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Chapter 2

Heuristics for counting rational points on diagonal quartic surfaces

If to do were as easy as to know what were good to do, chapels had been churches, and poor men's cottages princes' palaces

Portia, THE MERCHANT OF VENICE, Scene 1.2, lines 9-10

This chapter is concerned with finding heuristics for a conjecture in the style of Manin's Conjecture 1.2.15 for some K3 surfaces over $\mathbb{Q}$. In particular we will restrict ourselves to diagonal quartic surfaces. We use the circle method to obtain such heuristics. We do not take into account that such surfaces may have accumulating subvarieties, but we discuss these in relation to the circle method at the very end of this chapter.

In particular, the goal of this chapter is to prove the following main result, where we assume the generalized Riemann hypothesis (hereafter GRH). In fact, we do not need to assume GRH for all L-functions; just some of specific origin, as will come up in Proposition 2.2.5. Recall the singular integral $\mathfrak{J}(Q)$ from Definition 1.3.9 and the modified singular series $\mathfrak{S}^*(Q)$ from Definition 1.3.17. For diagonal quartic surfaces we have $n = d = 4$, so the singular integral and singular series together make up the contribution of the major arcs to counting points up to bounded height, provided that the major arcs do not overlap. In accordance to Lemma 1.3.5 and Remark 1.3.8 we implicitly assume $\delta < \frac{1}{5}$ to have been chosen.

THEOREM 2.0.1. *For $a_1, \ldots, a_4 \in \mathbb{Z} \setminus \{0\}$, let $F = \sum_{i=1}^{4} a_i x_i$ define a diagonal quartic surface X of Picard rank $\rho \geq 2$. Under the assumption* *of GRH, there exists a constant c_F such that as $Q \to \infty$ the contribution $\mathfrak{J}(Q)\mathfrak{S}^*(Q)$ equals*

$$c_F(\log Q)^\rho + o((\log Q)^\rho).$$

This theorem should be viewed in light of computational data produced by van Luijk and available on his website [Lui]. Indeed the heuristic in this theorem matches with the growth that the data seems to imply.

Remark 2.0.2. Some diagonal quartic surfaces (e.g. those with all a_i positive) have no rational points, so in general one should not expect c_F to be non-zero. Ideally, one would hope that a detailed treatment of c_F would show obstructions to it being positive. Such obstructions should be more complicated than just local obstructions as counterexamples to the Hasse principle are known for diagonal quartic surfaces (see for example [SD00] or [Bri06]).

2.1 Averages of multiplicative functions

In a recent preprint [GK17], Granville and Koukoulopoulos present a very strong theorem dealing with averages of multiplicative functions. Very similar theorems were first discovered by Wirsing using ideas of Selberg and Delange. In fact, the contents of this chapter were first proven using Wirsing's work [Wir61, Satz 1]. The downside of Wirsing's original theorem is that it only deals with non-negative multiplicative functions, restricting us to only apply the result to specific diagonal quartic surfaces. The new theorem of Granville and Koukoulopoulos however needs a good error term in one of its conditions. This is automatically provided by assuming GRH; see Proposition 2.1.11 and the remark following it. This assumption may be removed by knowing good zero-free regions for L-functions of varieties; see Remark 2.2.6.

2.1.1 A powerful result by Granville and Koukoulopoulos

In order to phrase the theorem, we need to introduce some notation. We will let Γ denote the classical Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t.$$

In particular, for positive integer z we have $\Gamma(z) = (z-1)!$.

DEFINITION 2.1.1. For any complex number α the multiplicative function τ_α is given on prime powers by

$$\tau_\alpha(p^\nu) = \alpha(\alpha+1)\cdots(\alpha+\nu-1)/\nu!.$$

In particular, on primes the function τ_α evaluates to α, and if α is a positive integer, then $\tau_\alpha(p^e) = \binom{\alpha-1+e}{e}$ holds.

DEFINITION 2.1.2. For a multiplicative function f, its associated Dirichlet series will be denoted by $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$. Fix some complex number α such that the function $(s-1)^\alpha L_f(s)$ is J times continuously differentiable in the half-plane $\Re(s) \geq 1$. For all $j \leq J$ we set[1]

$$c_j := \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}s^j}\bigg|_{s=1} \frac{(s-1)^\alpha L_f(s)}{s}.$$

THEOREM 2.1.3 (Granville–Koukoulopoulos). *Let f be a multiplicative function for which there exist $\alpha \in \mathbb{C}$ and $A \in \mathbb{R}_{>0}$ such that for $x \geq 2$, the function f satisfies*

$$\sum_{p\leq x} f(p)\log(p) = \alpha x + O\left(\frac{x}{(\log x)^A}\right), \tag{2.1}$$

where the sum is over the prime numbers at most x. Furthermore assume that there exists some $k \in \mathbb{R}_{>0}$ such that $|f| \leq \tau_k$ holds. If $J = \lceil A-1 \rceil$ is the largest integer smaller than A, then with notation c_j as above, $x \geq 2$ validates

$$\sum_{n\leq x} f(n) = x\sum_{j=0}^{J} c_j \frac{(\log x)^{\alpha-j-1}}{\Gamma(\alpha-j)} + O\left(x(\log x)^{k-1-A}(\log\log x)^{\mathbb{I}_{A=J+1}}\right).$$

The implied constant depends at most on k, A and the implied constant from (2.1). The dependence on A is twofold: from its size and its distance from the nearest integer.

