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Chapter 2

Elements of microwave technology
2.1 Introduction

This thesis concerns the field of microwaves. Within the electromagnetic (EM) spectrum, microwaves are those waves for which the wavelength $\lambda$ is approximately equal to the size of components used for transmission. Indeed, as microwaves have a frequency of $3$ to $300$ GHz \([1]\), their typical wavelength is approximately $\lambda = \frac{c}{f} = 10$ cm to $1$ mm, where $c$ is the speed of light. Therefore, circuit analysis in terms of lumped elements fails as the phase of the voltage and current over the element cannot be approximated as constant. On the other hand, the methods of geometrical optics fail, which assume that the wavelength is much shorter than the size of a component. This implies that microwave circuits must be described in terms of distributed elements.

In this chapter we cover the basics of microwave technology relevant for this thesis. We review microwave transmission lines, microwave reflections, the coplanar waveguide (CPW) and microwave CPW resonators.

2.2 Microwave transmission line theory

In microwave engineering, transmission lines are generally modelled as distributed-element circuits as depicted in figure 2.1. In this figure $L$, $R$, $C$ and $G$ are the inductance, resistance, capacitance and conductance of the line per unit length. $L$ and $R$ arise due to the self-inductance and finite conductivity of the line, whereas $C$ and $G$ are due to the proximity of the centre conductor and the ground that might induce dielectric losses. Here, we follow \([1]\).

![Lumped-element circuit representation of an incremental length of a microwave transmission line.](image)

Applying Kirchhoff’s law for voltage, $V$, to the circuit in figure 2.1 we find

$$ V(z,t) - R \Delta z I(z,t) - L \Delta z \frac{dI(z,t)}{dt} - V(z + \Delta z,t) = 0. $$

(2.1)

where $I(z,t)$ is the current in the line.

Similarly, Kirchhoff’s law for current gives

$$ I(z,t) - G \Delta z V(z + \Delta z,t) - C \Delta z \frac{dV(z + \Delta z,t)}{dt} - I(z + \Delta z,t) = 0. $$

(2.2)
Taking the limit $\Delta z \to 0$ yields the so-called telegrapher’s equations
\begin{align}
\frac{dV(z,t)}{dz} &= -RI(z,t) - L\frac{dI(z,t)}{dt}, \\
\frac{dI(z,t)}{dz} &= -GV(z,t) - C\frac{dV(z,t)}{dt},
\end{align}
(2.3) (2.4)

In case the propagating modes have an $\exp(-i\omega t)$-dependence on time, where $\omega$ is the angular frequency, equations (2.3) and (2.4) simplify to
\begin{align}
\frac{dV(z)}{dz} &= -(R - i\omega L) I(z), \\
\frac{dI(z)}{dz} &= -(G - i\omega C) V(z)
\end{align}
(2.5) (2.6)

from which the wave equations for voltage and current are easily derived as
\begin{align}
\frac{d^2V(z)}{dz^2} &= \gamma^2 V(z), \\
\frac{d^2I(z)}{dz^2} &= \gamma^2 I(z).
\end{align}
(2.7) (2.8)

Here, $\gamma \equiv ((R - i\omega L)(G - i\omega C))^{1/2}$ is the wave’s propagation constant. The wave number of the mode is given by $k = \text{Im}(\gamma)$ and is related to the wavelength as $\lambda = 2\pi/k$. The mode’s phase velocity is $v_{\text{ph}} = \omega/k$ and its damping coefficient is given by $r = -\text{Re}(\gamma)$.

Solving equations (2.7) and (2.8) yields
\begin{align}
V &= V_0^+ e^{\gamma z} + V_0^- e^{-\gamma z}, \\
I &= I_0^+ e^{\gamma z} + I_0^- e^{-\gamma z},
\end{align}
(2.9) (2.10)

where the amplitudes $V_0^{+,-}$ and $I_0^{+,-}$ are determined by the boundary values of the transmission line. When inserting equation (2.9) into equation (2.5), we find
\begin{equation}
I = \frac{\gamma}{R - i\omega L} (V_0^+ e^{\gamma z} - V_0^- e^{-\gamma z}).
\end{equation}
(2.11)

Comparing this result to equation (2.10) we are led, by Ohm’s law, to the concept of a characteristic impedance. This quantity is conceptually the same as the index of refraction in optical systems. From the comparison, we find
\begin{equation}
Z_c \equiv \sqrt{\frac{R - i\omega L}{G - i\omega C}}
\end{equation}
allowing us to rewrite equations (2.10) and (2.11) as
\begin{equation}
I = \frac{V_0^+}{Z_c} e^{\gamma z} - \frac{V_0^-}{Z_c} e^{-\gamma z}.
\end{equation}
(2.12) (2.13)

In particular, if we have a lossless transmission line, $R$ and $G$ equal 0. This implies $Z_c = \sqrt{L/C}$ and $k = \omega\sqrt{LC}$. The phase velocity equals $v_{\text{ph}} = 1/\sqrt{LC}$. 

