Radiative transfer in a spherical dust cloud

I. Exact results for isotropic scattering

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Received January 2, accepted August 23, 1986

Summary. Exact radiative transfer equations are used to derive the penetration of diffuse starlight into a spherical interstellar dust cloud and the diffuse reflection from the same cloud, both on the assumption of isotropic scattering.

We employ an old mapping theorem stating that the Milne equation for a homogeneous slab of isotropic scatterers with albedo \( a \) and its equivalent for a homogeneous spherical cloud have such forms that any symmetric solution in a sphere of optical radius \( r \) can be rigorously mapped on an antisymmetric solution in a slab of optical thickness \( b = 2r \).

The zero-order term, i.e. the radiation density of unscattered light in the sphere, is derived for radiation incident under one angle, for uniform incident radiation (\( U \)), and for incident radiation from a narrow layer of light sources just around the sphere (\( N \)).

Procedures are outlined for computing by means of the mapping theorem the full radiation density in the sphere and the spherical reflection function and its moments, using available analytical and numerical results for slabs. A surprising result is that the bimoments of the reflection function of the sphere, including its Bond albedo, are expressed exactly in terms of the well-studied moments of the \( X \) - and \( Y \)-functions for the slab. This result is related by reciprocity to an earlier result of Heaslet and Warming for a problem with internal sources.

A worked example for \( r = 0.25 \) illustrates the use of this scheme. Leading terms for small \( r \) and asymptotic expressions for large \( r \) are derived.

Finally, a graph of the albedo of the entire cloud (the Bond albedo) over the full range of \( a \) and for all cloud sizes is given. Each curve for fixed \( a \) reaches a maximum at some moderate optical depth of the cloud. Results for asymmetric phase functions will be presented later.

Key words: radiation transfer – interstellar clouds – interstellar dust

1. Introduction

Starlight falling on an interstellar dust cloud goes in part straight through, is in part diffusely reflected after one or several scattering events, and is in part absorbed inside the cloud. All these parts have consequences of astrophysical interest for reflection nebulae, or for the far infrared emission, or for the shielding of the inner part of the cloud from penetrating UV radiation. The correct interpretation of observational data requires model radiative transfer computations. Our recent interest in these models was kindled by the IRAS data on dust clouds (De Vries, 1986).

Many methods and codes have been used for such model computations. If we limit ourselves to homogeneous spherical clouds, as we shall do throughout this series of papers, notably the Monte Carlo method (Mattila, 1970; Sandell and Mattila, 1975) and the spherical harmonics method, leading to eigenfunction expansions (Flannery et al., 1980) deserve mention. The three key parameters are:

- \( a = (0, \ldots , 1) \) the albedo of single scattering by a dust grain
- \( g = 0, \ldots , 1 \) the asymmetry factor of the phase function
- \( b = 2r = (0, \ldots , \infty ) \) the optical depth along the diameter of the cloud

This parameter space is covered very incompletely in the published literature and the diffuse reflection is often treated separately from the penetration and absorption, or not at all. From unpublished material kindly made available by Dr. Mattila we found that the asymptotic behaviour for \( b \gg 1 \) followed clear empirical laws. These will be reported in a separate paper. But the precise theoretical understanding of these laws required too many intuitive guesses. This made us turn to the isotropic case (\( g = 0 \)) in search of some firm results useful as checks. The search turned out to be successful beyond expectation and the present paper reports the result.

2. Assumptions, definitions, notation

We assume that the scattering particles are contained in a spherical cloud of radius \( r \) and that the outside space is empty.

The mean free path is taken as the unit of length. Each distance inside the cloud thus is at the same time an optical path length. The scattering by each volume element is isotropic. The albedo \( a \) is constant throughout the cloud.

Various assumptions on the incident radiation field will be made but we shall confine the discussion (in this paper) to strict spherical symmetry. This means that the radiance is written as \( I(x, u) \), where \( x \) is the distance from the center of the cloud and \( u \) is the cosine of the angle with the outward radial direction. The values of \( I \) for negative \( u \) at the surface of the cloud, \( x = r \), constitute the incident radiation. We shall often write \( \mu_0 = -u \) in this case so that \( I(\mu_0) \).

Likewise we may (still at \( x = r \)) write \( \mu = u \) for positive \( u \) so that
\(I(\mu), 0 < \mu \leq 1\) is the emergent radiance. At the center \((x = 0)\) the radiance is isotropic. These conventions agree with the book ‘Multiple Light Scattering’ (Van de Hulst, 1980), to which we shall make frequent reference under the abbreviation MLS.

An arbitrary angular distribution of incident light \(I(\mu_0)\) is conceptually useful although it will not be encountered in practice. One practical situation is that of uniform illumination \(I(\mu_0) = 1\) (indicated in MLS by the symbol \(U\) of uniform) as will be caused by stars at infinity, illuminating the cloud equally from all directions. Another interesting assumption is \(I(\mu_0) = 1/(2\mu_0)\) (indicated in MLS by the symbol \(N\) of narrow layer) as will arise from a fictitious distribution of isotropically radiating sources (stars) placed in a narrow spherical shell all around the cloud immediately above the cloud’s surface. The same distribution will be mimicked by the first-order scattered light if the incident radiance is limited to grazing angles (\(\mu_0 = 0\)). \(U\) and \(N\) are the extremes for \(c = \infty\) and \(c = r\), respectively, of the angular distribution of the inward directed radiation at the surface of the cloud
\[
I(\mu_0) = \frac{r^2}{2(c - (c^2 - r^2)^{1/2})} \cdot \frac{1}{(c^2 - r^2 + r^2 \mu_0^2)^{1/2}}
\]  

caused by stars uniformly filling a spherical surface with radius \(c > r\) surrounding the cloud. The normalization in each case has been chosen so that
\[
\int_0^1 I(\mu_0) d\mu_0 = 1
\]

It is customary to speak of the integral (2) as the ‘flux integral’, although the actual flux, defined as the energy stream inward per unit time through a unit area of the surface is a factor \(\pi\) times this integral.

The mean (i.e., omnidirectional average) radiance at distance \(x\) from the center,
\[
J(x) = \frac{1}{2} \int_{-1}^1 I(x, u) du
\]

is proportional to the radiation density and will for brevity be referred to as such. The radiance reemitted isotropically in each direction by a volume \(dV\) is \(a J(x) dV\) and \(a J(x)\) is called the source function.