Proof. This is [GK17, Theorem 1]. □

[1]Notice that our notation is slightly different from that in [GK17] as we have surpressed some notation from the source since we will only need part of the conclusion of its main theorem.

REMARK 2.1.4.

- Implicit in the formulation of the previous theorem is that assumption (2.1) implies that the Dirichlet series associated to f is $\lceil A-1\rceil$ times continuously differentiable in the half-plane $\Re(s) \geq 1$.
- The Γ-function has poles at all non-positive integers, hence in the result of Theorem 2.1.3, for integer α all terms with $j \geq \alpha$ vanish. In particular, the theorem only yields an asymptotic for $\alpha \neq 0$, and only then if $k - A < \alpha$ holds.

For our purposes we only consider the $j = 0$ term from the conclusion of Theorem 2.1.3, which under the conditions in the remark above yields the dominating term.

The following lemma allows us to convert the conclusion of Theorem 2.1.3 into a form that we prefer. Notice that the case $a = -1$ below does not appear in the conclusion of the theorem – nor do any other cases with negative a.

LEMMA 2.1.5. *If $f : \mathbb{Z}_{>0} \to \mathbb{C}$ satisfies $\sum_{n\leq x} f(n) \sim cx(\log x)^a$ for some constants $a \in \mathbb{Z}$ and $c \in \mathbb{C}$, then for $a \neq -1$ we have*

$$\sum_{n\leq x} \frac{f(n)}{n} \sim \frac{c}{a+1}(\log x)^{a+1},$$

and for $a = -1$ we have

$$\sum_{n\leq x} \frac{f(n)}{n} \sim c \log\log x.$$

Proof. This is a simple application of Abel's partial summation formula (cf. Theorem1.3.1). We have

$$\begin{aligned}\sum_{n\leq x} \frac{f(n)}{n} &= \left(\sum_{n\leq x} f(n)\right)\frac{1}{x} + \int_1^x \left(\sum_{n\leq t} f(n)\right)\frac{1}{t^2}\mathrm{d}t \\ &\sim c(\log x)^a + c\int_1^x (\log t)^a \frac{1}{t}\mathrm{d}t.\end{aligned}$$

For $a \neq -1$ the integral evaluates to $\frac{1}{a+1}(\log x)^{a+1}$, whereas for $a = -1$ it evaluates to $\log\log x$, in either case giving the dominating term. □

2.1.2 Chebyshev-like functions

In his study towards the Prime Number Theorem, Chebyshev introduced the function

$$\psi(x) = \sum_{n \le x} \Lambda(n),$$

where

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^e \text{ is a non-trivial prime power,} \\ 0 & \text{otherwise} \end{cases}$$

is known as the von Mangolt function.

The function $\psi(x)$ relates to the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ which is the Dirichlet series of the constant multiplicative function 1. In studying averages of multiplicative functions, one often works with Chebyshev-like functions that we will now define. We generalize some definitions from §1.3.4.

Definition 2.1.6. Let $f : \mathbb{Z}_{>0} \to \mathbb{C}$ be a function. Its associated *Dirichlet series* is $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$. If f is multiplicative, the *von Mangoldt function* Λ_f associated with f is defined indirectly through its Dirichlet series as

$$-\frac{L'_f(s)}{L_f(s)} = \sum_{n=1}^{\infty} \Lambda_f(n) n^{-s}$$

and its associated *Chebyshev function* is

$$\psi_f(x) = \sum_{n \le x} \Lambda_f(n).$$

Lemma 2.1.7. *For a completely multiplicative function f we have*

$$L_f(s) = \prod_p \frac{1}{1 - f(p)p^{-s}}.$$

Proof. This follows from writing $L_f(s)$ as a product over primes

$$L_f(s) = \prod_p \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right)$$

and then recognizing the sums as geometric series. □

LEMMA 2.1.8. *The von Mangoldt function associated with a multiplicative function f is supported on the prime powers and for any prime number p it evaluates as $\Lambda_f(p) = f(p)\log(p)$. If f is completely multiplicative, the von Mangoldt function satisfies*

$$\Lambda_f(n) = \Lambda(n)f(n).$$

Proof. These properties may be found without proof on [IK04, page 17]. For the sake of completeness, we will give a proof here.

First, it is easily seen that $-L'_f(s)$ is the Dirichlet series of $f \cdot \log$. The convolution of two functions $f, g : \mathbb{Z}_{>0} \to \mathbb{C}$ is defined as

$$(f * g)(n) = \sum_{d|n} f(d)g(\tfrac{n}{d})$$

and it is well known (or easily computed) that the Dirichlet series of a convolution is the product of the two Dirichlet series. Hence we get $f * \Lambda_f = f \cdot \log$.

