1. The equations for the osculating elliptic elements of an asteroid relative to the sun in the coplanar motion of the sun, Jupiter and the asteroid may be reduced to the canonical form:

\[
\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}
\]

\((i = 1, 2).\)

The function \(F\) is a periodic function of the three variables \(y_1, y_2, y_3\) and \(l'\), the mean anomaly of Jupiter; this function may be expanded in the goniometric series:

\[
F = \sum C_{n_1 n_2 n_3} \cos (p_1 y_1 + p_2 y_2 + p_3 l')
\]

the summation is to be extended over all positive and negative integral values of \(p_1, p_2, p_3, q_0, q_0\), zero included. \(C_{n_0 o o}\) is of order zero with regard to Jupiter's mass, the remaining coefficients are at least of order one. The coefficients \(C\) are functions of \(x_1, x_2\) and of two parameters: the semi-axis and the eccentricity of Jupiter's orbit.

The variables \(y_1\) and \(y_2\) are linear functions with integral coefficients of \(l'\) and the mean longitude and longitude of perihelion of the asteroid; the variables \(x_1\) and \(x_2\) are functions of the semi-axis major and the eccentricity of the asteroid.

2. If the mean motions of Jupiter and the asteroid are approximately commensurable, some terms of the function \(F\) become of special importance, viz: those terms whose argument in elliptic motion moves very slowly. Apart from these terms those whose argument in elliptic motion is independent of \(t\) are always important. These two classes of terms form the critical and the secular parts of the function \(F\) respectively. The main features of the motion of the asteroid are obtained by restricting the function \(F\) to its critical and secular parts and solving the equations (1) with this restricted function \(F\). If we wish a still better approximation we remove from the function all non-critical and non-secular terms by a series of Delaunay-transformations or some equivalent procedure*). The result is a system of equations of the form (1), \(F\) being reduced to its secular and critical parts, but the coefficients in this reduced function \(F\) differ by quantities of higher order from the corresponding coefficients in the restricted function \(F\) from (2). We shall however, for simplicity's sake neglect these differences and consider the equations (1) with the restricted function \(F\) from equation (2).

3. The restricted function \(F\) may be considered as a periodic function of two arguments: a critical argument and a secular argument. Denoting these two arguments by \(\eta_1\) and \(\eta_2\), we perform a canonical transformation of variables, introducing for \(x_1\) and \(x_2\), \(\xi_1\) and \(\xi_2\), connected with the \(x\)'s by linear equations. By this transformation the arguments in the restricted function \(F\) become linear functions with integral coefficients of \(\eta_1\) and \(\eta_2\) only.

So we have the system of equations:

\[
\frac{d\xi_1}{d\eta_1} = \frac{\partial F}{\partial \eta_1}, \quad \frac{d\eta_1}{dt} = -\frac{\partial F}{\partial \xi_1}
\]

\[
F = \sum \Gamma_{\eta_1 \eta_2} \cos (q_1 \eta_1 + q_2 \eta_2)
\]

Each coefficient \(\Gamma\) is identical with a coefficient \(C\) from (2); in particular:

\[
\Gamma_{oo} = C_{oo}.
\]

4. The coefficients \(\Gamma\) are functions of \(\xi_1\) and \(\xi_2\) and two parameters: the semi-axis major \(a'\) and the eccentricity \(e'\) of Jupiter; they may be developed in powers of \(e'\). It is always possible so to introduce the arguments \(\eta_1\) and \(\eta_2\) that the coefficient \(\Gamma_{\eta_1 \eta_2}\) is a power series in \(e'\) beginning with a term in \(e'^{\|\eta_1\|}\); we shall always suppose this to be the case.

Consequently, if we wish to develop the solution

of equations (3) in powers of $\epsilon'$, the terms of degree zero can be obtained by solving the equations:

$$
\frac{d\xi'_i}{dt} = \frac{\partial F}{\partial n_i}, \quad \frac{d\eta'_i}{dt} = -\frac{\partial F}{\partial \xi'_i},
$$

$$
F = \sum_{i=1}^{n} \Gamma_{\epsilon',\epsilon} \cos q'_i \eta'_i.
$$

5. The solution of equations (5) may belong to one of two different cases: the variable $n_i$ oscillates about a mean value or it takes all values from $-\infty$ to $+\infty$.