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2.3 Microwave reflection

Although generally one wants to avoid reflections in the system, in practise every component or transmission line will reflect some power. In the following, we will discuss two cases in which we can use these reflections to characterise a set-up.

2.3.1 Non-impedance matched, dispersion-less transmission lines

As mentioned, in microwave technology the characteristic impedance plays the role of the optical index of refraction. This implies that a sudden change in characteristic impedance will, analogously, cause partial wave reflection in the transmission line. In case a dispersionless transmission line is not impedance-matched to its environment, we can determine its characteristics from these reflections.

Suppose we have a change in characteristic impedance as sketched in figure 2.2. Then, if a wave traverses the line from region 1 to region 2, the transmission and reflection coefficients are given by [1]

\[
\tau_{1\rightarrow2} = \frac{2Z_{c,2}}{Z_{c,1} + Z_{c,2}}, \quad (2.14)
\]

\[
\rho_{1\rightarrow1} = \frac{Z_{c,2} - Z_{c,1}}{Z_{c,1} + Z_{c,2}}, \quad (2.15)
\]

where the arrows over the regions indicate the direction of the incoming and outgoing wave.

Now consider the full network in figure 2.2. It consists of two leads with characteristic impedance \(Z_{c,1}\), the characteristic impedance of the environment. In between the leads is a transmission line of unknown impedance \(Z_{c,2}\) with a length \(l\). Due to reflections on the \(Z_{c,1}, Z_{c,2}\)-boundaries this network will behave as a Fabry-Pérot interferometer showing typical peaks and troughs (wiggles) in its transmission as function of frequency.

\[\text{Figure 2.2: An arbitrary transmission line with characteristic impedance } Z_{c,2} \text{ is coupled to an environment with impedance } Z_{c,1}. \text{ From the wiggles in the transmission spectrum we can infer the inductance and capacitance per unit length of the transmission line.}\]

In case losses are small (\(\omega L \gg R\) and \(\omega C \gg G\)) the inductance and capacitance per unit length, \(L_2\) and \(C_2\) respectively, can be estimated using three parameters
from the transmission spectrum. First one needs the period of the wiggles (the free spectral range), \( f \), which is linked to the phase velocity of the wave propagating in the transmission line as

\[
\frac{\lambda}{l} = \frac{1}{v_{\text{ph},2}} f = \frac{1}{2} \rightarrow v_{\text{ph},2} = 2lf. \tag{2.16}
\]

Secondly the difference in transmission between the wiggle peaks and troughs is required

\[
\Delta \tau \equiv \frac{|T|_{\text{max}}}{|T|_{\text{min}}} = 10^{\Delta T_{\text{dB(m)}}}/20. \tag{2.17}
\]

Here, \( \Delta T_{\text{dB(m)}} \) is the difference between the minimum and maximum of the wiggles in the frequency spectrum expressed in dB(m). Finally one needs the loss-coefficient \( r \), which can be estimated from the spectrum as

\[
r = \frac{-IL}{20l \log_{10} e} \approx \frac{-\Delta T_{\text{p-b}}}{20l 10^{\log e}} \tag{2.18}
\]

with \( IL \) the insertion loss in dB, which is approximately equal to the difference in transmission between a transmission peak of the device and a by-pass line, \( \Delta T_{\text{p-b}} \). In fact, since the loss diminishes the amplitude of the wiggles, this approximation yields a slight overestimation of \( r \), hence it is beneficial to estimate the loss and wiggle amplitude at a frequency (range), in which the losses are small. All these values can be obtained from the transmission spectrum.