We compute some values of Λ_f. Expanding $(f * \Lambda_f)(1) = f(1)\log(1) = 0$ and using $f(1) = 1$, we find $\Lambda_f(1) = 0$. Using this result, we move on to work out $(f * \Lambda_f)(p) = f(p)\log p$ for any prime number p and conclude $\Lambda_f(p) = f(p)\log p$.

Finally, taking p and q two different prime numbers, using the same strategy we conclude $\Lambda_f(pq) = 0$. Using this as a base case, one may apply induction to prove the same for numbers with more than two prime factors or higher exponents, hence Λ_f is supported on prime powers.

Having already proven $\Lambda_f(p) = f(p)\log p$, we may apply induction to the power of p to show that for completely multiplicative f we have $\Lambda_f(p^k) = f(p)^k \log p = f(p^k)\log p = f(p^k)\Lambda(p^k)$. The strategy is completely analogous to the first part of this proof. □

REMARK 2.1.9. From the last lemma and the definition of Λ it immediately follows that if f is a completely multiplicative function, then for $k \geq 1$ we have

$$\Lambda_f(p^k) = \Lambda(p^k)f(p^k) = f(p)^k \log(p)$$

as was already seen in the proof, but is worth stating separately.

The Chebyshev function associated with a multiplicative function f is often useful to study the sum $\sum_{p\le x} f(p)\log p$, provided one can bound the contribution of higher prime powers. Indeed, we have

$$\sum_{p\le x} f(p)\log p = \psi_f(x) - \sum_{k=2}^{\infty}\sum_{p^k\le x}\Lambda_f(p^k),$$

where the sum over k is actually a finite sum as for $k > \log_2(x)$, the second sum is empty.

LEMMA 2.1.10. *Let f be a multiplicative function for which there exists some $b \in \mathbb{R}_{>0}$ bounding from above every $|f(p)|$ for prime numbers p. For some $a \in \mathbb{Z}_{>0}$ define a completely multiplicative function f^* by*

$$f^*(p^e) = \begin{cases} 1 & \text{if } e = 0,\\ 0 & \text{if } e \ge 1, p \le a,\\ f(p)^e & \text{if } p > a.\end{cases}$$

Then for sufficiently large x we have

$$\left|\psi_{f^*}(x) - \sum_{a<p\le x} f(p)\log p\right| = \left|\sum_{k=2}^{\infty}\sum_{\substack{p\\ p^k\le x}}\Lambda_{f^*}(p^k)\right| \ll x^{1/2+\log_a(b)}.$$

Proof. By definition we have $\sum_{a<p\le x} f(p)\log p = \sum_{p\le x} f^*(p)\log p$ and as we have already remarked above, the equality of the statement follows. For the remainder of the proof, we will focus on the absolute value of the sum on the right-hand side of the equality, which we will call S.

After applying the triangle inequality, we begin by switching the order of summation and extending the sum over primes to all p satisfying $p^2 \le x$, thereby increasing its total value, i.e.

$$|S| \le \sum_{\substack{p\\ p^2\le x}} \log p \sum_{k=2}^{\lfloor \log_p(x)\rfloor} |f^*(p)|^k = \sum_{\substack{p>a\\ p^2\le x}} \log p \sum_{k=2}^{\lfloor \log_p(x)\rfloor} |f(p)|^k.$$

Without loss of generality we may assume $b \ge 2$. The sum over k is bounded from above by

$$\sum_{k=2}^{\lfloor \log_p(x)\rfloor} b^k \le b^2 \cdot \frac{x^{\log_p(b)} - 1}{b-1} \le b^2 x^{\log_a(b)}.$$

For sufficiently large x, the number of primes p with $p^2 \leq x$ is of the order $\frac{2x^{1/2}}{\log x}$ and for each of those we may trivially bound $\log p$ by $\frac{1}{2}\log x$. Hence the absolute value of S is bounded by $b^2 \cdot x^{1/2+\log_a(b)}$ as required. □

In order for Theorem 2.1.3 to be useful in the application that we have in mind, and following our proof, we will need (2.1) with an arbitrarily high exponent A. It is not unthinkable that this may be derived with some skilful application of zero-free regions for appropriate L-functions but in our main result of the chapter we opt to take the shortcut of assuming GRH.

Iwaniec and Kowalski dedicate Chapter 5 of their book [IK04] to a wide class of L-functions. They explicitly let their notation remain somewhat vague and suggestive, but they do give a list of requirements to what they call an L-function. For us it is enough to know that Dirichlet series, L-functions of cusp forms, and (sometimes conjecturally) L-functions of varieties fall in the class for which the following proposition is true. In particular, with the knowledge from §1.2.3, we see that the following proposition applies to $L(\mathrm{H}^2(X), s)$.

