2} - r \right)^{1/2} > M_{Alum} = \left( \frac{r_c}{r_c^2} \right)^{3/2}, \]
and then when condition (100) is satisfied, we do not have a singularity in the precursor.

Now, we can calculate \( \alpha_2 \) inserting (93) into (68) and taking \( u = u_2 \). We find
\[ \alpha_2 = \frac{\gamma_e p_{i2}}{J u_2 (\alpha_c (r_{**} - 1) + 1)}, \]
\[ (101) \]

where
\[ r_{**} = \frac{u_1^*}{u_2}. \]
\[ (102) \]
From (66), in the strong shock limit we have
\[ p_{i2} = J u_2 \frac{(r - 1) \beta}{(1 + \theta)(1 + \beta)} \]
and then
\[ \alpha_2 = \frac{\gamma_e \beta (r - 1)}{(1 + \theta)(1 + \beta) \left( \gamma_e + 1 \right) \frac{1}{2 \beta} (r_{**} - 1) + 1}. \]
\[ (103) \]

When \( \theta \to \infty \), \( \alpha_2 \approx \beta \theta \to 1 \).

As we will see further, \( r_{**} = r_c \) and when \( \theta \to 0 \), \( r_{**} \to 1 \). But taking into account (83) or (100) it is easy to verify we have always \( \alpha_2 < 1 \). Since we will take \( \delta \approx 10^{-4} \) and as \( \alpha_2 \approx 1 \) when \( \theta \ll 1 \), we always have \( \alpha_2 \approx \alpha_2' \).

#### 4.6.3. Derivation of the compression ratio \( r_{**} \)

The function \( \phi_c \) defined by (96) is completely determined if we calculate \( r_{**} \) defined by (102).

Using the simplification \( \alpha_2 \approx \alpha_2' \), we can calculate \( r_{**} \) inserting (95) into (67). We get
\[ r_{**} = \frac{\gamma_e + 1}{\gamma_e - 1 + 2 M_{**}^2}, \]
\[ (104) \]
where
\[ M_{**}^2 = \frac{(1 - \alpha_2') J u_1^*}{\gamma_e p_{c1} + \gamma_e \Delta p_c} \]
and
\[ \Delta p_c = \alpha_2 J (u_1 - u_0) - (1 - \alpha_2') J (u_1^* - u_0). \]
\[ (105a) \]
\[ (105b) \]

#### 4.6.4. Derivation of the compression ratio \( r_c \)

If we neglect \( \alpha_1 \), \( \alpha_1' \) and \( \alpha_2 \) compare to \( \alpha_2' \). The compression ratio \( r_c \) is given by the continuity of the pressures at \( u_0 \), using (84), (86), (93) and (95). We get
\[ p_{c0} = p_{c2} + (1 - \alpha_2') J (u_2 - u_0) \approx p_{c1} \]
\[ (106) \]
\[ p_{c0} = p_{c1} + J (u_1 - u_0) \approx p_{c2}. \]
\[ (107) \]
So in the strong shock limit, we have
\[ \theta = \frac{p_{c2}}{p_{c1}} \approx \frac{(1 - \alpha_2') (r_c - 1)}{r_c - r_c} \]
\[ (108) \]
and thus we obtain
\[ r_c \approx \frac{1 - \alpha_2' + \theta r}{1 - \alpha_2' + \theta} \]
\[ (109) \]
where \( \alpha_2' \) is given by (99). When \( \theta \to 0 \), \( r_c \to 1 \) and when \( \theta \to \infty \), \( r_c \to r_c \).
4.6.5. The minimum value of the heating ratio $\theta$ due to the magnetic corrections

When we take into account a finite value of $M_\Lambda$, the condition for the existence of the subshock i.e. $r_c > 1$ limits the range of $\theta$. When $\alpha_2 = 1$, the jump of $p_e$ disappears in the subshock. So the minimum value of $\theta$ can be obtained by writing the condition

$$\alpha_2 < 1$$

or

$$M_\Lambda^2 > \frac{r_e + 1}{r_e} \frac{r_e}{r_e - 1}. \tag{111}$$

Indeed, it is easy to see that condition (111) cannot be satisfied when $\theta = 0.01$, i.e. $r_c \approx 7$ and $r_e \approx 1$, if $M_\Lambda = 5$ for example.