Any of the quantities defined for an arbitrary value of \(a\) can be written as a power series in \(a\). For instance, the radiation density is
\[
J(x) = J_0(x) + a J_1(x) + a^2 J_2(x) + \cdots
\]

Here \(J_0(x)\) will be called the term due to zero-th order scattering because it describes the radiation density of photons that have suffered zero scatterings in succession.

### 3. Exact mapping theorem from a sphere onto a slab

Since much thought has gone into solving radiative transfer problems for plane-parallel layers (slabs) by a large variety of methods, a method to reduce the spherical problem to a plane one can be of great use.

A **mapping theorem** doing exactly that has been in the literature at least since Davison (1957, p. 96). It has been referred to in several places (e.g., Schmidt-Burgk, 1973) but has to my knowledge been used only twice in the solution of specific spherical transfer problems, both with internal sources (Heaslet and Warming, 1965; Gruschinske and Ueno, 1970). We briefly restate this mapping theorem, which is valid only under the restrictive assumptions made in Sec. 2, and then use it for solving the problem at hand.

The \(n\)-th order radiation density at an arbitrary point \(P\) in a convex, homogeneous cloud is made of photons that have suffered \((n - 1)\)st scattering at any point \(Q\) in the cloud as follows:
\[
J_n(P) = \int_{\text{total cloud}} J_{n-1}(Q) \frac{e^{-s}}{4\pi s^2} dV
\]

where \(s\) is the (optical) distance \(PQ\) and \(dV\) is the volume element around \(Q\). Summing all orders with albedo \(a\) we obtain by (4) and (5)
\[
J(P) = J_0(P) + a \int_{\text{total cloud}} J(Q) \frac{e^{-s}}{4\pi s^2} dV
\]

where \(J_0(P)\) represents the photons of the incident light that happen to have penetrated to the point \(P\) without scattering at all. We may switch as we please between formulations (5) and (6) but in any case the first step of the computation is to find the starter function \(J_0\).

In a plane-parallel layer, which we shall distinguish in this paper by the symbol \(\parallel\), Eq. (5) works out as the well-known Milne equation, written for separate orders as
\[
J_0(t) = \frac{1}{2} \int_0^{2\pi} \int_0^r J_{n-1}(t') E_1 |t - t'| \, dt'
\]

and for all orders combined as
\[
J(t) = J_0(t) + \frac{1}{2} \int_0^{2\pi} \int_0^r J(n)(t') E_1 |t - t'| \, dt'
\]

Here \(2\pi\) is the total optical thickness of the layer, \(E_1(x)\) is the exponential integral, and the optical depth \(\tau\) in the layer runs from 0 to \(2\pi\).

In a spherical cloud with radius \(r\) Eq. (5) works out to the form
\[
J_d(x) = \frac{1}{2\pi} \int_0^r J_{n-1}(y) E_1(x - y) - E_1(x + y) \, dy
\]

degenerating at \(x = 0\) to
\[
J_0(0) = \int_0^r J_{n-1}(y) e^{-s} \, dy
\]

and correspondingly, from Eq. (4):
\[
J(x) = J_d(x) + \frac{1}{2\pi} \int_0^r J(y) E_1(x - y) - E_1(x + y) \, dy
\]

Equation (9) will be referred to as the spherical Milne equation. It may be derived by taking \(P\) and \(Q\) at distances \(x\) and \(y\) from the center, seen under an angle \(\varphi\). Then \(s^2 = x^2 + y^2 - 2xy \cos \varphi\) and \(dV = 2\pi r^2 \, dy \, d(\cos \varphi)\). The integration over \(\varphi\) between \(\cos \varphi = -1\) and \(+1\) then becomes an integration over \(s\) between \(s_{\min} = |x - y|\) and \(s_{\max} = x + y\) and the rest follows easily.

It is important that Eqs. (7) and (9) are not only in a vague sense analogous but can be used for a rigorous mapping of the
spherical problem onto the plane problem and conversely. This is expressed in the following mapping theorem.

If \( J_\nu(x) \) is a solution of Eq. (7) on the interval \((0, 2r)\) then

\[
J_\nu(x) = \frac{1}{x} \left( x J_\nu\!\!\!\!\, (r - x) - J_\nu\!\!\!\!\, (r + x) \right)
\]

(12)

is a solution of Eq. (9) on the interval \((0, r)\). At \( x = 0 \), Eq. (12) breaks down and becomes

\[
J_\nu(0) = -2 [d J_\nu\!\!\!\!\, (x)/dx] \bigg|_{x=r}
\]

(13)

Conversely, any solution \( J_\nu(x) \) for the spherical problem may be mapped onto a solution

\[
J_\nu\!\!\!\!\, (x) = \frac{1}{2} x J_\nu\!\!\!\!\, (x) + G_\nu\!\!\!\!\, (x)
\]

(14)

for the plane problem, where \( x = r - \tau \) and \( G_\nu\!\!\!\!\, (x) \) is an arbitrary even function of \( x \).

The mapping theorem, Eq. (12), has here been expressed in terms of separate orders but may by Eq. (11) be equally well expressed for all orders combined. However, the term \( J_\nu(x) \) expressing the unscattered radiation cannot generally be obtained by mapping and has to be addressed separately.

We have not succeeded in tracing further historical roots of the mapping theorem. Davison does not present it as a new and striking result, nor does he give any reference or suggest its practical value in computation. One immediate consequence has caught wide attention (e.g. Case and Zweifel, 1967, p. 159–160). The eigenvalues of the homogeneous spherical Milne equation, or alternatively of the equation

\[
J_\nu(x) = \eta J_{\nu-1}(x)
\]

(15)

if numbered from small to large follow the rule

\[
\eta_1 < \eta_2 < \ldots < \eta_n < \ldots
\]

(16)

In particular, the lowest value \( \eta_1 \) for the sphere equals the second-lowest value for the plane slab, \( \eta_{3}^{slab} \). The series (4) behaves for large \( n \) as a geometric series with the ratio \( a_{n+1} = \nu a_n \) and if this ratio would be 1 (requiring \( a > 1 \)) criticality would be reached. As a numerical example, we may read from MLS p. 141 that a plane layer with total optical thickness 2 has \( \eta_{3}^{slab} = 0.783 \), \( \eta_{5}^{slab} = 0.503 \), \( \eta_{7}^{slab} = 0.352 \). Hence a sphere with radius 1 has \( \eta_1 = 0.503 \). A further example is given in Sect. 7.2.