Proposition 2.1.11. *Let $\frac{1}{2} \leq \sigma < 1$. The following statements are equivalent for an L-function:*

1. *There are neither zeros nor poles of $(s-1)^r L_f(s)$ in $\Re(s) > \sigma$, where r is a non-negative integer.*
2. *Let $r \geq 0$ be the order of the pole of $L_f(s)$ at $s = 1$. Then for all $\varepsilon > 0$ and $x \geq 2$ we have*

$$\psi_f(x) = rx + O\left(x^{\sigma+\varepsilon}\right),$$

the implied constant depending on f and $\varepsilon > 0$.

Proof. This is part of [IK04, Proposition 5.14]. □

Remark 2.1.12.

- The numbers r appearing in the two statements of the previous proposition are necessarily the same. Indeed, if the number r in the first statement is not the order of the pole of $L_f(s)$ at $s = 1$ then $(s-1)^r L_f(s)$ will either have a pole or a zero at $s = 1$.

- GRH asserts that the first statement in the proposition above is true for $\sigma = \frac{1}{2}$, and hence also the second one which is the statement that we want to use.

- Although the notation of the proposition seems to rely on some function f, we do not really need it for the result. Via $L'_f(s)/L_f(s)$ one may define $\Lambda_f(n)$ implicitly and from that also $\psi_f(x)$.

2.2 Evaluating the singular series

We now turn to the main goal of the chapter, namely evaluating the contribution of the major arcs to rational points on those diagonal quartic surfaces X as given in Theorem 2.0.1: defined by $F(\mathbf{x}) = \sum_{i=1}^{4} a_i x_i^4$ with $a_i \in \mathbb{Z} \setminus \{0\}$ such that X has Picard rank $\rho \geq 2$. Throughout the rest of the chapter we write S for the finite set of primes where X has bad reduction.

Section 2.1 provides the tools for evaluating the singular series. The singular integral will have to wait until §2.3.

In order to apply Theorem 2.1.3, we introduce a suitable multiplicative function f such that we have $\sum_{q=1}^{Q} \frac{f(q)}{q} = \mathfrak{S}^*(Q)$. This is provided by the following choice:

Definition 2.2.1. We denote $f(q) = \frac{S_q^*}{q^3}$ where S_q^* is given in Definition 1.3.17.

Indeed, Lemma 1.3.18 shows that f is multiplicative.

Before we proceed, we relate the function f to the geometry of X. Using the Lefschetz trace formula, which gives $\#X(\mathbb{F}_p) = p^2 + T_p \cdot p + 1$ where $T_p \cdot p$ is the trace of Frobenius on $\mathrm{H}^2_{\text{ét}}(X_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell)$, and moreover using that $N^*(p)$ counts the number of non-zero affine zeroes over $\mathbb{Z}/p\mathbb{Z}$ of the equation

$F(\mathbf{x}) = 0$, one finds

$$\begin{aligned} f(p) = \frac{S_p^*}{p^3} &= p\left(\frac{N^*(p)}{p^3} - 1\right) \\ &= p\left(\frac{\#X(\mathbb{F}_p)(p-1)}{p^3} - 1\right) \\ &= p\left(\frac{p^3 + (T_p - 1)p^2 - (T_p - 1)p - 1}{p^3} - 1\right) \\ &= (T_p - 1)\left(1 - \tfrac{1}{p}\right) \\ &= (T_p - 1) + O\left(\tfrac{1}{p}\right). \end{aligned}$$

The first non-trivial equality uses the result of Lemma 1.3.18 with $n = 4$.

The following lemma will be useful in checking the conditions of Theorem 2.1.3 for our chosen function f.

LEMMA 2.2.2. *For every prime p we have $|T_p| \leq 22$.*

Proof. The number $T_p \cdot p$ is the trace of Frobenius on $\mathrm{H}^2_{\text{ét}}(X_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell)$. By the Weil conjectures, each of the eigenvalues has absolute value p. Hence $|T_p \cdot p|$ is bounded by p times the dimension of $\mathrm{H}^2_{\text{ét}}(X_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell)$, which by comparison with singular cohomology is the second Betti number b_2. For K3 surfaces, the second Betti number equals 22, completing the proof. □

This description on prime values will be used to check that f satisfies the conditions of Theorem 2.1.3. We first assure ourselves of the assumption that there exists some $k \in \mathbb{R}_{>0}$ such that $|f| \leq \tau_k$ holds, leaving the more involved assumption (2.1) for later.

LEMMA 2.2.3. *There exists a number $k \in \mathbb{R}_{>0}$ validating $|f| \leq \tau_k$.*

Proof. Since for positive real k, both $|f|$ and τ_k are multiplicative and take values in $\mathbb{R}_{\geq 0}$ when applied to positive integers, we need only check the assertion on prime power values.