In practice, the minimum value of $\theta$ can be obtained computing $r_e$ and writing the condition $r_e > 1$. The minimum value of $\theta$ corresponding to given values of $M_\Lambda$ are given in Table 2 (See Sect. 4.7). The structure function $\phi$ is shown in Fig. 1b when $\theta \approx \theta_{\text{min}}$, i.e. $r_e \approx 1$.

4.6.6. Relation between $r_{**}$ and $r_e$

The continuity of the function $\phi$ at $u_0$, namely $\phi_r(u_0) = \phi_e(u_0)$ provides the relation between $r_{**}$ and $r_e$. We get

$$r_{**} = r_e + \frac{\alpha_e}{\alpha_r} \frac{1}{r_e}. \tag{112}$$

When $\delta \to 0$, $\alpha_e \to \infty$ and we have $r_{**} = r_e$.

For numerical application, when $\delta \approx 10^{-4}$ and $\theta \geq 10^{-3}$, (112) shows that we can take $r_{**} = r_e$.

4.7. The spectral index

We can now calculate the spectral index $\eta_0$ of the energy distribution of the relativistic electrons, or the loss free radio spectral index $\alpha_0 = \frac{1}{2} \eta_0$, as functions of $\theta$ and $1/M_\Lambda$.

The functions $\eta_0(\theta, 1/M_\Lambda)$, $\alpha_0(\theta, 1/M_\Lambda)$, $r_{**}(\theta, 1/M_\Lambda)$ and $r_0(\theta)$ are respectively given by (78), (109), (112) and (23c). We will take $\alpha_0 = 10^4$, i.e. $\delta \approx 10^{-4}$ and $\theta \geq 10^{-3}$.

From condition (111) and from $r_e > 1$ in (109) we deduce $\theta_{\text{min}}$ for a given value of $M_\Lambda$. Values of $\theta_{\text{min}}$ are given in Table 2.

The curves $\alpha_0(\theta, 1/M_\Lambda)$ and $\eta_0 = 3 \alpha_0 = \eta_0 + 2$ are plotted in Fig. 3.

The main difference with Fig. 2 is that even for $M_\Lambda = 4$, the loss-free radio spectral index is $\alpha_0 \approx 0.6$, value significantly lower than this obtained in the quasi parallel magnetized shock. The main practical difference for hot spots is that for $1 \leq \theta \leq 10$, $\alpha_0 \approx 0.5$ when the magnetic field is quasi perpendicular and $\alpha_0 \approx 0.8$ when the magnetic field is quasi parallel.

One of the most important result is the possibility to calculate $\eta_0$ using the linear theory result if $\theta \geq 10^{-1}$ for all the values of $M_\Lambda$, with $\theta \geq \theta_{\text{min}}$ for a given $M_\Lambda$.

### Table 2

<table>
<thead>
<tr>
<th>$M_\Lambda$</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>15</th>
<th>30</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{\text{min}}$</td>
<td>0.4</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
<td>0.02</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
</tr>
</tbody>
</table>

5. The radio spectral index as a diagnosis of different physical type of synchrotron sources

The main purpose of the theory developed in the previous paper and in this one is to provide a method of diagnosis based on the knowledge of the observed radio spectral index which allows the determination of the physical conditions. This is possible because the deviation of the loss free spectrum index from its ideal 1/2 depends of the heating ratio $\theta$ and on the Alfvénic Mach number.
introduced by the magnetic effects. In this section we will discuss briefly applications to supernova remnants, nuclei of galaxies and hot spots of extragalactic radio sources. A detailed study of these examples is not within the scope of this article and will be worked out elsewhere.