We shall suppose the first case to occur: the motion of $n_i$ will be librational.

Libration is a not uncommon feature in the satellite systems. The motion of Mimas, though not entering into the case explained just now, is of the same character. The same is true of the satellites of Jupiter.

An approximate value of $n_i$ is:

$$
n_i = \chi \sin \tau;
$$

$$
\frac{d\xi'_i}{dt} = \frac{\partial F}{\partial n_i} \frac{\xi'_i}{\partial \xi'_i} + \frac{\partial F}{\partial n_i} \frac{\eta'_i}{\partial \eta'_i} + \frac{\partial F}{\partial n_i} \frac{\eta'_i}{\partial \xi'_i} - \frac{\partial F}{\partial n_i} \frac{\xi'_i}{\partial \eta'_i},
$$

$$
F = \sum_{\epsilon'} F_{\epsilon'},
$$

$$
F_{\epsilon'} = \text{terms of degree } \epsilon' \text{ in } \epsilon' \text{ in the development of } F.
$$

The known terms in the right-hand members can be expanded in goniometric series of the form:

$$
\sum E_i \cos \left( (\omega + i \tau) \right);
$$

the coefficients $E_i$ are constants and contain $\epsilon'$ as a factor. The reason of the fact that $\omega$ occurs only with the coefficient $+1$ is the dependence of the degree of $\Gamma_{\epsilon',n_i}$ in $\epsilon'$ on the multiple $q'_i$ of $n_i$ in (3) (Cf. 4). The solution of equations (8) can be expanded in series of the form (9). The coefficients of $\sin (\omega \pm \tau)$ in $\delta n_i$ contain $\sqrt{(m'|M)}$ in the denominator; thus these coefficients are of the form:

$$
\frac{\chi \epsilon' K}{\sqrt{(m'|M)}},
$$

$K$ being a factor depending on the mean values of the eccentricity and semi-axis major of the asteroid.

If we had used DELAUNAY’s method of transformation we should have interpreted these terms as perturbations of the argument $\tau$ of the form:

$$
\frac{\epsilon' K'}{\sqrt{(m'|M)}} \sin \omega,
$$

where $K'$ is a constant analogous to $K$.

$x$ is an arbitrary constant, the amplitude of the libration; $\tau$ is a linear function of $t$; $\frac{dt}{dt}$ is of the order of the square root of the disturbing mass.

The approximate values of $\xi'_i$ and $n_i$ are:

$$
\xi'_i = \gamma \sin \cos \tau,
$$

$$
n_i = \omega = \text{linear function of } t;
$$

the constant $\gamma$ is of the order of the square root of the disturbing mass; the coefficient of $t$ in the linear function $\omega$ is of the order of the disturbing mass.

The value of $\xi'_2$ is constant, as $F$ does not contain $n_2$.

6. For the terms of $\xi'_i$, $n_i$ that are of the first degree in $\epsilon'$ viz: $\xi'_i, \delta n_i$ we have the system of linear equations with periodic coefficients:

$$
(11)
$$

The value of these terms in the mean longitude in the case Titan-Hyperion is $\star)$:

$$
(12) \quad + \Omega^0 \cdot 202 \sin (\tau + \omega) - \Omega^0 \cdot 202 \sin (\tau - \omega);
$$

the libration in mean longitude is $9^\circ$.

7. It may happen that the product $\frac{\chi \epsilon' K}{\sqrt{(m'|M)}}$ is large compared with unity. How must this be interpreted? The denominator is derived from a small analytical divisor $\star\star$). If the divisor makes the coefficient of the periodic term too large then the procedure used for the integration is not allowable. This procedure exists in developing the integral in powers of $\epsilon'$. So our conclusion must be: if the product $\frac{\chi \epsilon' K}{\sqrt{(m'|M)}}$ is large compared with unity, then the solution of equations (3) cannot be developed in powers of $\epsilon'$ in goniometric form. We must then abandon the expansion in powers of $\epsilon'$ and make an extensive investigation of the Delaunay-transformation that gives rise to the term (11).