4. The zero-order term

The zero-order term \( J_0(x) \), i.e., the radiation density of the unscattered radiation, acts as the starter to the recurrence scheme, Eq. (9), or as the known term in Eq. (11). No simple ‘mapping rule’ from the spherical onto the plane problem exists for this term. Radiation incident on a plane layer in the direction \( \mu_0 \) can with increasing attenuation reach arbitrary depth, while radiation incident with fixed \( \mu_0 \) on a sphere reaches in a straight line the minimum value

\[
x_{\min} = r(1 - \mu_0^2)^{1/2}
\]

(17)

and therefore a maximum optical depth from the surface, \( r - x_{\min} \). Accordingly, unscattered radiation can reach a point at distance \( x \) from the center only from the range of directions of incidence between \( \mu_0 = 1 \) and

\[
\mu_{0,\min} = (1 - x^2/r^2)^{1/2}
\]

(18)

The center \((x = 0)\) can be reached only by perpendicular incidence \((\mu_0 = 1)\).

We now wish to find \( J_0(x) \) in a sphere of radius \( r \), illuminated from outside by a radiation field \( I(\mu_0) \). The notation is explained in Fig. 1. With \( x \) and \( r \) fixed, all angles and the optical paths vary with the point \( Q \) at which the radiation enters the cloud. While \( s \) goes from \( r - x \) to \( r + x \), both \( \beta_0 \) and \( \beta_\theta \) swing (at unequal rates) from 0° to 180° while \( \mu_0 = \cos \beta_0 \) varies from 1 to \( \mu_{0,\min} \) and back. Any of these may be taken as the integration variable. We shall not spell out all the trigonometry. The basic equation is

\[
J_0(x) = \frac{1}{2} \int_{-1}^{1} I(\mu_0) e^{-s} \, d\mu_0
\]

(19)

where \( b_0 = \cos \beta_0 \) and \( \mu_0 \) and \( s \) must be expressed in \( r \) and \( b_0 \). In order to see how the various directions of incidence contribute we switch to \( \mu_0 \) as the integration variable, and find

\[
J_0(x) = \frac{1}{2x} \int_{\mu_{0,\min}}^{1} I(\mu_0) e^{-\mu_0} (e^{-x b_0} + e^{x b_0}) \frac{r^2 \mu_0 \, d\mu_0}{x b_0}
\]

(20)

with

\[
x b_0 = \left( x^2 - r^2(1 - \mu_0^2) \right)^{1/2}
\]

(21)

Note that \( b_0 \) becomes 0 and the integrand infinite as \( \mu_0 \) approaches \( \mu_{0,\min} \). This means that a relatively large contribution to \( J_0(x) \) comes from directions close to \( \mu_{0,\min} \) as we should expect.

The rather artificial assumption that the incident radiation comes from a very narrow range of directions about a fixed value \( \mu_0 \) and is normalized by Eq. 2 leads from Eq. (20) to

\[
J_0(x, \mu_0) = \frac{1}{4x} e^{-\mu_0} (e^{-x b_0} + e^{x b_0}) \frac{r^2}{x b_0}
\]

(22)

where \( x b_0 \) is again given by (21). This distribution function goes to \( \infty \) at \( x = x_{\min}, b_0 = 0 \). If \( \mu_0 = 1 \), the singularity is at the center.

If \( I(\mu_0) \) covers all \( \mu_0 \), it is better to adopt \( s = r \mu_0 - x b_0 \) as the integration variable giving with \( db_0/d s = -(r^2 - x^2 + s^2)/2x s^2 \) yet a third form equivalent to (19) and (20):

\[
J_0(x) = \frac{1}{2} \int_{-1}^{1} I(\mu_0) e^{-s} \left( \frac{r^2(x^2 + s^2) + 1}{2x s^2} \right) ds
\]

(23)

Two assumptions about \( I(\mu_0) \) are of special interest because they frequently appear in practical problems.

Uniform incidence (for which we shall use symbol \( U \) in accordance with MLS) has

\[
I(\mu_0) = 1
\]

(24)
This gives
\[
J_0^r(x) = \frac{1}{4x} \left\{ (r + x)E_2(r - x) - (r - x)E_2(r + x) + e^{-(r-x)} - e^{-(r+x)} \right\}
\]
where \(E_2(y)\) is the exponential integral. A further transformation may be made to
\[
J_0^r(x) = \frac{1}{2x} \left\{ E_2(r - x) - E_2(r + x) + rE_2(r - x) - rE_2(r + x) \right\}
\]
(26)

Incidence due to a narrow source layer (symbol \(N\)) is defined by
\[
I(\mu_0) = \frac{1}{2\mu_0} \frac{rs}{r^2 - x^2 + s^2}
\]
(27)
This gives the even simpler result
\[
J_0^N(x) = \frac{r}{4x} \left\{ E_2(r - x) - E_2(r + x) \right\}
\]
(28)
The Eqs. (17) to (28) are exact, i.e., valid for arbitrary \(r\) and \(x\). The functions derived are continuous at \(x = 0\) and reach at that point for arbitrary \(I(\mu_0)\) the limit
\[
J_0^r(0) = \left(\frac{1}{2}\right) e^{-r}
\]
(29)
found easily from Eq. (19). Equation (26) gives in the limit \(x = 0\) indeed \(J_0^r(0) = e^{-r}\) and Eq. (28) gives \(J_0^N(0) = \frac{1}{2} e^{-r}\).

The Eqs. (23), (22), (25), (26), (28) all happen to have the form (9) so that the appropriate starter function \(J_0^r(x)\) for mapping on the slab may immediately be read from the equation. The \(J_0^r(x)\) thus defined is the starter function for the plane slab but it is not generally the one corresponding to the assumed incident light distribution \(I(\mu_0)\) falling on the slab. In both examples it does, however, correspond to a different incident light distribution \(I^r(\mu_0)\). We find
\[
I(\mu_0) = 1 \text{ for sphere requires } I^r(\mu_0) = r + \mu_0 \text{ for slab (30)}
\]
\[
I(\mu_0) = \frac{1}{2\mu_0} \text{ for sphere requires } I^r(\mu_0) = \frac{r}{2\mu_0} \text{ for slab (31)}
\]
In the condensed notation to be introduced in sec. 6, the rule just found translates to
\[
I_0 = U \text{ (sphere)} \text{ requires } I_0^r = rU + W \text{ (slab) (30)}
\]
\[
I_0 = N \text{ (sphere)} \text{ requires } I_0^r = rN \text{ (slab) (31)}
\]

5. The full radiation density

Having thus completed the calculation of the starter function \(J_0\) we are on an established track. To go from there to the full radiation density \(J^r(r)\) for the slab means solving the Milne Eq. (8), for which a choice of methods is available. The final step then is to convert to the full radiation density for the sphere \(J(x)\) by
\[
J(x) = \frac{1}{x} \left\{ J^r(r - x) - J^r(r + x) \right\}
\]
(32)
which is based on the fact that now Eq. (12) is valid for all orders, including order 0.