5.1. The supernova remnants with a flat radio spectral index

A fraction of supernova remnants have a flat radio spectral index i.e. $\alpha \approx 0.25$ (Green, 1984). The relativistic electrons responsible for the radio emission of the shell are those accelerated by the spherical shock wave expanding in the interstellar medium after the explosion of the star. The shell corresponds to the downstream flow and the interstellar medium to the upstream flow. With typical values for the interstellar medium of $B \approx 2 \times 10^{-6}$ G and $n = 10^4$ cm$^{-3}$, the corresponding Alfvén velocity is $V_A \approx 3$ km s$^{-1}$. Thus the Alfvénic Mach number of the spherical shock wave can be far much greater than unity and the hydrodynamical assumption can be used. We can see from Figs 2 or 3, that the radio spectral index $\alpha_0 \approx 0.25$ can be obtained if $\theta \approx 0.1$, i.e. if the downstream pressure is dominated by the pressure of the relativistic particles. In the case of supernova remnants where the hydrodynamics are not relativistic, the downstream pressure can be dominated by the relativistic particles only if they are copiously injected in the process of repeated scattering through the shock, and/or if there is an additional source of cosmic rays in the shell. We do not know without further investigation, if the injection of the ambient cosmic rays could be sufficient. Alternately, the injection could be due to a central engine, i.e. a neutron star which could also produce a pressure of relativistic particles. Usually, one thinks the injection problem from the upstream medium. But for supernova remnants with a flat radio spectral index this would be from downstream, but this does not change the main body of the theory. We just suggest that such flat spectra could be the sign of the presence of a central neutron star in the shell. However this hypothesis would deserve some detailed investigation. In this case, as long as the downstream pressure is dominated by the relativistic particles and the Alfvén Mach number is great enough to justify the hydrodynamic assumption, the radio spectral index of the relativistic electrons accelerated by the spherical shock wave can be $\alpha_0 \approx 0.25$. Typical cases of such supernova remnants with $\alpha \approx 0.25$ containing a neutron star are the Crab and Vela and radio observations do indicate spectral indices such as those predicted by the theory.

5.2. The central radio source in nuclei of galaxies

Whereas at centimeter wavelength the spectra of nuclei of galaxies exhibit a low frequency cut-off probably due to the synchrotron self absorption, at millimeter wavelength, they often display a power law with an index $\alpha \approx 0.25$ corresponding to an optically thin synchrotron source (Kellerman and Pauliny-Toth, 1981).

If the radio source corresponds to the central source associated with the central engine, the radio spectral index can be explained in the same way as we did for supernova remnants with $\alpha_0 \approx 0.25$. Indeed, if we consider that the nucleus contains an engine producing high energy particles, then the pressure around it, is likely dominated by the relativistic particles and the radio source we observe can be due to the relativistic electrons produced by a spheroidal shock wave around the central engine. In such picture, the spectral index of the radio emission of the shell is governed by the downstream flow and is $\alpha_0 \approx 0.25$.

Let us remark that the hydrodynamical assumption is supported by radio observations of nuclei of galaxies for which $\beta$ is much greater than 1 (Kellerman and Pauliny-Toth, 1981).

