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Introduction

O, for a muse of fire, that would ascend
The brightest heaven of invention

Chorus, HENRY V, Prologue, lines 1-2

This thesis deals with three questions from arithmetic geometry. Even though it seems difficult to indicate one overarching topic, there do exist links between every pair of questions. Chapters 2 and 3 both deal with counting rational points, Chapters 2 and 4 are about K3 surfaces, and Chapters 3 and 4 both involve the concept of local solubility and obstructions to global solubility.

In Chapter 1 we give some necessary background from geometry and number theory that will allow us to study the topics in the later chapters. In particular, we give a quick treatment of the circle method, which is commonly used to address counting questions from geometry. When studying a polynomial $F \in \mathbb{Z}[X_1, \ldots, X_n]$, then the integral

$$\int_0^1 \exp(2\pi i \alpha F(\mathbf{x})) \mathrm{d}\alpha$$

tests if F has a zero at some point $\mathbf{x} \in \mathbb{Z}^n$. Indeed, since $F(\mathbf{x})$ is an integer, the integral evaluates to 0 unless $F(\mathbf{x}) = 0$ holds, in which case the integral equals 1. When for some bound B, we sum such integrals over all $\mathbf{x} \in (\mathbb{Z} \cap [-B, B])^n$, we may switch the order of the sum and the integral. The circle method describes a quite general way in which this integral can be approximated well enough so that we get valuable information out of it for large B. Moreover, this process generalizes to multiple polynomials. For now, it suffices to know that the main term will be a product of an integral (which should be easier to compute) and a series.

Chapter 2 is devoted to producing evidence towards a precise Manin type conjecture for K3 surfaces, predicting the number of rational points up to bounded height. If $\mathbf{x} = (x_0 : \ldots : x_n) \in \mathbb{P}^n(\mathbb{Q})$ is any point, we may arrange its coefficients so that each of the x_i is an integer, and the $(n+1)$-tuple has trivial greatest common divisor. If we do that, then we call $H(\mathbf{x}) = \max_i |x_i|$ the height of the point $\mathbf{x}$. A useful property is that for any given bound B, there are only finitely many points in $\mathbb{P}^n(\mathbb{Q})$ that have height at most B. The same is true for any subvariety of $\mathbb{P}^n_{\mathbb{Q}}$. Hence, for any given subvariety, we may count this number $N(B)$ of points for varying B. In the case of Fano varieties, Manin stated a rather general conjecture for the shape of $N(B)$, involving some geometric invariants. In particular, Manin's conjecture asks us to restrict ourselves to open subsets since there might exist so-called accumulating subvarieties. These may contain 'too many' rational points and we should ignore those. Many examples of Fano varieties have been studied in the literature, and at least for surfaces there are many examples where $N(B)$ involves a power of B and a power of $\log(B)$. The exponent of the logarithm is $\rho - 1$, where ρ is the rank of the Picard group of the variety.

Leaving the world of Fano varieties, we study diagonal quartic surfaces, which are examples of K3 surfaces, and we employ the circle method to count zeroes of the defining equation. Experts will immediately recognize that in this case the emergent error terms will exceed the desired main term. We focus on the main term and we speculate that there should be a connection between the error terms and accumulating subvarieties on the surface. Such behaviour has been observed in the literature for other types of varieties, but a detailed treatment supporting such speculation seems out of reach of our methods. The main result of this chapter gives heuristic support for hitherto unexplained data obtained in computer experiments by van Luijk some years ago. In particular we find that no power of B occurs and in contrast to the Fano varieties discussed above, that the power of $\log(B)$ ought to be ρ instead. Apart from this result, the approach to obtain these heuristics could be viewed as the main contribution of this chapter: we use averaging results of multiplicative functions to study the series coming out of the circle method, and we exploit the Tate conjecture to determine the exponent of the logarithm via the L-function of the surface. This gives $\rho-1$ factors of $\log(B)$, while the last logarithm is obtained from the remaining factor that comes in the form of an integral.