*) On the perturbations in the motion of Hyperion proportional to the first power of Titan’s eccentricity, Amsterdam Proceedings XXI 9.

The case is quite analogous to that of small numerical divisors: the critical terms of the classical planetary theories. There, if such a critical term exceeds a certain limit, expansion in powers of the disturbing mass is not allowed.

8. The theory of the Trojans *) is especially suitable to elaborate the terms in question, firstly as expansions in powers of small quantities find in this theory a more general application than in the case of other commensurabilities; secondly, as the argument in the perturbative function subject to libration does not contain the longitude of perihelion.

The preceding explanations need some modifications for the Trojans. The function \( F \) in equation (3) cannot be expanded in goniometric form, but must be developed as a power-series in \( \nu_s - \frac{1}{2} \pi \), where \( \nu_s \) represents the difference of mean longitude between the asteroid and Jupiter. The consequence of this modification is a certain loss of symmetry, but essentially the results of the preceding sections subsist.

The following sections contain the details of the computation of the terms in question.

9. Let \( a, e, l, g, h, v, r, p, \Delta \) denote respectively

\[
\frac{a'}{\Delta} = 1 - \frac{1}{2} \alpha + \frac{5}{8} \alpha^2 \ldots
\]

(16)

\[+ \left[ - \frac{1}{2} \alpha - \beta - \frac{5}{8} \alpha^2 + \frac{5}{8} \alpha \beta + \ldots \right] \frac{3}{4} \sqrt{3} \gamma
\]

\[+ \left[ \frac{5}{8} - \frac{1}{8} \alpha - \frac{1}{4} \beta - \frac{5}{8} \gamma + \frac{5}{8} \alpha^2 - \frac{5}{8} \alpha \beta + \frac{5}{8} \alpha \gamma + \frac{5}{8} \beta^2 + \ldots \right] y^a
\]

+ 

The formulae of elliptic motion afford the developments of \( \alpha, \beta, \gamma \) and \( y \) in powers of \( e, e' \). The results are, to the second degree in \( e \) and \( e' \) inclusive:

\[
\alpha = \frac{1}{2} e^2 \cos l - e' \cos l' + 2 \sqrt{3} e \sin l - 2 \sqrt{3} e' \sin l' + 2 e^2 + 2 e^3 - e^2 \cos 2 l - e^3 \cos 2 l'
\]

\[+ \frac{1}{4} \sqrt{3} e^3 \sin 2 l - \frac{1}{2} \sqrt{3} e^3 \sin 2 l' - \frac{3}{4} e^3 \cos (l' - l) - \frac{3}{4} e^3 \cos (l' - l) - 2 e e' \sin (l' - l)
\]

\[
\beta = - e \cos l - e' \cos l' + \frac{1}{2} \sqrt{3} e \sin l - \frac{3}{2} \sqrt{3} e' \sin l' - \frac{1}{2} e^2 - \frac{1}{2} e^2 + \frac{1}{2} e^2 \cos 2 l + \frac{1}{2} e^2 \cos 2 l' + \frac{1}{2} e' \cos (l' - l) + \frac{1}{2} e' \cos (l' - l) + \frac{3}{4} \sqrt{3} e' \sin (l' - l)
\]

\[
\gamma = - e \cos l - e' \cos l' - 2 \sqrt{3} e \sin l + 2 \sqrt{3} e' \sin l' - \frac{1}{2} e^2 - \frac{1}{2} e^2 + \frac{1}{2} e^2 \cos 2 l + \frac{1}{2} e^2 \cos 2 l' + \frac{1}{2} e' \cos (l' - l) - 2 \sqrt{3} e \sin (l' - l)
\]