The actual choice of method will depend on requirements and available means. A computer code for solving the Milne equation will serve for arbitrary \(I(\mu_0)\) by (20), (22) or (23). Distributed incidence, \(U\) or \(N\), may not require going through the zero order separately, because existing codes or ready tables (moments of point-direction gain, see MLS p. 208) may be available for the inputs \(N, U\) and \(W\) required by (30) and (31). For very small \(r\) or very large \(r\) expansions may be more appropriate. Any of these methods is eligible. They may be used separately or combined. Some illustrations are found in the final sections of this paper.

6. The spherical reflection function, its moments and bimoments

6.1. Definitions, condensed notation

A condensed notation is employed in the remainder of this paper. Functions of \(\mu\) or \(\mu_0\), or both, are written as bare symbols (mostly capitals) and a symbolic multiplication of two such functions stands for an integration over \(\mu\), or \(\mu_0\) over the interval (0, 1), always with an extra factor 2\(\mu\), or 2\(\mu_0\), in the integrand. Thus
\[
FG \text{ stands for } \int_0^1 F(\mu)G(\mu)2\mu d\mu
\]
(33)
Further, three symbols are introduced for simple functions of \(\mu\) (or \(\mu_0\))
\[
N = 1/(2\mu), \quad U = 1, \quad W = \mu
\]
(34)
Symbolic multiplication of any \(F = F(\mu)\) by these symbols means calculating a moment of order 0, 1, or 2. The reflection function \(R(\mu, \mu_0)\), or any other function of these two variables, may be subject to front multiplication by \(N\), \(U\) or \(W\), meaning an integration over \(\mu\), which yields a moment (e.g. \(UR\)) that is a function of \(\mu_0\). Rear multiplication stands for integration over \(\mu_0\) and results in a moment (e.g. \(RU\)) which is a function of \(\mu\). Multiplication at both sides results in a bimoment (e.g. \(URU\)).

This condensed notation is the same as used in MLS p. 69. Its use in the remaining sections of this paper saves about 90 integral signs and many hundreds of times writing \(\mu\) or \(\mu_0\); compare Eqs. (35) and (35)’ below. However, the main advantage of this notation is that the structure of the equations, particularly those with many double integrations, remains more clearly visible.

We now turn to defining a spherical reflection function. The slab permits a clear distinction between reflection function \(R(\mu, \mu_0)\) describing the radiance emerging in direction \(\mu\) at the illuminated side and the transmission function \(T(\mu, \mu_0)\) doing the same for the opposite side. The full relations are
\[
I^{\text{in}}(\mu) = \int_0^1 R(\mu, \mu_0)I^{\text{in}}(\mu_0)2\mu_0 d\mu_0
\]
(35)
and
\[
I^{\text{out}}(\mu) = \int_0^1 T(\mu, \mu_0)I^{\text{out}}(\mu_0)2\mu_0 d\mu_0
\]
(36)
but in condensed notation they are written as
\[
I^{\text{in}} = R^{\text{in}}, \quad I^{\text{out}} = T^{\text{in}}
\]
(35)(36)
A zero-order term in \(T(\mu, \mu_0)\) describes the light penetrating the slab without scattering. It contains, for a slab with thickness 2\(r\),
the attenuation factor $\exp(-2r/\mu)$. The reflection function has no zero-order term.

The sphere permits no such distinction. The radiance emerging from any point of the surface with direction $\mu$ equals

$$ R_{\text{out}}(\mu) = R_{\text{in}}(\mu)e^{-2r/\mu} + \int_0^{2\mu} J(X) e^{-s} ds $$

(37)

Here $r$ is the radius, $a$ the albedo, $s$ is the optical depth along the line of sight, $2\mu$ is the length of the chord, $R_{\text{in}}(\mu)$ incident radiance at the far end of the chord, and $J(X)$ the radiation density at any internal point as calculated in the preceding section.

The linearity of (37) combined with the linearity of (11) and (20) or (23) guarantees that (37) may be written in the form

$$ R_{\text{out}}(\mu) = \frac{1}{2} \int_0^\mu R(\mu, \mu_0) R_{\text{in}}(\mu_0) d\mu_0 $$

(38)

This equation, identical in form to (35), will be regarded as the definition of the spherical reflection function $R(\mu, \mu_0)$. In condensed notation:

$$ R_{\text{out}} = R_{\text{in}} $$

(39)

We consistently call the zero-order term, corresponding to the first term at the right side of (37), zero-order reflection, in spite of the fact that it refers to light that has travelled through the sphere without any scattering at all.

The radically different attenuation factors in the zero-order terms of the slab and sphere problems make it all the more surprising that there should be any useful relationship at all. We now give a systematic (but not exhaustive) description of suggested procedures to compute $R(\mu, \mu_0)$ and its moments and bimoments.

6.2. The reflection function

Since the zero-order term is written separately, the second term of (37) describes the diffusely reflected light. This is true for an arbitrary $I(\mu_0)$ if we specify $J_{\text{d}}(x)$ by (20) and then compute $J(x)$ by (32), but upon specifying the particular form (22) of $J_{\text{d}}(x)$ the same procedure yields the properly normalized diffuse reflection function $R(\mu, \mu_0) - R_{\text{d}}(\mu, \mu_0)$. In either case, the integration over $s$ may be transformed into an integration over $x$ by referring back to Fig. 1 and Eqs. (17) and (21), now dropping the index 0 of $b_0$ and $\mu_0$. The same $x$ occurs twice along the path with $s = ry - xb$ and $s = ry + xb$, respectively. This gives two terms in the $x$-integration. Assuming (22) for the specification of $J_{\text{d}}(x)$, we thus obtain from the second term of (37):

$$ R(\mu, \mu_0) - R_{\text{d}}(\mu, \mu_0) = ae^{-r}/\mu \int_{x_{\text{min}}}^r J(x)(e^{-xb} + e^{-xb}) dx/b $$

(39)

Input and output, Eqs. (22) and (39), contain apart from the index 0 the same functions of $x$, $b$, and $\mu$. This expresses the principle of reciprocity, just as it does for the slab (MLS, Ch. 7) and makes $R(\mu, \mu_0)$ a symmetric function. This symmetry is further illustrated by the explicit form of the first-order term

$$ aR_{\text{1}}(\mu, \mu_0) = \frac{ar^2}{4} e^{-r}(\mu + \mu_0) \int_{x_{\text{min}}}^r (e^{-xb} + e^{-xb})(e^{-xb} + e^{-xb}) dx/xb \cdot xb_0 $$

(40)

where $x_{\text{min}}$ is the higher of the two $x_{\text{min}}$ values defined for $\mu$ and $\mu_0$. No further simplification seems possible.

