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Preface

In the first half of the 1960s Bryan Birch and Peter Swinnerton-Dyer conceived a conjecture on certain arithmetic and geometric invariants of elliptic curves, which is now known as the Birch and Swinnerton-Dyer conjecture (BSD).

**Conjecture ([BiSw65])**. Let \( E / \mathbb{Q} \) be an elliptic curve of algebraic rank \( r \). Let \( L(E, s) \) be its \( L \)-function, \( R_E \) its regulator, \( \Sha(E) \) its Tate-Shafarevich group and \( \Omega_E \) its real period. For each prime \( p \), let \( c_p \) be the Tamagawa number of \( E \) at \( p \). Then \( L(E, s) \) has an analytic continuation, \( \Sha(E) \) is finite, \( L(E, s) \) has a zero of order \( r \) at \( s = 1 \), and

\[
\lim_{s \to 1} (s - 1)^{-r} L(E, s) = \frac{R_E \cdot \Omega_E \cdot |\Sha(E)| \cdot \prod_p c_p}{|E(\mathbb{Q})_{\text{tors}}|^2}.
\]

The conjecture, which has been the result of many computations with elliptic curves, has been generalised later to abelian varieties over general number fields by Tate ([Tate66]).

Due to work of Holmes ([Holm12]) and Müller ([Müll14]), it became possible to compute the regulator for Jacobians of hyperelliptic curves over \( \mathbb{Q} \) of higher genus. Due to work of Dokchitser ([Dokc04]), it became possible to evaluate the \( L \)-function for the same Jacobians. The computation of \( \Sha \) was still difficult, even for elliptic curves. This lead us to the following question.

**Question.** Is it possible to numerically verify, except for the computation of \( \Sha \), the Birch and Swinnerton-Dyer conjecture for hyperelliptic curves over \( \mathbb{Q} \)?

In [FLSSSW01], Flynn, Leprévost, Schaefer, Stein, Stoll and Wetherell already showed that this is possible for some curves of genus 2, but they relied on modular methods to do so.

In chapter [1] we describe the implementation and results of an algorithm in Magma to do this verification without using modular methods, and also for hyperelliptic curves of genus greater than 2. The main contribution of the author is an implementation of an algorithm to compute the Tamagawa numbers, and a better theoretical understanding of the different sheaves of differentials on the regular model of the curve, resulting in an algorithm to compute the real period.

It has already been known due to Tate ([Tate66]) and Milne ([Mil72]) that the Birch and Swinnerton-Dyer conjecture is compatible with isogenies and Weil restrictions. The
Jacobian of an elliptic curve over a quadratic number field is an abelian surface over $\mathbb{Q}$, and most abelian surfaces are Jacobians. So we asked ourselves the following question.

**Question.** Is it possible to numerically verify, except for $\mathbb{Q}$, the Birch and Swinnerton-Dyer conjecture for an elliptic curve over a quadratic number field, using the methods for genus 2 curves over $\mathbb{Q}$?

Unfortunately, this turned out to be more difficult than expected, but we did manage to find an example of an elliptic curve over $\mathbb{Q}(\sqrt{5})$ for which we could verify the Birch and Swinnerton-Dyer conjecture, using the methods for genus 2 curves over $\mathbb{Q}$. In chapter 2, we prove the results and explain the methods used to find this example.

The last two chapters of this thesis treat ordinary reduction for curves. While proving that for all positive integers $M$ and $g$ and any number field $K$, a proportion of 100% of hyperelliptic curves of genus $g$ over $K$, ordered by height, have at least $M$ primes of ordinary good reduction (see chapter 4), we found that the hardest part was to actually prove that there exist ordinary hyperelliptic curves of genus $g$ in each characteristic.

For hyperelliptic curves, this matter has been resolved by Glass and Pries in [GlPr05]. Instead of hyperelliptic curves, which are $\mathbb{Z}/2\mathbb{Z}$-covers of $\mathbb{P}^1$, one could also look at Galois covers of $\mathbb{P}^1$ with fixed Galois group $G$, and ask the following question, for which the methods of Glass and Pries do not give an easy answer.

**Question.** Are there Galois covers $C \to \mathbb{P}^1$ with Galois group $G$ in characteristic $p$, such that $C$ is ordinary?

The inverse Galois problem has already been studied extensively for $\mathbb{P}^1$, but the methods used for this, mainly based on rigid geometry, does not seem to give any information on whether the curves obtained are ordinary or not.

In chapter 3, we show how one can partially answer the question for ordinary curves. We construct Galois covers of ordinary semi-stable curves, and use deformation theory to deform them into Galois covers of ordinary smooth curves. We provide several classes of examples for which this construction gives a positive answer to the question. The content of this chapter has appeared, in a shortened form, in the International Journal of Number Theory, see [vB18].

In chapter 4, we will apply the results of chapter 3 to prove the aforementioned result on the proportion of hyperelliptic curves having at least $M$ primes of ordinary good reduction.

The reader is advised that the chapters of this thesis are written to be independent. The author intends to also publish the first two chapters separately, in shortened form, as articles. The content of each chapter is supposed to be self-contained. It could happen that notations between the different chapters differ slightly.