We first consider the values of $|f|$ on primes. By Lemma 2.2.2, $|T_p|$ is bounded by 22, so for every prime p, the value $|f(p)|$ is bounded by 23. Recalling Corollary 1.3.22, we see that for almost all primes p, we have $f(p^e) = 0$ for all $e \geq 2$. Hence we only need to further consider a finite set of primes T, and moreover, by Lemma 1.3.19, only a finite set of prime powers $P = \{p^e : p \in T,\ f(p^e) \neq 0\}$. Over this finite set, $|f|$ takes a maximum value. Moreover, for fixed e, the value of $\binom{k+e-1}{e}$ as a function of k is unbounded. Hence there exists some positive integer K such that for all $p^e \in P$ we have

$$|f(p^e)| \leq \binom{K+e-1}{e} = \tau_K(p^e).$$

Hence $|f(p^e)| \leq \tau_{\max\{23,K\}}(p^e)$ holds for all prime powers p^e and therefore the desired inequality $|f| \leq \tau_{\max\{23,K\}}$ holds on all positive integers. Hence we retrieve the statement of the lemma with $k = \max\{23, K\}$. □

REMARK 2.2.4. Since the number k in the lemma above is ineffective, and the result of Theorem 2.1.3 is only useful for $k - A < \alpha$, we need to verify condition (2.1) for arbitrarily large A.

To verify condition (2.1) we need to evaluate

$$\sum_{p \leq x} f(p) \log p = \sum_{p \leq x} T_p \log p - \sum_{p \leq x} \log p + O\left(\sum_{p \leq x} \tfrac{1}{p} \log p\right) \tag{2.2}$$

where we have used that the implicit constants in $f(p) = (T_p - 1) + O\left(\frac{1}{p}\right)$ are universally bounded by 23.

We consider this sum in three parts: first, evaluation of

$$\sum_{p \leq x} \tfrac{1}{p} \log p = \log x + O(1)$$

is standard in analytic number theory and is known as Mertens' first theorem.

Then, the middle term $\sum_{p \leq x} \log p$ comes up in the proof of the Prime Number Theorem (Theorem 1.3.2), and the validity of $\sum_{p \leq x} \log p \sim x$ is in fact equivalent to it by application of Abel's summation formula from Theorem 1.3.1.

In order to evaluate $\sum_{p \leq x} T_p \log p$ we will use an L-function involving T_p.

Consider the function g given on primes by $p \mapsto T_p$ and extended to have domain $\mathbb{Z}_{>0}$ by complete multiplicativity. Its Dirichlet series becomes

$$L_g(s) = \prod_p \frac{1}{1 - T_p p^{-s}}$$

by Lemma 2.1.7.

PROPOSITION 2.2.5. *Assuming GRH and writing r for the order of the pole of $L_g(s)$ at $s=1$, for any $A>0$ and $x \geq 2$ we have*

$$\sum_{p \leq x} T_p \log p = rx + O\left(\frac{x}{(\log x)^A}\right)$$

and moreover

$$\sum_{p \leq x} f(p) \log p = (r-1)x + O\left(\frac{x}{(\log x)^A}\right).$$

Proof. We apply Lemma 2.1.10 to the completely multiplicative function g defined through $p \mapsto T_p$. Indeed by Lemma 2.2.2 it is applicable with $b = 22$. We may use $a = 22^3$ as this makes the resulting power of x in the conclusion of Lemma 2.1.10 equal to $\frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1$. In fact, any $a > 22^2$ would also have sufficed.

Having fixed a, the sum $\sum_{p \leq a} T_p \log p$ is bounded, hence with notation of Lemma 2.1.10 we have

$$\sum_{p \leq x} T_p \log p = \psi_{g^*}(x) + O\left(x^{5/6}\right).$$

The Dirichlet series of g^* and $L_g(s)$ are not equal, but their Euler products only differ for primes $p < a$. These are finite in number, so the order of the pole at $s = 1$ is not affected. Hence, by Proposition 2.1.11 we have

$$\psi_{g^*}(x) = rx + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Combining these two estimates, we conclude

$$\sum_{p \leq x} T_p \log p = rx + O\left(x^{\max\left\{\frac{5}{6}, \frac{1}{2}+\varepsilon\right\}}\right).$$

Realizing that saving a power of x gives a stricter error term than saving any power of $\log x$, we conclude the proof of the first equality upon choosing any $\varepsilon < \frac{1}{2}$.

The second equality immediately follows by recombining the three terms in (2.2). Indeed the error $O\left(\frac{x}{(\log x)^A}\right)$ also applies to the middle sum $\sum_{p \leq x} \log p \sim x$. $\square$

REMARK 2.2.6. In the proposition above, we did not really need to assume the full power of GRH: by Proposition 2.1.11, a zero-free region $\Re(s) > \sigma$ for any $\sigma > \frac{1}{2}$ would have sufficed.

2.2.1 Identification of the logarithmic exponent

The last step in calculating the singular series is to identify the constant r appearing in Proposition 2.2.5. So far, $L_g(s) = \prod_p (1 - T_p p^{-s})^{-1}$ seemed to have appeared out of nowhere, and a priori it is not obvious what the order r of the pole would be. We will now explain how $L_g(s)$ is related to the variety X and how r is related to the Picard group of X. Recall the L-function $L(\mathrm{H}^2(X), s)$ from Definition 1.2.19.