5.3. Hot spots similar to those of Cygnus A

It has been shown (Pelletier and Roland, 1986) that the radio properties of Cygnus A hot spots can be explained either if the jet velocity is classical with a downstream pressure dominated by the thermal classical component or if the jet reaches a relativistic velocity (i.e., $u > c/\sqrt{3}$) with a downstream pressure dominated by the relativistic protons. The improvement of the theory by taking into account the magnetic corrections brings a new constraint on the heating parameter $\theta$ for the hot spots. In the absence of a spectral theory of magnetized shocks in relativistic magnetohydrodynamics, we suppose here that the jet velocity is classical. Assuming a quasi perpendicular magnetic field we have estimated (Pelletier and Roland, 1986) the parameter $\beta$ of the hot spots and found $\beta \approx 2$, which corresponds to a Mach number in the jet $M_\alpha \approx 6$ (see (72)). The high frequency radio spectral index of Cygnus A hot spots is $\alpha_0 \approx 1.0$ and taking into account synchrotron losses (Kardashev, 1962), we deduce $\alpha_0 \approx 0.5$ as the low frequency radio spectral index behind the shock. From Fig. 3 and Sect. 4 of this article, we deduce for Cygnus A hot spots $\theta \approx 6$, i.e. the dominant pressure is the classical one. This is an important consequence of Sect. 4, in which it has been shown that, for classical hydrodynamics with a moderate Alfvenic Mach number, then the dominant pressure in the downstream flow is necessarily the classical one. For Cygnus A hot spots, with a jet velocity $v_\parallel \approx 0.3c$ and a density inside hot spots $n_\parallel \approx 10^{-3}$ p cm$^{-3}$, the corresponding mean energy per particle for the classical thermal protons is $E_{th,p} \approx 15$ MeV. About one percent of the electrons have an energy $E \geq E_{th,e} \approx 100$ MeV and are responsible for the synchrotron emission of the hot spots and we deduce from $\theta \approx 6$ that the thermal electrons have a mean energy per particle $E_{th,e} \lesssim 3$ MeV, i.e. they constitute a relativistic gas. This conclusion agrees with the low internal depolarisation observed for Cygnus A hot spots (Dreher et al., 1987 and references quoted), because the contribution of these thermal electrons to the medium birefringence varies as the inverse square of their relativistic mass (Jones and O'Dell, 1977).

5.4. The very steep spectrum hot spots

A number of hot spots have a very steep radio spectral index, namely $\alpha_0 \approx 1.3$ (see i.e. Roland et al., 1982). These radio sources are associated with very distant galaxies $z \approx 1$ and their angular dimension is quite small. The separation of the two hot spots is very often smaller than 10" and most of the flux of the hot spots arises from regions smaller than 0.1"-0.3". If we take into account redshift corrections, this means that the two hot spots are close to the parent galaxy and that their diameter is less than 1 kpc. The extreme case of these sources is 1518+047 studied by Mutel et al. (1985) and Mutel and Hodges (1986). Although its redshift is not known, the linear separation between the two hot spots is of the order of a few 100 pc and the hot spots diameter is about 10 pc.

It has been proposed by Roland et al. (1982), that they correspond to a stage of a rapid fading of the activity of the central engine which power the two hot spots.
Let us simply note we can explain the difference of the radio spectral index of the very steep spectrum hot spots and those of Cygnus A. Indeed as long as the hot spot is close to the galaxy and the jet is sufficiently collimated the magnetic field remains parallel to it. Consequently the radio spectral index can be \(x_0 \approx 0.8\) (see Fig. 2) and if we take into account synchrotron losses the radio spectral index is \(x_r \approx 0.8 + 1/2 = 1.3\) over the radio range.

Note that, if the very steep spectrum hot spots are those with the magnetic field quasi parallel to the jet, there exists an important difference between their physical parameters and the physical parameters of hot spots powered by jets with quasi perpendicular magnetic field. From (23a), when the magnetic field is quasi parallel, we have \(\beta_2 \approx 50\) with \(M_A \approx 6\) and \(r \approx 4\), but when the magnetic field is quasi perpendicular, we have \(\beta_2 \approx 2\) with the same values, i.e. \(M_A \approx 6\) and \(r \approx 4\) (see Eq. (72)).