The transmission spectrum of the Fabry-Pérot interferometer is well-known to be

\[
T = \frac{\tau_{12} \tau_{23} e^{i(k-r)l}}{1 - \rho_{22} \rho_{23} e^{2i(k-r)l}}, \tag{2.19}
\]

omitting the arrows from equations (2.14) and (2.15). Taking the absolute value of this equation one arrives at

\[
|T|^2 = \frac{|\tau_{12}|^2 |\tau_{23}|^2 e^{-2rl}}{1 + \rho_{22}^2 |\rho_{23}|^2 e^{-4rl} - 2\rho_{22} \rho_{23} e^{-2rl} \cos (2kl)}. \tag{2.20}
\]

Neglecting the \( R \)- and \( G \)-contributions to the characteristic impedance (equation (2.12)), \( \tilde{\tau}_{\tilde{m} \tilde{n}} \in \mathbb{R} \) and \( \rho = \pm \rho_{22} = \pm \rho_{23} \in \mathbb{R} \), where the \( \pm \)-signs arise as we do not know whether \( Z_{c,1} > Z_{c,2} \) or \( Z_{c,1} < Z_{c,2} \) one finds from equation (2.20)

\[
\Delta \tau \equiv \frac{|T|_{\text{max}}}{|T|_{\text{min}}} = \frac{1 + \rho^2 e^{-2rl}}{1 - \rho^2 e^{-2rl}}. \tag{2.21}
\]

Solving for \( \rho \) yields

\[
\rho = e^{rl} \sqrt{\frac{\Delta \tau - 1}{\Delta \tau + 1}}. \tag{2.22}
\]
Finally, from equation (2.15)

\[ \rho = \pm \frac{Z_{c,2} - Z_{c,1}}{Z_{c,1} + Z_{c,2}} \]  

and one can solve for the unknown impedance as

\[ Z_{c,2} = \frac{\mp 1 - \rho}{\rho \mp 1} Z_{c,1}. \]  

(2.24)

To obtain \( L_2 \) and \( C_2 \) one can apply

\[ Z_{c,2} = \sqrt{\frac{L_2}{C_2}}, \quad v_{ph,2} = \frac{1}{\sqrt{L_2 C_2}} \]  

(2.25)

which holds for lossless networks. Whence

\[ L_2 = \frac{Z_{c,2}}{v_{ph,2}}, \quad C_2 = \frac{1}{Z_{c,2} v_{ph,2}}, \]  

(2.26)

or, taking the results from equations (2.16), (2.22) and (2.24) together

\[ L_2 = \frac{-Z_{c,1} e^{rt} \sqrt{(\Delta \tau - 1) / (\Delta \tau + 1)} \pm 1}{2 f_w e^{rt} \sqrt{(\Delta \tau - 1) / (\Delta \tau + 1)} \mp 1} \]  

(2.27)

\[ C_2 = \frac{-1}{2 Z_{c,1} f_w e^{rt} \sqrt{(\Delta \tau - 1) / (\Delta \tau + 1)} \mp 1} \]  

(2.28)

It should be noted that this calculation always yields two separate values for \( L_2 \) and \( C_2 \), corresponding to the high-impedance and low-impedance solution to the problem. If the line is really mismatched, it is possible to rule out one of the solutions on physical grounds. However, if the line impedance is close to the impedance of the environment, this is no longer possible.

As an example, consider the (theoretical) case in which we put a 80 Ω-line with a length of 10 cm in a 50 Ω-environment. Within the line the phase velocity equals \( 1 \times 10^8 \) m/s and the line has a frequency-dependent loss coefficient \( r = 0.2 f / [\text{GHz}] \) m\(^{-1}\). If we were given such a line and only know its length, we can use the theory presented in this section to estimate the line parameters using a measured frequency spectrum. The frequency spectrum as obtained from equation (2.19) is depicted in figure 2.3. From this figure, we estimate \( f_w \approx 500 \) MHz and \( \Delta T_{[\text{dB}]} = 0.790 \) dB and \( IL \approx 0.86 \) dB at \( f = 4.5 \) GHz (the first transmission peak). This yields, using equations (2.18), (2.17), (2.27) and (2.28), \( r = 0.99 \) m\(^{-1}\) (input 0.90 m\(^{-1}\)), \( L_2 = 0.808 \) or 0.309 \( \mu \)H/m and correspondingly \( C_2 = 124 \) or 232 pF/m. From the obtained \( L_2s \) and \( C_2s \) we estimate \( Z_{c,2} = 80.8 \) or 30.9 Ω and \( v_{ph} = 1 \times 10^8 \) m/s.