PROPOSITION 2.2.7. *The order of the pole of $L(\mathrm{H}^2(X), s)$ at $s = 2$ is the Picard rank of X.*

Proof. As was already seen in §1.2.3, this is part of the Tate conjecture, which is known for K3 surfaces and hence in particular for X. □

PROPOSITION 2.2.8. *The shifted Dirichlet series $L_g(s-1)$ and $L(\mathrm{H}^2(X), s)$ have poles of the same order at $s = 2$.*

Proof. First notice that indeed both L-functions have a pole at $s = 2$. We compare the p-adic factors for both Euler products

$$L_g(s-1) = \prod_p \frac{1}{1 - T_p p^{1-s}}$$

and

$$L(\mathrm{H}^2(X), s) = \prod_{p \notin S} \frac{1}{\det(1 - \mathrm{Frob}_p \, p^{-s} \mid \mathrm{H}^2_{\text{ét}}(\overline{X_p}, \mathbb{Q}_\ell)}.$$

Note that there are only finitely many bad primes $p \in S$, so their appearance will not affect the order of the pole. Let α_j for $j = 1, \ldots, b_2 = 22$ be the eigenvalues of Frob_p on $\mathrm{H}^2_{\text{ét}}(\overline{X_p}, \mathbb{Q}_\ell)$. Since one obtains the expression $\det(1 - \mathrm{Frob}_p \, p^{-s} \mid \mathrm{H}^2_{\text{ét}}(\overline{X_p}, \mathbb{Q}_\ell)$ as the characteristic polynomial of the Frobenius endomorphism, read backwards, with p^{-s} substituted, this determinant equals $\prod_{j=1}^{22}(1 - \alpha_j p^{-s})$, which is

$$1 - (T_p \cdot p)p^{-s} + \sum_{i<j} \alpha_i \alpha_j p^{-2s} - \sum_{i<j<k} \alpha_i \alpha_j \alpha_k p^{-3s} + \ldots + \left(\prod_{j=1}^{22} \alpha_j \right) p^{-22s}.$$

Let us study the fraction between p-adic factors for the two L-functions

$L(\mathrm{H}^2(X), s)$ and $L_g(s-1)$ for $p \notin S$:

$$\frac{1 - (T_p \cdot p)p^{-s} + \sum_{i<j} \alpha_i \alpha_j p^{-2s} - \sum_{i<j<k} \alpha_i \alpha_j \alpha_k p^{-3s} + \dots}{1 - T_p p^{1-s}}$$
$$= 1 + \frac{\sum_{i<j} \alpha_i \alpha_j p^{-2s} - \sum_{i<j<k} \alpha_i \alpha_j \alpha_k p^{-3s} + \dots}{1 - T_p p^{1-s}} =: F(p, s)$$

Each of the α_j has modulus p, so for every l, the term in the numerator involving p^{-ls} has modulus at most $\binom{22}{l} p^{l-ls}$. Moreover, for any $s > \frac{3}{2}$ and $p > 44^2$, the denominator is larger than $\frac{1}{2}$. Hence for $s > \frac{3}{2}$ and $p > 44^2$ the expression $F(p, s)$ is $1 + O(p^{-2(s-1)})$. We need to study $\prod_{p \le t} F(p, s)$ as $t \to \infty$ and subsequently $s \to 2$. Writing C for the implicit constant in the bound for $F(p, s)$, we have

$$\prod_{44^2 < p \le t} F(p, s) \le \exp\left(C \sum_{44^2 < p \le t} p^{-2(s-1)} \right). \tag{2.3}$$

Since for every positive ε, the sum $\sum_{p \le t} p^{-(1+\varepsilon)}$ converges absolutely as $t \to \infty$, so does the sum in the exponential above for any $s > \frac{3}{2}$ and in particular for $s = 2$. Therefore, for fixed s, the product of $F(p, s)$ converges absolutely as $t \to \infty$. Moreover, since the inequality (2.3) is uniform in s, we may switch the limits $t \to \infty$ and $s \to 2$ to conclude that $\prod_p F(p, 2)$ has a finite, non-zero value. This confirms the statement of the proposition. □

COROLLARY 2.2.9. *Assuming GRH, and denoting the Picard rank of X by ρ, there is a constant c such that we have $\mathfrak{S}(Q) \sim c(\log Q)^{\rho-1}$.*

Proof. We take the $j = 0$ term from Theorem 2.1.3. Proposition 2.2.5 tells us to use $\alpha = r - 1$ and Propositions 2.2.7 and 2.2.8 verify $r = \rho$. Now we apply Lemma 2.1.5 with $a = \rho - 2$.

Lemma 2.2.3 provides us with an ineffective k to be used in the assumptions of Theorem 2.1.3. In order for the error term in this theorem not to dominate, we need to take $A > k - \alpha = k + 1 - \rho$. Indeed, the error term in Proposition 2.2.5 allows such a choice of ineffective A. □

REMARK 2.2.10. Following our proof, we have to exclude the case $\rho = 1$ from our main result. The specific place where the proof falls short is the case $0 = \alpha = \rho - 1$ in Theorem 2.1.3.