6.3. The moments of the reflection function

Precisely as with the slab, moments of the reflection function may be obtained by integration over $\mu_0$ yielding a function of $\mu$ or by integration over $\mu$ yielding a function of $\mu_0$. The symmetry of $R(\mu, \mu_0)$ makes these functions identical, apart from the argument. See further explanation in MLS. In order to find $RU$ it seems simplest to start with input $U$, take $J_{\text{d}}(x)$ from (26) or equivalently $J_{\text{d}}(x)$ from (30), compute the full radiation density by any of the options suggested in Sect. 5, and find the output in direction $R(\mu)$. The alternative route starting from (22) and ending with the output $U$ to find $UR$ is also possible but seems less attractive. Finding $RN$ (or $NR$) proceeds along exactly the same lines using (28) or (31).

It is well known that all moments of the reflection and transmission functions of a slab may be expressed in terms of the $X$- and $Y$-functions. Such a reduction is not possible for the sphere.

6.4. The bimoments of the reflection function

The real bonus of the mapping theorem comes in computing the bimoments $NRN$, $URN = NRU$, and $URU$ of which

$$ URU = \frac{1}{2} \int_0^\mu R(\mu, \mu_0) d\mu_0 d\mu_0 d\mu_0 $$

(41)

is the fraction of the total incident energy which is returned (with or without scattering) by the cloud back into space. This is the physical interpretation for uniform incident radiation from all sides but for symmetry reasons it is also correct in the case of illumination by one distant star.

The starter terms for inputs $N$ and $U$ were worked out in Eqs. (28) and (26), respectively, and were translated into equivalent inputs for the slab in Eqs. (31) and (30). It should not be a surprise that the output integration, which results if (37) is subjected to integration over $\mu$, leads to exactly the same functions of $x$ and, hence, to equivalent outputs again given by (31) and (30). The zero orders for both the spherical and the slab problem are excluded from this relation but can be computed separately by elementary means.

We thus find that the bimoments of the spherical reflection function do not require new numerical integrations but can be expressed in terms of the bimoments for the slab. The formal derivation requires a more cumbersome notation and will be presented in a separate paper (Van de Hulst, 1978a). The result is so simple that it can almost be guessed and reads:

$$ N(R - R_0)N = N(R - R_0)N + T^\Omega $$

$$ N(R - R_0)U = N(R - R_0)U - T^\Omega $$

(42)

$$ U(R - R_0)U = (U + r^{-1}W)(R - R_0)U + T^\Omega $$

$$ (U - R_0)U = (U + r^{-1}W)(R - R_0)U + T^\Omega $$

(43)

(44)

The left sides refer to the sphere of radius $r$ and the right sides to the slab with thickness $2r$, both sides in condensed notation. The only remaining job is to find the bimoments for the slab and the zero-order bimoments at both sides.

The zero-order moments and bimoments for the sphere are found by elementary means to be, with $2r = b$:

$$ R_0N = \frac{1}{2\mu} e^{-bu} $$

$$ R_0U = e^{-bu} $$

(45)

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\[ NR_0 N = \infty \]
\[ NR_0 U = \frac{1}{b} (1 - e^{-b}) \]
\[ UR_0 U = \frac{2}{b^2} (1 - e^{-\gamma}) - \frac{2}{b} e^{-b} \]

The zero-order moments with \( N \) and \( U \) for a slab are given in MLS, Display 9.1 (p. 194–195) and had to be supplemented with the \( W \)-moment giving the set
\[
\begin{align*}
T^{\|}_N &= \frac{1}{2 \mu} e^{-b/a} \\
T^{\parallel}_U &= e^{-b/a} \\
T^{\perp}_W &= \mu e^{-b/a}
\end{align*}
\]

from which the 6 necessary bimoments (three known from Display 9.1, three new) are expressed in terms of exponential integrals as
\[
\begin{align*}
NT^{\|}_N &= \frac{1}{2} E_1(b) \\
UT^{\parallel}_U &= 2 E_2(b) \\
UT^{\perp}_W &= E_2(b) \\
WT^{\|}_N &= E_2(b) \\
WT^{\parallel}_U &= 2 E_4(b)
\end{align*}
\]

Note that the zero-order terms of sphere and slab cannot possibly cancel, first because they have very different functional forms, and secondly because when brought to the right side of Eqs. (42)–(44) they both have positive sign.

All moments of \( R^{\parallel} \) and \( T^{\parallel} \) are expressible in terms of the functions \( X(\mu) \) and \( Y(\mu) \) and all bimoments correspondingly in terms of the moments of the \( X \) - and \( Y \)-functions \( \alpha_\nu \) and \( \beta_\nu \). Since only the difference \( R^{\parallel} - T^{\parallel} \) occurs in (42)–(44) the results look simpler when expressed in terms of
\[
\begin{align*}
C(\mu) &= X(\mu) + Y(\mu), \quad \gamma_\nu = \alpha_\nu + \beta_\nu, \quad \gamma^*_{-1} = \alpha^*_{-1} + \beta_{-1} \\
D(\mu) &= X(\mu) - Y(\mu), \quad \delta_\nu = \alpha_\nu - \beta_\nu, \quad \delta^*_{-1} = \alpha^*_{-1} - \beta_{-1}
\end{align*}
\]