From our initial input we know that we would have to choose the first solution, which is indeed close to \( Z_{c,2} = 80 \) Ω. However, in practice we should make a decision based on the design of our device as this will constrain \( L \) and \( C \).
2.3. Microwave reflection

Figure 2.3: Theoretical frequency spectrum of a 10 cm-long, 80 Ω-transmission line. To obtain estimations of the necessary parameters for estimating the line characteristics of this transmission line, we can make a linear fit of the spectrum and subtract it from the data.

2.3.2 Reflection planes

Each connection in a microwave set-up reflects some power creating reflection planes. If one is interested from what part of the set-up reflections arise the following method can be used. First the transmission needs to be measured as a function of frequency. Although this spectrum might seem flat, each pair of reflection planes will, as in the last section, create a Fabry-Pérot cavity. From the specifications of the different components, we know the characteristic impedance (usually 50 Ω) and the phase velocity (usually 2c/3, where c is the speed of light). In order to find the periodicity of the Fabry-Pérot interference pattern, one takes the Fourier transform of the appropriately windowed spectrum. This yields $f_w$ as in last section, from which we can easily obtain the line length between the reflection planes as

$$l_{culp} = \frac{v_{ph}}{2f_w}$$

(2.29)

where “culp” refers to the culprits of the reflection.

Due to the scalloping loss of the window function and the loss in the transmission line(s) connecting the culprits, the wiggle amplitude resulting from the Fourier transform does not correspond to the actual wiggle amplitude. Therefore, this method does not provide a means to estimate the reflection coefficients of the reflection planes. Moreover, care should be taken that the calibration of the measurement apparatus is off during the measurement, as this will yield spurious results. An example is depicted in figure 2.4, where we have placed a NbTiN-transmission line of 15 cm in between two 1.5 m-long VNA cables using
female-female adapters. In the Fourier transform resulting from the frequency spectrum we can discern a peak at 15 cm, 1.5 m and 1.65 m corresponding to the line lengths in the set-up.

### 2.4 Coplanar waveguides

As a basic structure for the travelling-wave parametric amplifier (TWPA) discussed in chapter 5, we will use a superconducting coplanar waveguide (CPW), which is schematically depicted in figure 2.5. These transmission lines support quasi-transverse electromagnetic (TEM) modes. Their relevant characteristics are discussed in, among others, [2] and references therein, which we will review shortly.

Assuming the CPW has a low loss, the impedance is well-described by $Z_c = \sqrt{L/C}$. In ordinary CPWs, the inductance has a geometric and a kinetic contribution summing to the total inductance. The geometric inductance is given by

$$L_g = \frac{\mu_0}{4} \frac{E(\bar{w})}{E(\bar{w}')},$$

where $\mu_0$ is the permeability of free space and $E$ is the elliptic function of the first kind with arguments $\bar{w} = w_c/(w_c + 2w_g)$ and $\bar{w}' = \sqrt{1 - \bar{w}^2}$. $w_c$ is the width
2.4. Coplanar waveguides

Figure 2.5: Schematic representation of a coplanar waveguide. The CPW consists of a centre conductor in between two conducting ground planes on an isolating substrate. It supports quasi-TEM modes of EM-radiation.

of the CPWs centre conductor and \( w_g \) is the width of the gaps, see figure 2.5. This holds as long as the material around the CPW has a relative permeability \( \mu_r \approx 1 \). The kinetic contribution, due to the inertia of the Cooper pairs carrying the modes through the transmission line, is found from multiplying the sheet inductance

\[
L_S = \mu_0 \lambda \coth \left( \frac{d}{\lambda_m} \right)
\]

with a geometric factor \( g \) with unit \( m^{-1} \). Here, \( d \) is the thickness of the material. \( \lambda_m = \sqrt{\hbar \sigma / \pi \mu_0 \Delta_{sc}(0)} \) is the magnetic penetration depth, where \( \sigma \) is the material’s normal state sheet resistivity and \( \Delta_{sc}(0) = 1.76k_B T_c \) is the superconducting gap energy, assuming \( T \ll T_c \). The geometric factor contains two terms, one due to the contribution of the centre conductor and the other due to the contribution of the ground planes to the kinetic inductance. These contributions are given by

\[
g_c = \frac{1}{4w_c \tilde{\omega}^2 E^2(\tilde{\omega})} \left( \pi + \ln \left( \frac{4\pi w_c}{d} \right) - \tilde{\omega} \ln \left( \frac{1 + \tilde{\omega}}{1 - \tilde{\omega}} \right) \right), \tag{2.32}
\]