Chapter 3 is joint work with Efthymios Sofos and concerns Serre's problem about fibrations. In the 1990s, Serre studied examples of conic bundles, where he looked at the number of fibres containing a rational point. More precisely, he studied the number of rational points in the base up to a bounded height, say B, such that the fibres over these points contain a rational point themselves. Serre was only able to prove upper bounds, but for his specific examples it has since been shown that these upper bounds are in fact asymptotically correct. In recent years more examples of fibrations, not necessarily conic bundles, have been studied. Most notably, Loughran and Smeets have provided a framework into which results in this area should fit. Complete examples in the literature are rare, and our contribution lies in giving a wide class of conic bundles for which we can not only prove asymptotic results, but where we are also able to study the leading constant of such asymptotics. In particular, we look at bundles that can locally be described as follows. We take two polynomials f_1 and f_2 in n variables and of even degree d, subject to some more conditions that are outlined in Chapter 3. Now consider the variety $\mathcal{B}$ defined by $f_2(t_1, \ldots, t_n) = 0$ as a subvariety of $\mathbb{P}^{n-1}_{\mathbb{Q}}$. Over the affine patch where we take $t_i = 1$, we define a conic bundle

$$\begin{aligned} x^2 + y^2 =& f_1(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n) z^2, \\ & f_2(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n) = 0. \end{aligned}$$

These bundles glue together into a conic bundle $\phi : X \to \mathcal{B}$. Now let $N(B)$ be the number of rational points of $\mathcal{B}$ up to height B, the fibre over which has a rational point itself. Then we prove that there is a constant c_ϕ such that we have

$$N(B) = c_\phi \frac{B^{n-d}}{(\log B)^{1/2}}$$

up to an error term with a logarithmic exponent slightly bigger than $1/2$. Moreover, our methods allow us to prove a Hasse principle on smooth fibres. To obtain our results, we use the circle method together with sieving techniques to study the series that appears. The conditions on the fibre in order for it to contain rational points appear through an indicator function detecting everywhere local solubility. Since our fibres are conics and therefore themselves satisfy the Hasse principle, this guarantees global solubility. Considerable effort goes into writing the leading constant c_ϕ as a product of recognizable factors, and we indicate how this fits conjectural expectations first described by Loughran.

It is known that the Hasse principle does not hold for K3 surfaces, and it has been conjectured that the Brauer–Manin obstruction has enough strength to explain this fact. In Chapter 4, which is written together with

Victoria Cantoral-Farfán, Yunqing Tang, and Sho Tanimoto, we study Brauer groups of Kummer surfaces. In particular, by a celebrated result of Skorobogatov and Zarhin we know that for any K3 surface over the field of rational numbers, its Brauer group (modulo constants) is finite. Our work gives explicit upper bounds for the size that this group may attain, at least when dealing with Kummer surfaces. For an abelian surface A over a number field k, with Kummer surface X, it is known that $\mathrm{Br}(\overline{A})$ and $\mathrm{Br}(\overline{X})$ are isomorphic as Galois modules, and we will bound the transcendental Brauer group $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ as a subgroup of $\mathrm{Br}(\overline{X})^\Gamma$, where Γ denotes the absolute Galois group of k.

The proof of Skorobogatov and Zarhin's result relies on an exact sequence of cohomology groups, namely

$$0 \to \left(\mathrm{NS}(\overline{A})/\ell^n\right)^\Gamma \xrightarrow{f_n} \mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_{\ell^n})^\Gamma \to \mathrm{Br}(\overline{A})^\Gamma_{\ell^n} \to \\ \to \mathrm{H}^1(\Gamma, \mathrm{NS}(\overline{A})/\ell^n) \xrightarrow{g_n} \mathrm{H}^1(\Gamma, \mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_{\ell^n})),$$

where A is an abelian surface over a number field k, ℓ is a prime number, and the subscript ℓ^n indicates ℓ^n-torsion. We use this exact sequence and we study the cokernel of f_n and the kernel of g_n. Bounds on their sizes imply bounds on the size of $\mathrm{Br}(\overline{X})^\Gamma_{\ell^n}$, and we provide bounds independent of n. There are only finitely many ℓ for which such bounds are non-trivial, so we should also bound the size of such ℓ. This last point in particular relies on effective versions of Faltings' finiteness theorem for abelian varieties.

Although the existence of upper bounds like ours is not new, our methods have the advantage of being more explicit than those that were already known, especially for Kummer surfaces of minimal Picard rank. In particular, if one is given a curve C of genus 2 over a number field k, for which the Jacobian $\mathrm{Jac}(C)$ has Picard rank 1 and Faltings height h, let δ denote the discriminant of $\mathrm{NS}(X)$, where X is the Kummer surface of $\mathrm{Jac}(C)$. In order to obtain an upper bound for the transcendental part $\#\mathrm{Br}(X)/\mathrm{Br}_1(X)$, one needs only apply our formula with inputs $[k : \mathbb{Q}]$, h, and δ. Moreover, we prove that the algebraic part $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ has order at most 2. Although our bounds are explicit, they are unlikely to be sharp. We compute the example of the curve C given by

$$y^2 = x^6 + x^3 + x + 1.$$

The Brauer group of its associated Kummer surface turns out to have an order at most $2^{10} \cdot 10^{805050}$.