Summary

Here's the scroll,
The continent and summary of my fortune

Bassanio, The Merchant of Venice, Scene 3.2, lines 132-133

This PhD thesis concerns the topic of 'arithmetic geometry', that is, the interplay between arithmetic on the one hand – integer numbers, their addition and multiplication, and their fractions – and geometry on the other – shapes and their intersections. We address three different questions and each of the questions in some way is about counting how big some set is or can be.

In arithmetic geometry we are interested in so called *polynomial* equations: we restrict ourselves to only use fractions of integers and any number of indeterminates – sometimes also called variables – and the rules we may use are just addition, multiplication, and exponentiation with positive integer powers. That means no square roots, logarithms, or trigonometric functions for example. That may feel like a relief (we may forget half of our secondary school mathematics), but the questions that arise turn out to be surprisingly difficult. Some of them date back at least to the ancient Greeks, if not further!

So what *are* the questions that we want to study? Take for example the equation

$$x^2 + y^2 = 1. \tag{5.1}$$

You may remember from that half of secondary school mathematics that we did not forget in the last paragraph that this is the equation of a circle in a plane. In other words: if we label our axes x and y and we draw all the points in the plane that have x- and y-coordinates which satisfy the equation, our drawing will take the shape of a circle. This is where we see the geometry appear naturally. How about the arithmetic? Well, we cannot individually draw *all* the points on a circle – there are simply too many of them. Let us separate them into two sets. One set, which we will

call S, will contain all those points whose coordinates x and y are fractions of integer numbers, for example $(1,0)$ lies in S, and so does $(\frac{3}{5},\frac{4}{5})$. The other set will contain all the other points, for example $(\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2})$, and we will ignore it. Now the set S is still infinite, but let us impose a further restriction on its elements. What if we don't take all possible fractions, but we only use those whose numerator and denominator are small, let's say less than some number B that we may freely choose? This set is finite so we can count the number N of elements of this set, which depends on what we choose for B.

Question 1: How exactly does this number N depend on B?

In this case the answer is not too difficult to find and it turns out to be a linear function in B. But if we replace equation (5.1) by a more difficult equation, say

$$x^4 + y^4 - 3z^4 = 1, \tag{5.2}$$

then the question has become a whole lot more difficult. With the extra variable z, the dimension of the corresponding geometric object is raised to two (whereas the circle has dimension one), and with the higher exponent 4, the shapes get more complicated. For equations of this new shape, no definitive answer has been proven to date. In Chapter 2 we give evidence (which is not synonymous to 'proof') that the answer should be some explicit power of $\log B$. Indeed, to understand the answer, we need to relearn about logarithms!

Chapter 3 deals with a similar question, and to explain this question, we need to learn about modular arithmetic. In essence, this is arithmetic like on a clock: every 12 hours the time on the clock is repeated. How can we phrase this in mathematics? On the clock we know that 13 equals 1, 14 equals 2 and so on. Mathematically we equate a number with its remainder upon division by 12. Indeed, we have $13 = 1 \times 12 + 1$, and $14 = 1 \times 12 + 2$, and so on. There is no need to stop at 24: we also have $35 = 2 \times 12 + 11$, so we equate 35 with 11. Moreover, mathematically there is nothing special about the number 12. We could imagine a clock with any number of 'hours'.

We now replace (5.1) by a different equation than before. For example we may also look at the ellipse whose equation is

$$x^2 + 3y^2 = 2. \tag{5.3}$$

Using the modular arithmetic just introduced, it is not hard to prove that this equation has no solutions where x and y are fractions of integers. Equations (5.1) and (5.3) are very similar, in fact we can write down a *family* of equations of which both are a member. In this case, equations (5.1) and (5.3) are both members of the family defined by

$$x^2 + (1 + 2t)y^2 = 1 + t. \tag{5.4}$$

We get members of this family when we choose values for the parameter t. For example, if we set $t = 0$ then we recover equation (5.1), and for $t = 1$ we obtain equation (5.3).

Question 2: Given some family, can we count how many of its members have fractional solutions?

This number could be infinite, so we need our question to be phrased more carefully. For example, we may restrict ourselves to members with a t that is a fraction of integers not exceeding some number B. Again we are confronted with a question that can be described in simple terms, and again the answer is quite difficult to prove. In fact, there are reasons to believe that there exists some deeper meaning that covers the answers to Questions 1 and 2, but we seem quite far from understanding this meaning. In Chapter 3 we look at families of some prescribed shape and we answer this counting question in full. Like in Chapter 2, the answer depends on B and we find a formula for the number that we wanted to count. There are two noteworthy observations about this: in the literature such formulas are quite rare – normally one is only able to give upper bounds – and the formula involves a complicated constant that we have unravelled. The way that this constant is built up provides further evidence for the deeper meaning that we alluded to above.

Equation (5.3) has no fractional solutions because of problems arising from modular arithmetic. One may wonder if these are the only problems that may occur, and this is exactly what Yuri Manin did in the 1970's. He gave a construction that may explain the existence of *obstructions* to fractional solutions; this construction involves a set that we call the *Brauer group*. For many simple geometric objects, Manin's construction accounts for all obstructions, but this need not always be the case. Recall the equations of the shape (5.2) that we studied in Chapter 2. Their geometric objects are examples of what we call K3 surfaces. In recent years people have started to wonder if Manin's construction is strong enough to explain all obstructions to fractional solutions for K3 surfaces, and this question remains open. In order to work towards an answer, we studied these

Brauer groups for some type of K3 surfaces in Chapter 4. It is known for K3 surfaces that Brauer groups only have finitely many elements, but the theorem that shows this does not tell us how many.

Question 3: How big can a Brauer group of a K3 surface get?

Our result gives a recipe that takes as ingredients only a few basic numerical values attached to the surface whose Brauer group one wants to study. However, our method does not give the exact answer but only an upper bound. We were not the first ones to give such upper bounds, but our result has the benefit of being easy to compute. There is, however, no reason to assume that our upper bounds are in any way *sharp*, which means that these upper bounds may be far above the actual size.

In conclusion, the title of this thesis goes against the main strength of mathematics: to describe complex phenomena with no room for ambiguity. The title can be separated in two different ways. Reading it as "Counting points on (K3 surfaces and other arithmetic-geometric objects)" emphasizes that in each chapter we focus on counting some quantities, while reading it as "(Counting points on K3 surfaces) and other arithmetic-geometric objects" shows that we are mainly interested in K3 surfaces, but that the thesis also contains other results. In this case the ambiguity does not hurt: both readings are correct.