(17)

\[
\alpha^2 = \frac{1}{2} e^2 + \frac{1}{2} e^2 - \frac{1}{2} e^2 \cos 2 l - \frac{1}{2} e^2 \cos 2 l' - \frac{1}{2} e^3 \cos 2 l + \frac{1}{2} \sqrt{3} e^3 \sin 2 l + \frac{1}{2} \sqrt{3} e^3 \sin 2 l' + 13 e^3 \cos (l' - l) - 4 e e' \cos (l' - l) + 4 \sqrt{3} e e' \sin (l' - l)
\]

\[
\alpha \beta = \frac{1}{2} e^2 + \frac{1}{2} e^2 - \frac{3}{2} e^2 \cos 2 l - \frac{3}{2} e^2 \cos 2 l' + \frac{3}{4} \sqrt{3} e^3 \sin 2 l + \frac{3}{4} \sqrt{3} e^3 \sin 2 l' + 3 e^3 \cos (l' - l) - 3 e^3 \cos (l' - l) + \frac{3}{4} \sqrt{3} e^3 \sin (l' - l)
\]

\[
\alpha \gamma = - \frac{1}{2} e^2 - \frac{1}{2} e^2 + \frac{1}{2} e^2 \cos 2 l + \frac{1}{2} e^2 \cos 2 l' - 11 e e' \cos (l' - l) + 13 e e' \cos (l' - l)
\]

\[
\beta^2 = \frac{1}{2} e^2 + \frac{1}{2} e^2 - \frac{1}{2} e^2 \cos 2 l - \frac{1}{2} e^2 \cos 2 l' - \frac{1}{2} \sqrt{3} e^3 \sin 2 l + \frac{1}{2} \sqrt{3} e^3 \sin 2 l' + \frac{1}{2} e^2 \cos (l' - l) - \frac{1}{2} e^2 \cos (l' - l) + \frac{3}{4} \sqrt{3} e^3 \sin (l' - l)
\]

\[
+ \frac{1}{2} e^3 \cos (l' - l) - \frac{1}{2} e^3 \cos (l' - l) + \frac{3}{4} \sqrt{3} e^3 \sin (l' - l)
\]

*) Recently the subject has been treated by E. W. Brown: Transactions of the astronomical Observatory of Yale University, Vol. 3, part 1. I think that the critical terms occur at the integration of his equation (59.1). W. M. Smart (Mem. R. A. S. LXXII) started an investigation of the motion of this group of asteroids, using Delaunay's method, but his results are as yet incomplete. The expansions of the succeeding sections have been carried out independently from his developments.
Substitution in (16) furnishes the required development of $a'/\Delta$. However, terms with arguments $l, l', 2l, 2l', l' + l$ are of no importance (cf. 2). Omitting these terms the resulting expansion of $a'/\Delta$ is:

$$
\frac{a'}{\Delta} = 1 + \frac{3}{3} \varepsilon^2 + \frac{3}{3} \varepsilon^2 - \frac{3}{3} \varepsilon^2 \cos (l' - l) + \frac{3}{3} \sqrt{3} \varepsilon \sin (l' - l)
$$

$$
+ \left[ \frac{3}{3} \sqrt{3} \varepsilon^2 - \frac{3}{3} \varepsilon^2 + \frac{3}{3} \varepsilon \cos (l' - l) - \frac{3}{3} \varepsilon \sin (l' - l) \right] y
$$

$$
+ \left[ \frac{3}{3} \sqrt{3} \varepsilon^2 + \frac{3}{3} \varepsilon^2 - \frac{3}{3} \varepsilon \cos (l' - l) + \frac{3}{3} \varepsilon \sin (l' - l) \right] y^2
$$

(18)

The factors of the powers of $y$ are complete to the third order of the eccentricities inclusive, as the arguments in this class of terms are of the form $3l, 2l \pm l', l \pm l', 3l'$.

The development (18) shows some peculiarities, that may be used as a partial check on this expansion. Firstly, the coefficients of $\varepsilon^2$ and $\varepsilon^3$ must be equal on account of the perfect symmetry in these variables.