2.3 Evaluating the singular integral

In this section we will show that in the case of diagonal quartics, the singular integral contributes a factor of $\log(B)$ to the counting function.

Recall that the singular integral is

$$\mathfrak{J}(Q) = \int_{-Q}^{Q} \int_{[-1,1]^n} e(\theta F(\mathbf{x})) \mathrm{d}\mathbf{x} \mathrm{d}\theta$$

for some small power $Q = B^{\delta}$.

Our first step is to evaluate the integral

$$\begin{aligned} I(\theta) &= \int_{[-1,1]^4} e\left(\theta \sum_{i=1}^{4} a_i x_i^4\right) \mathrm{d}\mathbf{x} \\ &= \prod_{i=1}^{4} \int_{-1}^{1} e\left(\theta \cdot a_i x_i^4\right) \mathrm{d}x_i \\ &= \prod_{i=1}^{4} 2 \int_{0}^{1} e\left(\theta \cdot a_i x_i^4\right) \mathrm{d}x_i, \end{aligned}$$

hence we focus on the 1-dimensional integral that appears fourfold. We split the calculation into two cases: where $\theta \cdot a_i > 0$ and where $\theta \cdot a_i < 0$ hold.

For $\theta \cdot a_i > 0$ we substitute $u = \theta a_i x_i^4$, transforming the integral into

$$\frac{1}{4(\theta \cdot a_i)^{1/4}} \int_{0}^{\theta a_i} e(u) u^{-3/4} \mathrm{d}u.$$

For $\theta \cdot a_i < 0$ we substitute $u = -\theta a_i x_i^4$, transforming the integral into

$$\frac{1}{4(-\theta \cdot a_i)^{1/4}} \int_{0}^{-\theta a_i} e(-u) u^{-3/4} \mathrm{d}u.$$

Definition 2.3.1. For any $\sigma \in (-1, 0) \subset \mathbb{R}$ and $t \in \mathbb{R}_{>0}$ we introduce notation

$$\begin{aligned} i_{\sigma}(t) &= \int_{0}^{t} e(u) u^{\sigma} \mathrm{d}u, \\ j_{\sigma}(t) &= \int_{0}^{t} e(-u) u^{\sigma} \mathrm{d}u = \overline{i_{\sigma}(t)}. \end{aligned}$$

Using this notation, we have found the validity of

$$\begin{aligned} I(\theta) &= \frac{1}{16} \prod_{i:\theta a_i > 0} \frac{1}{(\theta a_i)^{1/4}} i_{-3/4}(\theta a_i) \cdot \prod_{i:\theta a_i<0} \frac{1}{(-\theta a_i)^{1/4}} j_{-3/4}(-\theta a_i) \\ &= \frac{1}{16|\theta|} \prod_i \frac{1}{|a_i|^{1/4}} \prod_{i:\theta\cdot a_i>0} i_{-3/4}(\theta a_i) \prod_{i:\theta\cdot a_i<0} j_{-3/4}(|\theta a_i|). \end{aligned}$$

2.3.1 The integral over theta

We are left with calculating $\int_{-R}^{R} I(\theta)\mathrm{d}\theta$. Without loss of generality we may assume $R > 1$, and we have

$$\begin{aligned} \int_{-R}^{R} I(\theta)\mathrm{d}\theta &= \int_{-R}^{-1} I(\theta)\mathrm{d}\theta + \int_{-1}^{1} I(\theta)\mathrm{d}\theta + \int_{1}^{R} I(\theta)\mathrm{d}\theta \\ &= \int_{-1}^{1} I(\theta)\mathrm{d}\theta \\ &\quad + \frac{1}{8\prod_i |a_i|^{1/4}} \int_1^R \frac{1}{\theta} \Re\left\{ \prod_{i:a_i>0} i_{-3/4}(\theta a_i) \prod_{i:a_i<0} \overline{i_{-3/4}(-\theta a_i)} \right\} \mathrm{d}\theta, \end{aligned} \tag{2.4}$$

where we have used $I(-\theta) = \overline{I(\theta)}$.

From

$$I(\theta) = \prod_{i=1}^{4} 2 \prod_{i=1}^{4} \int_0^1 e(\theta \cdot a_i x_i^4)\mathrm{d}x_i$$

we see $|I(\theta)| \leq 2^4 \int_0^1 |e((\theta \cdot a_i x_i^4)|\mathrm{d}x_i = 16$, hence we may estimate

$$\int_{-1}^{1} I(\theta)\mathrm{d}\theta = O(1).$$

The integral over the interval $(1, R)$ requires further study.