\[
g_g = \frac{\tilde{\omega}}{4w_c \tilde{\omega}^2 E^2(\tilde{\omega})} \left( \pi + \ln \left( \frac{4\pi (w_c + 2w_g)}{d} \right) - \frac{1}{\tilde{\omega}} \ln \left( \frac{1 + \tilde{\omega}}{1 - \tilde{\omega}} \right) \right), \tag{2.33}
\]

yielding a kinetic inductance of

\[
L_k = (g_c + g_g) L_S
\]

which is accurate to within 10\% for \( d < 0.05w_c \) and \( \tilde{\omega} < 0.8 \). This yields a total inductance of \( L = L_g + L_k \).

The capacitance of the CPW is given by

\[
C_g = 4\varepsilon_0 \varepsilon_{\text{eff}} \frac{E(\tilde{\omega}')}{E(\tilde{\omega})}
\]

where \( \varepsilon_0 \) is the vacuum permittivity and \( \varepsilon_{\text{eff}} \) is the effective relative permittivity of the CPW given by the average relative permittivity of the material below
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Figure 2.6: Sonnet-implementation for calculating the resonator coupling. The resonator (in green) is coupled to a feedline (in purple) between ports 1 and 2. In order to calculate the coupling strength we need the magnitude of the scattering parameter $S_{13}$ at the resonator’s frequency.

and on top of the conducting material. In case one of these is vacuum, $\epsilon_{\text{eff}} = (1 + \epsilon_r) / 2$, where $\epsilon_r$ is the relative permittivity of the substrate.

For a CPW that has a centre conductor width of $12 \mu m$, gaps of $5 \mu m$, made out of 200 nm-thick NbTiN ($T_c = 15.1 \, \text{K}$, $\sigma = 1.06 \, \mu \Omega \text{m}$) on a silicon substrate ($\epsilon_r = 11.45$) at mK-temperatures, $L = 470 \, \text{nH/m}$ and $C = 177 \, \text{pF/m}$ indeed yielding $Z_c \approx 50 \, \Omega$.

2.5 Microwave resonators (CPW)

To obtain a high-gain, large-bandwidth TWPA, some form of dispersion engineering is necessary in order to reach the phase-matching condition. Therefore, as a last section of this chapter, we will review CPW microwave resonators that can be used for this purpose. Here we follow [2, 3] and references therein.

CPW resonators come in two types. Either both ends are open or closed, in which case we have a $\lambda/2$-resonator or one of the ends is closed yielding a $\lambda/4$-resonator. The advantage of the latter is that they are shorter than $\lambda/2$-resonators. The resonant frequencies of such resonators are given by

$$f_{r,n}^0 = \frac{m}{n l_r \sqrt{L_r C_r}}. \quad (2.36)$$

Here, $n$ determines the resonator type, $n = 2$ for a $\lambda/2$-resonator and $n = 4$ for a $\lambda/4$-resonator. $m$ is the order of the resonance. For $\lambda/2$-resonators, $m \in \{1, 2, 3, 4, \ldots\}$ and for $\lambda/4$-resonators $m \in \{1, 3, 5, 7, \ldots\}$.

Upon coupling a resonator to a transmission line the resonator is loaded and as
a result the resonance frequency shifts. This shift is given by

\[ \delta f_{r,n} = -\sqrt{\frac{2}{\pi Q_c}} f_{r,n}^0 \]  

(2.37)

such that the resonance frequency of a loaded resonator is \( f_{l,r,n} = f_{r,n}^0 + \delta f_{r,n} \). In equation (2.37), \( Q_c \) is the coupling quality factor, which depends mainly on the magnitude of the coupling capacitance and the frequency. It can be determined as

\[ Q_c = \frac{2\pi n}{m |S_{13}|^2} \]  

(2.38)

where \( S_{13} \) is the scattering parameter from port 3 to port 1 at \( f = f_{r,n}^0 \), see figure 2.6. It can be obtained from Sonnet-simulations.

The width of the resonance, \( \Delta f_{l,r} \), is determined by the loaded quality factor, which can be calculated from

\[ \frac{1}{Q_l} = \frac{\Delta f_{l,r}}{f_{l,r}} = \frac{1}{Q_c} + \frac{1}{Q_i}. \]  

(2.39)

Here, \( Q_i \) is the internal quality factor of the resonator, which can be calculated as \( Q_i = k/2r \).

References