Secondly, if $e = e'$ and $l = l'$ the formula for $\Delta^a$ is:

$$
\Delta^a = r^a [2 - 2 \cos \left( \frac{1}{2} \pi + y \right)]
$$

and

$$
\frac{r'^2}{a'^3} \cos (\psi - \psi') = \frac{1}{2} \left( 1 + \gamma \right) - \frac{3}{2} V_3 (1 + \beta) \gamma - \frac{1}{2} \left( 1 + \gamma \right) \gamma y^2 + \ldots .
$$

(19)

and retaining only the constant parts and terms with arguments $l' - l$:

$$
\beta \left( \frac{a'^3}{r'^3} \right) = -\frac{1}{2} \varepsilon^2 - 2 \varepsilon^2 + \varepsilon e' \cos (l' - l) - \frac{1}{2} \sqrt{3} e \sin (l' - l)
$$

$$
\gamma \left( \frac{a'^3}{r'^3} \right) = -\frac{1}{2} \varepsilon^2 - 2 \varepsilon^2 + \varepsilon e' \cos (l' - l) + \frac{1}{2} \sqrt{3} e \sin (l' - l).
$$

Hence we derive:

$$
- a' \frac{r \cos (\psi - \psi')}{r^a} = -\frac{1}{2} \frac{1}{2} \varepsilon^2 + \frac{1}{2} \varepsilon^2 - \frac{1}{2} \varepsilon e' \cos (l' - l) - \frac{1}{2} \sqrt{3} e \sin (l' - l)
$$

$$
+ \left[ \frac{3}{2} V_3 - \frac{3}{2} \varepsilon^2 - \frac{3}{2} \sqrt{3} e \cos (l' - l) - \frac{3}{2} e \sin (l' - l) \right] y
$$

$$
+ \left[ \frac{3}{2} - \frac{1}{2} \varepsilon^2 + \frac{1}{2} \varepsilon^2 + \frac{1}{2} \varepsilon e' \cos (l' - l) + \frac{1}{2} \sqrt{3} e \sin (l' - l) \right] y^2.
$$

(20)

Adding both parts of the perturbative function we get:

$$
a' \left[ \frac{1}{\Delta} - \frac{r \cos (\psi - \psi')}{r^a} \right] = \frac{1}{2} + \frac{1}{2} \varepsilon^2 + \frac{1}{2} \varepsilon^2 - \frac{1}{2} \varepsilon e' \cos (l' - l)
$$

$$
+ \left[ - \frac{3}{2} \varepsilon V_3 \varepsilon^2 - \frac{3}{2} \varepsilon \cos (l' - l) - \frac{3}{2} \varepsilon \sin (l' - l) \right] y
$$

$$
+ \left[ \frac{3}{2} \varepsilon - \frac{3}{2} \varepsilon^2 + \frac{1}{2} \varepsilon^2 - \frac{3}{2} \varepsilon e' \cos (l' - l) + \frac{1}{2} \sqrt{3} e \sin (l' - l) \right] y^2.
$$

(21)

10. Introducing the variables $x, y$ and $u, w$ by the transformation:

$$
l + g + h - l' - g' - h' = \frac{1}{3} \pi + y
$$

$$
g + h - g' - h' = -w
$$

$$
V(a/a') = 1 + x
$$

$$
(1 - \sqrt{1 - e^2}) V(a/a') = u
$$

the differential equations of motion are:

$$
\frac{dx}{dt} = \frac{\partial H}{\partial y} = \frac{\partial H}{\partial y} = -\frac{\partial H}{\partial x} - u'
$$

$$
\frac{dy}{dt} = \frac{\partial H}{\partial w} = \frac{\partial H}{\partial w} = -\frac{\partial H}{\partial u}
$$

$$
H = \frac{\sqrt{M}}{2 (1 + x)^{3/2}} + \frac{m'}{M'} \left[ \frac{\alpha'}{\Delta} - \frac{r a'}{r^a} \cos (\psi - \psi') \right]
$$

$u'$ is the mean motion of Jupiter; the constant of gravitation is taken $= 1.$
The function $H$ is supposed to be restricted to its secular and critical parts. Denoting the terms of degree $p$ in $\epsilon$ by $H_p$, the function $H$ can be expanded in an infinite series:

$$H = \sum_{p=0}^{\infty} H_p.$$  \hspace{1cm} (28)

11. The terms of the variables $x, y, u, w$ of degree zero in $\epsilon'$ are determined by the equations:

$$\begin{align*}
\frac{dx}{dt} &= \frac{\partial H_0}{\partial x}, \\
\frac{dy}{dt} &= \frac{\partial H_0}{\partial y} - \frac{\partial H_0}{\partial x}s' \\
\frac{du}{dt} &= 0, \\
\frac{dw}{dt} &= \frac{\partial H_0}{\partial w}.
\end{align*}$$  \hspace{1cm} (29)