LEMMA 2.3.2. *For every $\sigma \in (-1, 0)$, there exists a constant c_σ such that the function $i_\sigma(t)$ equals $c_\sigma + O(t^\sigma)$ for $t \geq 1$.*

Proof. We write

$$i_\sigma(t) = \int_0^\infty e(u)u^\sigma \mathrm{d}u - \int_t^\infty e(u)u^\sigma \mathrm{d}u$$

and we prove that $\int_0^\infty e(u)u^\sigma \mathrm{d}u =: c_\sigma$ converges and that the second integral can be estimated by $O(t^\sigma)$.

We split each of the integrals into their real and imaginary parts

$$\int_a^b e(u)u^\sigma = \int_a^b \cos(2\pi u)u^\sigma \mathrm{d}u + i\int_a^b \sin(2\pi u)u^\sigma \mathrm{d}u$$

and we argue on the real parts; the calculation on the imaginary parts is completely analogous.

We first bound the integral over $u > t$; we apply integration by parts:

$$\int_t^\infty \cos(2\pi u)u^\sigma \mathrm{d}u = \left[\frac{1}{2\pi}\sin(2\pi u)u^\sigma\right]_t^\infty - \frac{\sigma}{2\pi}\int_t^\infty \sin(2\pi u)u^{\sigma-1}\mathrm{d}u.$$

The latter integral is bounded from above by $\int_t^\infty u^{\sigma-1}\mathrm{d}u = O(t^\sigma)$.

The convergence of $\int_0^\infty e(u)u^\sigma$ is proven by splitting the positive real line into the two parts $(0,1)$ and $\mathbb{R}_{\geq 1}$. It is easily seen that $\int_0^1 \cos(2\pi u)u^\sigma \mathrm{d}u$ converges: it is bounded from above by $\int_0^1 u^\sigma \mathrm{d}u$ which clearly converges for $\sigma > -1$. The integral over $\mathbb{R}_{\geq 1}$ converges by substituting $t = 1$ in the previous calculation. □

Lemma 2.3.2 clearly also applies to the function $j_\sigma(t)$ with constant $\overline{c_\sigma}$. Write $n \leq 4$ for the number of coefficients a_i that are positive and write $c := \Re\left\{c_{-3/4}^n \overline{c_{-3/4}}^{4-n}\right\}$. The last integral in (2.4) is well approximated by $c\int_1^R \frac{1}{\theta}d\theta$, which provides the logarithm that we were out to find.

Proposition 2.3.3. *With the constant c as given above, the singular integral evaluates as*

$$\mathfrak{J}(Q) = \int_{-Q}^Q I(\theta)\mathrm{d}\theta = \frac{c}{8\prod_i |a_i|^{1/4}}\log(Q) + O(1).$$

Proof. The proof is no more than following the arguments and calculations in this section in a linear fashion. □

2.3.2 The proof of the main theorem

Proof of Theorem 2.0.1. The proof of the theorem is now a simple combination of the statements of Corollary 2.2.9 and Proposition 2.3.3 with Q a sufficiently small power of B. □

One might notice that we did not specify any choice for δ in the proof of the theorem. Indeed, we did not make any, other than those mentioned for the machinery to work (cf. Remark 1.3.8). Any actual choice will influence the result in the sense that $Q = B^\delta$ will be affected. Secondarily, through the logarithm that appears in the statement of the theorem, the leading constant c_F will depend on said choice, after switching to the variable B. The overall shape of the major arc contribution however, will not.

2.4 Minor arcs and bad subvarieties

As a variation on the concept of *Hardy–Littlewood systems* where the circle method counts rational points in accordance with Manin's conjecture, Vaughan and Wooley [VW95] have introduced what they call *quasi Hardy–Littlewood systems (QHL models)*. The circle method may not work for QHL models in the sense that the minor arcs give a contribution that is not necessarily dominated by the contribution of the major arcs, but the major arcs nonetheless contribute exactly the rational points away from accumulating subvarieties. The contribution of accumulating subvarieties is found in the minor arcs. Vaughan and Wooley observe that many varieties are QHL models; explicit examples include the zero locus of $x_1x_2 = x_3x_4$ (worked out in [VW95] and the zero locus of $x_1^4 + x_2^4 + x_3^4 = y_1^4 + y_2^4 + y_3^4$ (attributed to Wooley in [Con16, p. 14]). In this light it must also be recorded that in [HB98], Heath-Brown was succesfull in separating out the contribution of accumulating subvarieties, using a modified version of the circle method.

After introducing their terminology, Vaughan and Wooley immediately confess that their notion needs to be adapted to include information on the possible failure of the Hasse principle. As originally stated, diagonal quartic surfaces lie outside the range of expected QHL models. However, neither do diagonal cubic surfaces satisfy the original definition of QHL models, but Browning has produced a heuristic showing that for such surfaces the major arcs indeed give the contribution as predicted by Manin's conjecture, albeit with a leading constant that is different from the one predicted by Peyre [Bro09, Chapter 8]. A further adaptation to the notion of QHL models is not unthinkable and it may not be unreasonable to believe that in the case of diagonal quartic surfaces the major arcs indeed reveal the rational points away from accumulating subvarieties.