The merits of using these sums and differences have been discussed in a wider context by Hovenier (1978). The resulting moments (two from MLS, Display 9.1, one newly derived) are
\[
\begin{align*}
(R^{\parallel} - T^{\parallel}) N &= \frac{1}{2 \mu} \{ D(\mu) - 1 \} \\
(R^{\parallel} - T^{\parallel}) U &= 1 - (1 - 2 a \delta) C(\mu) \\
(R^{\parallel} - T^{\parallel}) W &= - \mu + \frac{1}{2} a \delta C(\mu) + (1 - \frac{1}{2} \sigma \gamma) \mu D(\mu)
\end{align*}
\]

and the corresponding bimoments (three from Display 9.1, three new) are
\[
\begin{align*}
N(R^{\parallel} - T^{\parallel}) N &= \frac{1}{2} \delta^*_{-1} \\
U(R^{\parallel} - T^{\parallel}) N &= \delta_0 - 1 \\
W(R^{\parallel} - T^{\parallel}) \delta_1 &= \delta_{-1} - \frac{1}{2} \\
U(R^{\parallel} - T^{\parallel}) U &= 1 - (2 - a \delta) \gamma_1 \\
W(R^{\parallel} - T^{\parallel}) U &= \frac{1}{2} - (2 - a \delta) \gamma_2 \\
W(R^{\parallel} - T^{\parallel}) W &= - \frac{1}{2} + (2 - a \sigma) \delta_3 + a \delta_1 \gamma_2
\end{align*}
\]

The Eqs. (42)–(44) with the appropriate substitutions from (46), (48) and (51) give the bimoments of the spherical reflection function \( R \). The terms not containing the moments permit substantial simplification by repeated use of the recurrence relation
\[
nE_{\nu+1}(b) = e^{-b} - b E_{\nu}(b)
\]

(six terms thus cancel in \( URU \)). The final expressions, valid for arbitrary \( a \) and \( b \) are
\[
\begin{align*}
N(R - R_0) N &= \frac{1}{2} E_1(b) + \frac{1}{2} \delta^*_{-1} \\
N R U &= -1 + \delta_0 + \frac{1}{2} \delta_1 \\
U R U &= 1 + \frac{8}{3 b} (2 - a \delta_0) \gamma_1 + \frac{4}{b} \gamma_2 \\
&\quad + \frac{4}{b^2} (2 - a \delta_0) \delta_3 + \frac{4}{b^2} a \delta_1 \gamma_2
\end{align*}
\]

where \( \gamma_\nu, \delta_\nu \) defined in (49) depend on \( a \) and \( b \) through equations well documented in the literature.

I am very grateful to a referee for reminding that Heaslet and Warming (1965) in solving a different problem, involving homogeneously distributed internal sources, derived by the same mapping method of slab onto sphere equations resembling (53)–(55). This led to a systematic exploration of the reciprocity relations for spherical clouds (Van de Hulst, 1987b). The gratifying conclusion is that the calculation of \( 1 - URU \) by Eq. (55), which signifies the fraction of incident radiation absorbed inside the cloud, is indeed the exactly reciprocal problem to Heaslet and Warming’s and that (after some trivial reductions) the results are identical.

We have one more severe check. Conservation of energy requires that conservative scattering \( (a = 1) \) should give for any \( b \)
\[
URN = URU = 1
\]

Although it seems incredible that the complicated expression (55) should ever simplify so drastically, four known relations among the moments of the \( X \) - and \( Y \)-functions for \( a = 1 \) (MLS p. 228–229), which read with the notation (49):
\[
\begin{align*}
\gamma_0 &= 2 \\
\beta_0 \gamma_2 + \gamma_1 \delta_1 &= \frac{1}{3} \\
\delta_1 &= b \beta_0 = b(1 - \frac{1}{3} \delta_0)
\end{align*}
\]

lead after an amusing game of ‘patience’ indeed to Eqs. (56).

In some situations the following expressions for the first-order terms of the bimoments are useful. They are given here without proof:
\[
\begin{align*}
NR_1 N &= \frac{4}{r} \int_0^r \{ J_0^2(x) \}^2 x^2 \, dx = N(R^{\|} - T^{\|}) N \\
NR_1 U &= \frac{4}{r} \int_0^r \{ J_1^2(x) \}^2 x^2 \, dx = (U + r^{-1} W)(R^{\|} - T^{\|}) N \\
UR_1 N &= \frac{4}{r} \int_0^r \{ J_1^2(x) \}^2 x^2 \, dx = (U + r^{-1} W)(R^{\|} - T^{\|}) (U + r^{-1} W)
\end{align*}
\]

Since the first-order bimoments for the slab problem can be exactly expressed in terms of \( G \) and \( G' \) functions (MLS p. 13–15, 194–195 + extensions), Eqs. (58)–(60) may be reduced (still with \( b = 2r \)) to
\[
\begin{align*}
NR_1 N &= \frac{1}{4} \{ G_{11}(b) - G'_{11}(b) \} \\
UR_1 N &= \frac{1}{2} \{ G_{12}(b) - G'_{12}(b) \} + \frac{4}{b} \{ G_{13}(b) - G'_{13}(b) \}
\end{align*}
\]
\[ UR_1 U = G_{22}(b) - G'_{22}(b) + \frac{4}{b} \{ G_{23}(b) - G'_{23}(b) \} \]
\[ + \frac{4}{b^2} \{ G_{33}(b) - G'_{33}(b) \} \]  

(63)

7. Some special cases

The derivations presented in the preceding sections were supported by frequent checks on special cases. We used in particular three assumptions about the optical radius of the cloud, namely \( r \ll 1, r = 0.25 \) (numerical checks), and \( r \gg 1 \). A summary of these checks is presented below.

7.1. Spheres with small optical depth

Most of the preceding results simplify greatly in the limit \( r \ll 1 \). The larger fraction of the incident radiation then penetrates the sphere without scattering. If we neglect terms of the order \( r^2 \), second and higher-order scattering may be neglected and the first order scattering by each volume element is as if no attenuation occurs. Uniform incidence \( U \) then gives \( J_0 = 1 \) throughout, so that the emerging radiance simply equals the length of the chord \( R_1 U = 2r\mu \), which gives \( UR_1 U = \text{(volume of sphere)/(projected area of sphere)} = \frac{4}{3} r \) and \( NR_1 U = r \). The same results follow with more effort from (59)–(60) or from (62), (63). The remaining part emerges without scattering: \( R_0 U = 1 - 2r\mu \), \( UR_0 U = 1 - \frac{3}{r} \), \( NR_0 U = 1 - r \). The result \( NR_1 N = (\pi^2/12) r \) from (61) is found ready on MLS p. 228. Eqs. (26) and (28) give

\[ J_0^N(x) = \frac{r}{4x} \ln \frac{r + x}{r - x} - \frac{1}{2} \frac{r + r^2}{4} + O(r^3) \]
\[ J_0^U(x) = 1 - \frac{r^2 - x^2}{4x} \ln \frac{r + x}{r - x} - \frac{1}{2} \frac{r + r^2}{2} - \frac{1}{6} x^2 + O(r^3) \]  

(64)

(65)

7.2. Sphere with \( r = 0.25 \)

This practical example is near the small-\( r \) limit: 61 per cent of the incident photons can travel the full diameter without being scattered.