$H_0$ is independent of $w$; thus the first two equations form a system of two variables $x$ and $y$ with one parameter, the constant value of $u$. A solution of this system is:

$$x = \text{const.}, \quad y = \text{const.}$$  \hspace{1cm} (30)

These constant values are the solutions in terms of $u$ of the two equations:

$$\begin{align*}
\frac{\partial H_0}{\partial y} &= 0, \\
\frac{\partial H_0}{\partial x} + s' &= 0.
\end{align*}$$  \hspace{1cm} (31)

The second equation written out is:

$$\frac{1}{(1 + x)^{y}} - 1 = \frac{m'}{M} P_s,$$  \hspace{1cm} (32)

where $P_s$ is a power series in $m'/M$, $x$, $y$ and $u$. This equation enables us to express $x$ as a power series in $y$, $u$ and $m'/M$. This series has $m'/M$ as a factor. From (25) we derive the connection between $\epsilon$ and $u$:

$$\frac{1}{e^u} \left[ 1 + \text{terms in } e^u \right] = \frac{u}{1 + x}.$$  \hspace{1cm} (33)

Using this equation and the expansion (25), and eliminating $x$ by (32), we derive the constant value of $y$ from (31):

$$y = \frac{1}{2} \sqrt{3} u \left[ 1 + P_s \right],$$  \hspace{1cm} (34)

where $P_s$ is a power-series in $m'/M$ and $u$ which has no term of degree zero.

12. The oscillations $\delta x, \delta y$ with respect to the constant values (30) must be derived as far as the first degree in $\delta x$ and $\delta y$ from the linear equations:

$$\begin{align*}
\frac{d\delta x}{dt} &= \frac{\partial^2 H_0}{\partial x^2} \delta x + \frac{\partial H_2}{\partial y} \delta y, \\
\frac{d\delta y}{dt} &= \frac{\partial^2 H_0}{\partial x^2} \delta x - \frac{\partial H_2}{\partial x} \delta y.
\end{align*}$$  \hspace{1cm} (35)

Substituting *):

$$\delta y = \exp (\alpha t) \quad \delta x = \gamma \exp (\alpha t),$$

$\alpha$ and $\gamma$ result from the equations:

$$\begin{align*}
\alpha \gamma &= \frac{\partial^2 H_0}{\partial x^2} \gamma + \frac{\partial H_0}{\partial y}, \\
\alpha &= -\frac{\partial^2 H_0}{\partial x^2} \gamma - \frac{\partial H_0}{\partial x} \delta y.
\end{align*}$$  \hspace{1cm} (36)

These equations allow the determination of $\alpha$ and $\gamma$:

$$\begin{align*}
\left( \frac{\partial^2 H_0}{\partial x^2} \right)^* &= \frac{\partial^2 H_0}{\partial x^2} \frac{\partial^2 H_0}{\partial y}, \\
\gamma &= \frac{\partial^2 H_0}{\partial x^2} + \alpha.
\end{align*}$$  \hspace{1cm} (37)

The first equation expanded is:

$$-\alpha^2 = \frac{\partial^2 H_0}{\partial x^2} \frac{m'}{M} [1 + P_s],$$  \hspace{1cm} (38)

where $P_s$ is a power-series in $m'/M$ and $u$ having no term of degree zero. Two values of $\alpha$, numerically equal and of opposite sign, are possible. Each value of $\alpha$ furnishes a solution $\delta x, \delta y$. The complete solution is equal to the sum of the particular solutions, each multiplied by an arbitrary constant. Choosing these constants in an expedient way and supposing $\alpha/\sqrt{-1}$ positive, we put:

$$\delta y = \frac{x}{2\sqrt{-1}} \left[ \exp \left( x (t-t_0) \right) - \exp \left( - x (t-t_0) \right) \right],$$

$x$ and $t_0$ being the constants of integration.