6. Conclusion

The introduction of moderate magnetic corrections to the theory of mixed shocks give rise to new constraints to the structure of the shock and interesting corrections to the radio index of the synchrotron spectrum. Whereas we found a wide range of possible mixed shocks in our previous work (Pelletier and Roland, 1986) in the limit \(M_A \rightarrow \infty\) with a heating ratio \(\theta\) comprised between \(\theta_{\text{min}} \approx \delta\) and \(\theta_{\text{max}} \approx \delta^{-1}\), our present analysis shows that the finite Alfvénic Mach number effects involve an avoidable heating, significantly greater than the unmagnetized case. When the magnetic field is quasi parallel, the streaming instability in the precursor is responsible for the increase of the heating ratio \(\theta\), for instance \(\theta_{\text{min}} \approx 1\) for \(M_A = 5\) (see Table 1). But when the magnetic field is quasi perpendicular, one cannot avoid that a heating occurs with the growth of the magnetic pressure, typically we get \(\theta_{\text{min}} \approx 0.3\) for \(M_A = 5\) (see Table 2). Consequently the efficiency of the shock in converting kinetic power into luminosity is lowered.

One of the most interesting result of our nonlinear theory is that it predicts a wide range of validity of the approximation of the spectrum index formula by the linear result, even when the back reaction of the relativistic pressure is important. However the relevant compression ratio must be used, taking account of the structure of the magnetized mixed shocks, parametrized by \(\theta\) and \(M_A\).

Precisely the loss free synchrotron index \(x_0\) at radio frequencies is given by the following simple formula which is a valid approximation as long as \(\theta \geq 10^{-1}\) and \(M_A \geq 4-5\)

\[
x_0 = \frac{3}{2} \frac{1}{r_\text{rs} - 1},
\]

(114)

where \(r_\text{rs}(\theta, M_A)\) is the compression ratio experienced by the scattering centers.

In a quasi perpendicular shock \(r_\text{rs}\) is simply the usual global compression ratio \(r\)

\[
r(\theta, M_A) = \frac{r_0}{1 + \frac{r_0 - 3}{2(r-1)} M_A^2}
\]

(115)

with \(r_0(\theta) = \frac{7 + 4\theta}{1 + \theta}\).

In a quasi parallel shock

\[
r(\theta, M_A) = \frac{r_0}{1 + \frac{16}{3M_A r_0 - 1} \left(1 - \frac{r_\text{elo}}{r_0}\right)^{3/2}}
\]

(116)

with \(r_\text{elo} = 1 + \frac{1}{\theta r_0}\).

which brings "anomalous" corrections in \(M_A^{-1}\).

It turns out that the corrections in the quasi parallel shock are more sensitive than in the quasi perpendicular shock. In the latter case the spectral index remains between 0.25 and 0.55 if \(\theta \geq 10^{-1}\) and is close to 0.5 as soon as \(\theta \geq 3\), whereas in the former, it can spread from 0.25 to 0.8. For small \(\theta\), the nonlinear correction proportional to \(a^{-1}(\theta, M_A)\) (Eq. (7)) becomes important for \(x_0\) given by

\[
x_0 = \frac{3}{2} \frac{1}{a(r_\text{rs} - 1)}
\]

(117)

and are calculated numerically. For \(M_A \rightarrow \infty\) and \(\theta \rightarrow 0\), \(a \sim 7/6\) and our solution tends to the result obtained by Drury et al. (1982). But our graphs in Fig. 2 and Fig. 3 indicates how large the Alfvénic Mach number must to reach this limit.

Let us mention that we did not analyse the case of a relativistic jet for which we have a relativistic shock (Peacock, 1981; Webb, 1985; Webb et al., 1987 and Kirk and Schneider, 1987).

Regarding the astrophysical objects our theory of the spectral index is a suitable key to get a diagnosis of the medium of the synchrotron source.

Our first example is the young supernova remnants with a flat radio spectral index, i.e. \(x_0 \approx 0.25\). They are spherical shocks propagating in the interstellar medium with a very high Alfvénic Mach number. Since the observed spectral index is 0.25, according to our results (Fig. 2 or Fig. 3), this can be explained only by a dominating relativistic pressure in the downstream shell flow, which suggests a relativistic particle supply by a central engine, i.e. a neutron star.

Our second example is the active galactic nuclei. There is an obvious source of relativistic particles whose pressure dominates necessarily the thermal pressure since the spectral index of 0.25 has been observed and confirms Kellerman and Pauliny-Toth's conclusion about the large value of \(\beta\) for the nuclear region.