The successive orders converge rapidly with the dominant eigenvalue \( \eta_n = \eta_0^N = 0.175 \) (MLS p. 141). There is a misprint in this table: \( \eta_0^U \) should read 0.4474; this was pointed out to me by Dr. G. Brussaard). When enough orders have been computed, the remainder of the series may be estimated at \( \eta_n/(1 - \eta_n) = 0.21 \) times the last computed term. Generally we found (Table 1) that 60–100% passes without scattering and that the orders 2 and beyond contain 1–8% of the total, or up to 25% of the scattered radiation. The fourth order usually influences only the fourth decimal.

The results of many checks have been collected in Table 1.

| Table 1. Numerical results for sphere with \( r = 0.25 \), \( a = 1 \) |
|---------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                                | Order 0         | Order 1         | Order 2 \( \infty \) | Order 1 \( \infty \) | Order 0 \( \infty \) |
| \( J_0^N(x) \) \( x \) = 0.25 | \( \infty \) .0771 | .0123           | .0894           | \( \infty \)           |
| \( x \) = 0.20                | .5759           | .1099           | .0196           | .1205           | .6964           |
| \( x \) = 0.15                | .4669           | .1062           | .0240           | .1302           | .5971           |
| \( x \) = 0.10                | .4187           | .1085           | .0268           | .1353           | .5540           |
| \( x \) = 0.05                | .3960           | .1098           | .0281           | .1379           | .5339           |
| \( x \) = 0                   | .3894           | .1100           | .0286           | .1386           | .5280           |
| \( J_0^U(x) \) \( x \) = 0.25 | .8935           | .0903           | .017            | .1065           | 1.0000          |
| \( x \) = 0.20                | .8358           | .137            | .027            | .1642           | 1.0000          |
| \( x \) = 0.15                | .8077           | .158            | .034            | .1923           | 1.0000          |
| \( x \) = 0.10                | .7908           | .170            | .039            | .2092           | 1.0000          |
| \( x \) = 0.05                | .7815           | .176            | .042            | .2185           | 1.0000          |
| \( x \) = 0                   | .7788           | .179            | .042            | .2212           | 1.0000          |
| \( RN \) \( \mu = 0 \)        | \( \infty \) .075 | .004            | .081            | 4.837           |
| \( \mu = 0.1 \)               | 4.7561          | .143            | .014            | .157            | 1.596           |
| \( \mu = 0.3 \)               | 1.4390          | .172            | .024            | .196            | .975            |
| \( \mu = 0.5 \)               | .7788           | .189            | .035            | .224            | .724            |
| \( \mu = 0.7 \)               | .5033           | .197            | .045            | .242            | .596            |
| \( \mu = 0.9 \)               | .3542           | .197            | .051            | .250            | .553            |
| \( \mu = 1 \)                 | .3033           | .197            | .051            | .250            | .553            |
| \( RU \) \( \mu = 0 \)        | 1.0000          | 0               | 0               | 0               | 1.0000          |
| \( \mu = 0.1 \)               | .9512           | .043            | .0055           | .0488           | 1.0000          |
| \( \mu = 0.3 \)               | .8607           | .1219           | .0174           | .1393           | 1.0000          |
| \( \mu = 0.5 \)               | .7788           | .1901           | .0311           | .2212           | 1.0000          |
| \( \mu = 0.7 \)               | .7047           | .2484           | .0469           | .2953           | 1.0000          |
| \( \mu = 0.9 \)               | .6376           | .2972           | .0652           | .3624           | 1.0000          |
| \( \mu = 1 \)                 | .6065           | .3182           | .0753           | .3935           | 1.0000          |
| \( NRN \) \( \mu = 0 \)       | \( \infty \) .1543 | .0246           | .1789           | \( \infty \)           |
| \( \mu = 0.1 \)               | .7869           | .1799           | .0331           | .2131           | 1.0000          |
| \( \mu = 0.3 \)               | .7216           | .2327           | .0456           | .2784           | 1.0000          |
Consequently, (22) takes the form

\[ J_0(x, \mu_0) = \frac{1}{4\mu_0} e^{-r/\mu_0} \left[ 1 + \left( \frac{1}{\mu_0} - 1 \right) \left( \frac{\tau}{2\mu_0} \right) \right] + O(r^{-2}) \]  

(67)

which yields by the appropriate integration over \( \mu_0 \)

\[ J_0^0(x) = \frac{1}{4} E_1(\tau) \left( 1 + \frac{\tau}{r} \right) + O(r^{-2}) \]  

(68)

\[ J_0^1(x) = \frac{1}{2} E_2(\tau) + \frac{\tau E_2(\tau) + E_3(\tau)}{2r} + O(r^{-2}) \]  

(69)

The same expressions can be obtained directly from (28) and (26). Alternate forms of the terms in \( r^{-1} \) may be written by Eq. (52) but the forms given have the advantage that multiplying by \( x \), as is necessary in the mapping procedure, Eq. (12), amounts to multiplying by \( r \) and crossing out one first order term. The product \( xJ_0^0(x) \) has no first-order term.

Exploration of the consequences of (67) for the asymptotic forms of the spherical reflection function and its moments will be left to later. We now discuss only the bimoments, for which either Eqs.(42)–(44) or Eqs. (53)–(55) may be taken as the starting point. Since we wish to derive exact asymptotic expressions covering all \( a \) and all \( b \), there is no escape from the complexities of the asymptotic expressions for slabs given in MLS, Display 9.2 (p. 200). A distinction must be made between conservative and non-conservative scattering.