Denoting $x/\sqrt{-1}$ by the symbol $v$ the value of $\delta y$ is:

$$\delta y = x \sin v (t-t_0).$$  \hspace{1cm} (40)

The corresponding value of $\delta x$ is:

$$\delta x = x \sqrt{\frac{m'}{M}} P_4 \cos v (t-t_0) + \frac{m'}{M} P_5 \sin v (t-t_0),$$  \hspace{1cm} (41)

where $P_4$ and $P_5$ are power-series in $m'/M$ and $u$.

13. The terms in the variables $x$ and $y$ of the second degree in $x$ can be derived from a system of linear equations **). Denoting these terms by the symbols $\delta_x$ and those of the preceding section by $\delta_y$, we have:

$$\text{*) The exponential function of a variable } x \text{ will be denoted by } \exp x, \text{ as the symbol } e \text{ is reserved for the eccentricity.}
\text{**) The development of the solution of (29) in powers of } x \text{ is not necessary, and unsuitable for numerical purposes.}$$
\[
\begin{align*}
\frac{d \delta_x x}{dt} &= \frac{\partial H_0}{\partial x} \delta_x x + \frac{\partial H_0}{\partial y} \delta_x y + \frac{1}{4} \frac{\partial^3 H_0}{\partial x^3} \delta_x x^3 + \frac{\partial^3 H_0}{\partial y^3} \delta_x y^3,
\frac{d \delta_y y}{dt} &= - \frac{\partial H_0}{\partial x^2} \delta_x x - \frac{\partial H_0}{\partial y^2} \delta_x y - \frac{1}{4} \frac{\partial^3 H_0}{\partial x^3} \delta_x x^3 - \frac{\partial^3 H_0}{\partial y^3} \delta_x y^3,
\end{align*}
\]

where \( P_\tau \) is a power series in \( m'/M, x, y \) and \( u \), without a term of degree zero, we derive:

\[
\text{constant part of } \delta_x y = \frac{1}{4} \sqrt{3} x^2 [1 + P_u];
\]

\( P_u \) is a power series in \( m'/M \) and \( u \) without a term of degree zero.

14. The integration of the equation:

\[
\frac{dw}{dt} = - \frac{\partial H_0}{\partial u}
\]

completes the solution of the system (29).

Substitution of the values of \( x \) and \( y \) in the right hand member of (47) makes \( \frac{dw}{dt} \) equal to a periodic series with argument \( \nu (t-t_0) \) and of order \( m'/M \). The constant term of this series is equal to:

\[
P_\nu = \frac{m'}{M} \nu^2 [1 + P_\nu];
\]

In the succeeding sections we shall consider the dependence of the variables \( x, y, u, w \) on the constants of integration in two ways. Firstly we consider \( x, y, u, w \) as functions of \( \nu, \alpha, t, \epsilon \) and the additive constant in \( w \). Secondly we consider \( x, y, u, w \) as functions of \( \nu, \alpha, \tau \) and \( w \).

16. The terms proportional to the first power of \( \nu \) in the variables \( x, y, u, w \) can be derived from a system of linear differential equations with periodic coefficients. Denoting these terms by the symbol \( \delta \) the equations are:

\[
\begin{align*}
\frac{d \delta x}{dt} &= \frac{\partial H_0}{\partial x} \delta x + \frac{\partial H_0}{\partial y} \delta y + \frac{\partial H_0}{\partial x^2} \delta x^2 + \frac{\partial H_0}{\partial y^2} \delta y^2,
\frac{d \delta y}{dt} &= - \frac{\partial H_0}{\partial x^2} \delta x - \frac{\partial H_0}{\partial y^2} \delta y - \frac{\partial H_0}{\partial x^3} \delta x^3 - \frac{\partial H_0}{\partial y^3} \delta y^3,
\frac{d \delta u}{dt} &= \frac{\partial H_0}{\partial x^2} \delta x - \frac{\partial H_0}{\partial y^2} \delta y - \frac{\partial H_0}{\partial x^3} \delta x^3 - \frac{\partial H_0}{\partial y^3} \delta y^3.
\end{align*}
\]
17. The solution of these equations can be derived by the method of variation of arbitrary constants
denoted by Greek letters.
The equations enter into the general form:

\[
\frac{dp}{dt} = Ap + Bq + P
\]

\[
\frac{dq}{dt} = Cq - Aq + Q; \tag{51}
\]

\(A, B, C, P, Q\) are known functions of \(t\). For the third equation (50) furnishes \(\delta u\); then the equations for \(\delta x\) and \(\delta y\) form a system of the type (51); if these equations have been solved, \(\delta w\) results by a quadrature from the last equation (50).

