COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

On the temperature-density relation in Eddington's theory about the interior of a star, by J. Woltjer Jr.

EDDINGTON's theory *) about the interior of a star started from two assumptions: the constancy of the coefficient of absorption and of the mean rate of generation of energy per unit mass within a sphere of any radius concentric with the surface of the star. In a series of papers on the absorption of radiation EDDINGTON **) made the coefficient of absorption the object of an extended investigation. The resulting value of this coefficient appeared to be not at all constant, but to vary in the main part as the first power of the density and the third power of the reciprocal temperature. However by a happy coincidence this large deviation from a constant value was in the interior of a star without significance, as in the original theory the relation between temperature and density was exactly of the form needed to effect a compensation, viz: the density varied as the third power of the temperature.

Nevertheless it is of some interest to derive the temperature-density relation starting from a value of the coefficient of absorption that depends on temperature and density in the way indicated, the more so as EDDINGTON's relation was an approximation ***) and it was not evident a priori in which way the character of the approximation should be changed on account of the new assumption about the absorption coefficient. The object of this paper is to supply this lacuna; the result is a confirmation of EDDINGTON's temperature-density relation.

The fundamental equations of the theory may be resumed as follows:

I. gas-law: \( \rho = \frac{R}{m} \rho T \);

II. law of radiation pressure: \( \Pi = \frac{1}{3} a T^3 \);

III. equation of stationary radiation: \( \frac{dT^4}{dr} = - \frac{3x}{ac} F \);

IV. equation of hydrostatic equilibrium: \( \frac{d(p + \Pi)}{dr} = - g \rho \);

V. hypothesis about the rate of generation of energy:

\[
F = \frac{n Q}{4\pi r^2} (\text{a constant})
\]

\( \rho \) = gas-pressure, \( \rho \) = density, \( T \) = absolute temperature, \( m \) = molecular weight, \( R \) = absolute gas-constant, \( \Pi \) = radiation-pressure, \( a \) = energy density of black body radiation for \( T = 1^\circ \), \( x \) = mean absorption-coefficient, \( c \) = velocity of light, \( F \) = resulting energy radiation in outward direction per unit of time and area, \( g \) = acceleration of gravity, \( n Q \) = mean rate of generation of energy per unit of time and mass within a sphere of radius \( r \), \( M_r \) = mass within a sphere of radius \( r \), \( Q \) = ratio of total radiation of the star (luminosity) to mass; \( r \) is the independent variable, the distance to the centre of the star.

Introducing the constant of gravitation \( f \), III, IV and V can be combined into the relation:

\[
\frac{dT^4}{dr} = \frac{3x}{4\pi acf} \frac{d(p + \Pi)}{dr} ;
\]

or, introducing I and II:

\[
\frac{dT^4}{dr} = \frac{3x}{4\pi acf} \frac{d}{dr} \left[ \frac{R}{m} \rho T + \frac{1}{3} a T^3 \right].
\]

As \( x \) is supposed constant, the only variable part of the factor in the right hand member is \( x \); if we neglect the variability of \( x \) and introduce the boundary condition: \( T = 0 \) if \( \rho = 0 \), we see at once that \( \rho \) is proportional to \( T^3 \), EDDINGTON's relation. However this relation is an approximation *) on account of the approximative character of the boundary condition.

*) EDDINGTON l.c.

---

© Astronomical Institutes of The Netherlands • Provided by the NASA Astrophysics Data System
I now introduce the law of variation of the absorption coefficient:

\[ x \text{ is proportional to } \rho T^{-3}. \]

Denoting quantities in the centre of the star by the index \( c \) we may write:

\[ x = x_c \frac{T^3}{\rho_c T_c^3}. \]

Introducing the variables:

\[ X = \frac{\rho T}{\rho_c T_c}, \quad Y = \frac{T^4}{T_c^4}, \]

and putting:

\[ \frac{3}{a m} \frac{\rho_c}{T_c^3} = \alpha, \quad 4\pi c_f \frac{Q}{\rho c} \frac{x_c}{T_c} = \gamma, \]

(2) reduces to:

\[ azdX + dY = \frac{Y}{X} \gamma dY. \]

Abbreviating:

\[ A = \frac{1}{2\gamma} + \frac{1}{4\gamma^2}, \quad B = -\frac{1}{2\gamma} + \frac{1}{4\gamma^2}, \]

the solution of (7) is:

\[ \left( \frac{1-A}{Z-A} \right)^B \left( \frac{1+B}{Z+B} \right)^A = X^{A+B}. \]

(9)

\( A \) and \( B \) are always positive; the integration constant involved is determined by the condition \( X = 1, Z = 1 \).

The main features of the functional relation (9) may easily be developed.

Firstly, suppose \( 1 - A > 0 \); then for large values of \( Z \), \( X \) is very small; if \( Z \) decreases, \( X \) increases until for \( Z = A \), \( X \) becomes infinite; if \( Z < A \) the curve has no real branch.

Secondly, suppose \( 1 - A \leq 0 \); then the real branch of the curve is contained between the parallel lines \( Z = -B \) and \( Z = A \); the curve intersects the \( X \)-axis in a point with the abscissa:

\[ \left( \frac{A-1}{A} \right)^B \left( \frac{1+B}{B} \right)^A \]

i.e. if \( T \) is zero, \( \rho T \) is not zero which is physically impossible.

Thirdly, suppose \( 1 - A = 0 \); then \( Z = 1 \) for every value of \( X \).

Returning to the only possible cases \( 1 - A \geq 0 \) we might be inclined to consider the functional relation (9) as largely deviating from EDDINGTON's relation: \( Z = 1 \) for all values of \( X \). So it would be if all values of \( A \) and \( B \) could be admitted. However, at the boundary of the star (\( \rho = 0, X = 0 \)) \( T \) ought to be of the order of magnitude of the effective temperature of the star; supposing \( T \) large at the centre, this means that if \( X = 0 \), \( Y \) should be very small. Thus we may impose the condition:

\[ \lim_{X \to 0} XZ \text{ is small.} \]

(10)

From (9) we derive for small values of \( X \):

\[ (1 - A)^B \quad (1 + B)^A \quad Z^{\frac{(A+B)}{A}} = X^{A+B} \]

(11)

hence:

\[ \lim_{X \to 0} XZ = (1 - A)^B (1 + B)^A \]

(12)

both exponents are smaller than unity; the second factor is always larger than unity; thus if the limit shall be small we must have

\[ 1 - A \text{ nearly zero.} \]

(13)

But if this is the case the curve (9), for values of \( X \) ranging from a small positive value to unity, reduces very nearly to the straight line \( Z = 1 \); so we see that (accepting the constancy of \( x \) and the variation of the coefficient of absorption with density and temperature according to (3)) with a large degree of approximation in the interior of a star \( \rho \) is proportional to \( T^3 \).