Conservative scattering \((a = 1)\) yields, with \( b = 2r, \ s = (b + 1.42)^{-1} \) (as in MLS p. 229), by straight substitution into (42)–(44):

\[ N(R - R_0)N = 1.0674 - 2.0000s + O(b^{-2}) \]  

(70)

\[ N(R - R_0)U = 1.0000 - 1.0000s + O(b^{-2}) \]  

(71)

\[ U(R - R_0)U = 1.0000 + O(b^{-2}) \]  

(72)

Adding from (46) \( NR_0U = s + O(b^{-2}) \) and \( UR_0U = O(b^{-2}) \) restores the balance of the two last expressions to 1 in accordance with (56).

In non-conservative scattering \((a < 1)\) the key quantity is the diffusion exponent \( k(a) \). The earlier decision to neglect exponential would reduce \( f = \exp(-kb - kq_x) \) to zero and all bimoments for the finite slab to those for the halfspace. This is a correct approximation whenever the diffusion stream dies out before the other side of the slab is reached, i.e., when \( kb > 1 \). However, this approximation would also spoil the smooth transition from \( a < 1 \) to \( a = 1 \) (compare the discussion in MLS, p. 684ff). Hence, with a view to obtain expressions useful throughout the nearly conservative domain we retain the factor \( f \). The asymptotic results then become

\[ N(R - R_0)N = \frac{1}{2} a^* \left[ a^* - \frac{2k}{a^*} \right] + O(b^{-2}) \]  

(73)

\[ N(R - R_0)U = (a_0 - 1) \]  

\[ - \frac{4\sqrt{1 - a}}{a^*} + \frac{-1 + 2x_1}{b} + O(b^{-2}) \]  

(74)

\[ U(R - R_0)U = (1 - 2x_1) \sqrt{1 - a} \]  

\[ - \frac{8(1 - a)}{kad \cdot a^*} + \frac{2ax_1^2}{b} + O(b^{-2}) \]  

(75)

7.3. Spheres with large optical depth; asymptotics

The problem which triggered this study, namely the asymptotic behaviour for \( r \gg 1 \), forms our next task. The limit is simple: at \( r = \infty \) the sphere becomes a halfspace (semi-infinite atmosphere), \( r - x \) becomes \( \tau \), the \( X \)-function becomes the \( H \)-function, the transmission terms in the slab problem and the zero-order term in the spherical reflection function vanish, and so do all functions with argument \( r + x \). The familiar radiation density and reflection function for a halfspace are thus readily recovered.

The asymptotic approach to these limits will now be derived. We set \( r - x = \tau \) and retain terms proportional to \( r^{-1} \) but neglect terms of order \( r^{-2} \) and exponential terms \( e^{-\tau} \) where \( c \) is a positive number. By expanding (21) we arrive at

\[ r\mu_0 - x\mu_0 = \frac{\tau}{\mu_0} + \left[ 1 - \frac{1}{\mu_0} \right] \frac{\tau^2}{2\mu_0} + O(r^{-2}) \]  

(66)
All terms in these expressions depend on \(a\). The \(z_i\) are the moments of the \(H\)-functions. The first terms on the right side are the bimoments for a halfspace (MLS p. 173). The second terms go to 0 if \(kb \gg 1\). In the opposite limit \(kb \ll 1\), which is reached only for a very close to 1, the second terms are proportional to \((b + q_n)^{-1}\) and we may approximate \(k^2 = 3(1 - a)\) (MLS p. 152) and \(z_4 = 2/\sqrt{3}\). The second and third terms then are readily combined and all three equations converge to the corresponding ones in (70)–(72).

8. The Bond albedo

The Bond albedo probably is the quantity with the most direct physical interest. A curious dilemma in terminology arises here. In order to find the Bond albedo, should we divide the total amount of scattered radiation by the total amount falling on the cloud, or by the amount actually intercepted by the cloud, i.e., not transmitted without any scattering? A planet does not need this distinction and gives simply Bond albedo = \(U / R U\). For an optically thick cloud, likewise, the difference is \(<1\%\) if \(b > 12\).

That the difference is substantial for a partially transparent cloud may be illustrated by an example. Let \(b = 2\) and \(a = 0.8\). We then have \(URU = 0.297\) and \(URU = 0.775\) with the resulting values

- **first option**: Bond albedo = \(U(R - R_0)U = 0.478\)
- **second option**: Bond albedo = \(U(R - R_0)U/(1 - UR_0U) = 0.680\)

It would seem that generally, in astronomical applications, the second option is the more relevant quantity to deal with. This also has the property that it goes to the individual particle albedo \(a\) as \(b \rightarrow 0\). Its value for clouds of moderate optical depth (and for asymmetric phase functions, see a forthcoming paper) is of key importance in interpreting IRAS data on interstellar dust clouds (De Vries, 1986). Which option is more appropriate for a model of a comet nucleus consisting of loosely packed particles (Greenberg, 1986) remains to be discussed.

We present in Fig. 3 a plot of the Bond albedo as defined in the first option over the full range of \(a\) and \(b\). The plotted quantity is the total fraction of diffusely reflected energy \(U(R - R_0)U\). The zero-order term, i.e., the energy penetrating the sphere without being scattered, is independent of \(a\) and may be read from the same figure as the amount necessary to complete the \(a = 1\) value to 1. The difference between any \(a\)-curve and the \(a = 1\) curve is the energy fraction absorbed in the cloud. The fact that the curves for \(a < 1\) go through a maximum and reach the asymptotic limit on a downward slope is in agreement with the positive sign of the third term in (75).

Figure 3 is based on the following data: slopes for small \(r\) from sec. 7.1; limits and slopes for large \(r\) from (75); points for \(r = 0.25\) from Table 1 with summation as in Eq. 4; points for \(r = 0.75, 1\) and \(1.5\) (\(b = 1.5, 2\) and \(3\)) from Eq. (55) with subtraction of Eq. (46) using \(z_i\) and \(\beta_i\) from Loskutov (1973); finally, Eq. (63) to find the main term for \(a = 0\) and the position of the maximum near \(r = 1.0\) as \(a\) goes to 0. In spite of the paucity of computed points the curves can hardly be wrong by more than 2 per cent anywhere.

**Acknowledgements.** I wish to thank Harm Hading for suggesting this problem, Kalevi Mattila for letting me have his unpublished data, Teije de Jong for the reference to Davison's work, Ivan Kuščer for conversations about the historical roots of the mapping theorem, Joop Hovenier, Mayo Greenberg and an unknown referee for useful comments, and the Institute for Advanced Study at Princeton where part of this work was conceived for their hospitality.

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