Our third example is the hot spots similar to those of Cygnus A. We previously demonstrated that the assumption of a classical velocity for the jets is consistent with the shock model explaining the observed luminosity produced by the conversion of a small fraction of the incoming kinetic power. Assuming a quasi perpendicular magnetic field, it turns out that the Alfvénic Mach number is moderate: about 6. Since the spectrum steepens at frequency higher than 1 GHz because of synchrotron losses, we explain the observed index equal to one, by the usual change of slope \(x_0 = x_1 = x_0 + 1/2\) which yields \(x_0 \approx 0.5\), and implies, according to our theory, that the heating ratio \(\theta\) is about 6. Consequently, the dominant pressure in hot spots is the pressure of a classical gas i.e. the thermal protons which have a mean energy per particle \(E_{\text{th, p}} \approx 15\) MeV.
Our fourth example is the hot spots with very steep spectrum. These large $z$ radio sources are probably sources with quasi parallel shocks. They have a very steep spectrum with $\alpha \approx 1.2-1.3$ and thus $\alpha_0 \approx 0.7 - 0.8$ requires a moderate Alfvénic Mach number.

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Appendix A

Modification of the global compression ratio $r$ by the streaming instability. From (19)-(22) we get

$$d(p_e + p_r) + Jdv = 0$$

(A1)

$$\int_{v_2}^{v_1} (\gamma_e \rho_e dv + \nu_d p_e) = (\gamma_e - 1) \int_{v_2}^{v_1} Vdp,$$

(A2)

$$\int_{v_2}^{v_1} (\gamma_r \rho_r dv + \nu_d p_r) = \int_{v_2}^{v_1} Vdp,$$

(A3)

Thus (A2) and (43) imply

$$\left[ \frac{\gamma_r \rho_e}{\gamma_e - 1} + \frac{\gamma_r \rho_r}{\gamma_r - 1} \right] v_2^{v_1} - \int_{v_2}^{v_1} v(\nu_d p_e + p_r) = \frac{\gamma_r}{\gamma_e - 1} \int_{v_2}^{v_1} Vdp.$$  

(A4)

Inserting (A1) into (A2) and using

$$\mu = \left( \frac{\gamma_r \rho_e}{\gamma_e - 1} + \frac{\gamma_r \rho_r}{\gamma_r - 1} \right) \frac{p}{p},$$

with $p = p_e + p_r$, we obtain

$$\nu_2 \nu_1 - \nu_2 \nu_2 \frac{\gamma_r}{\gamma_e - 1} \int_{v_2}^{v_1} Vdp,$$

(A5)

which can be written as follows:

$$r \left[ 1 + \frac{\nu_2 \mu_2}{r - 1} \right] - \frac{2r}{r - 1} \frac{\gamma_r}{\gamma_e - 1} \int_{v_2}^{v_1} Vdp.$$  

(A6)

We will calculate the integral (A6) at the first order in $M^{-1}$ using (27) which implies

$$dp_r = -(1 - \alpha + \alpha_1) Jdv = -(1 - \alpha + \alpha_1) f \left( 1 - \frac{dv}{d\nu} \right)$$

(A7)

and for $\nu_0 \leq \nu_1 \leq \nu$, one has $V = \nu_1 (\nu/\nu_1)^{1/2}$. Thus

$$\int_{\nu_2}^{\nu_1} Vdp_r = -\int_{\nu_2}^{\nu_1} \nu dv + 0(M^{-2})$$

$$= \frac{\nu_1}{3} \left( 1 - \left( \frac{\nu_0}{\nu_1} \right)^{1/2} \right) + 0(M^{-2}).$$  

(A8)

Inserting (A8) into (A7), we get a first order correction to the strong shock limit $r_0 = 2 \mu_2 - 1$:

$$r \approx \frac{r_0}{1 + \frac{16}{3M} \frac{r_0}{r_0} \left( 1 - \frac{r_0}{v_0} \right)^{2/3}}.$$  

(A9)

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