Suppose the system (51) to have been solved if \(P = Q = 0\); let

\[
\begin{align*}
\delta p &= \delta p_s, & \delta q &= \delta q_s, \\
\delta p &= \delta p_s, & \delta q &= \delta q_s
\end{align*} \tag{52}
\]

be two particular solutions in this case. The general solution of (51) can then be expressed by the formula:

\[
\int (Pq_s - Qp_s) \frac{dt}{D} = \frac{1}{D} \int \left[ \left( \frac{\partial^2 H_s}{\partial y \partial u} \delta u + \frac{\partial^2 H_s}{\partial y} \right) \delta y + \left( \frac{\partial^2 H_s}{\partial x \partial u} \delta u + \frac{\partial^2 H_s}{\partial x} \right) \delta x \right] dt
\]

\[
= \frac{1}{D} \int \left[ \frac{\partial H_s}{\partial x} - \frac{\partial H_s}{\partial \omega} \right] dt + \frac{1}{D} \int \delta u \left( \frac{\partial H_s}{\partial u} \right) dt. \tag{55}
\]

The relation:

\[
\int \delta u \left( \frac{\partial H_s}{\partial u} \right) dt = - \int \delta u \left( \frac{\partial^2 H_s}{\partial x \partial u} \right) dt = - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} + \int \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} dt = - \frac{\partial u}{\partial x} + \int \frac{\partial w}{\partial x} \frac{\partial H_s}{\partial u} dt
\]

reduces the preceding formula to:

\[
\int (Pq_s - Qp_s) \frac{dt}{D} = \frac{1}{D} \int \frac{\partial H_s}{\partial x} dt - \frac{\partial u}{\partial x} \frac{\partial w}{\partial x}. \tag{57}
\]

In these formulae \(H_s\) is considered as a function of \(u, t, \varepsilon, \tau, \omega\) and the additive constant in \(\omega\). An analogous formula can be derived for the integral:

\[
\int (-Pq_s + Qp_s) \frac{dt}{D} = - \frac{1}{D} \int \frac{\partial H_s}{\partial x} dt + \frac{\partial u}{\partial x} \frac{\partial w}{\partial x}. \tag{58}
\]

Collecting results we get:

\[
\begin{align*}
\delta x &= \frac{\partial x}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt - \frac{\partial x}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt + \frac{1}{D} \left( \frac{\partial x}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} - \frac{\partial x}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} \right) \delta u, \\
\delta y &= \frac{\partial y}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt - \frac{\partial y}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt + \frac{1}{D} \left( \frac{\partial y}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} - \frac{\partial y}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} \right) \delta u. \tag{59}
\end{align*}
\]

18. The results of the preceding section are readily applicable to the equations (50). If we consider \(x, y\), as functions of \(u, t, \varepsilon\) in the way explained in section 15, we may put:

\[
\begin{align*}
\delta p &= \frac{\partial x}{\partial \varepsilon} \delta x, & \delta q &= \frac{\partial y}{\partial \varepsilon} \delta y, \\
\delta p &= \frac{\partial x}{\partial \varepsilon} \delta x, & \delta q &= \frac{\partial y}{\partial \varepsilon} \delta y
\end{align*} \tag{54}
\]

\[
P = \frac{\partial^2 H_s}{\partial u \partial x} \delta u + \frac{\partial H_s}{\partial u} \delta x, \quad Q = -\frac{\partial^2 H_s}{\partial u \partial x} \delta u - \frac{\partial H_s}{\partial u} \delta x.
\]

Remembering the fact that \(H_s\) is a function of \(x, u, \tau, \omega, \varepsilon\) of \(x, u, \varepsilon\), and the additive constant in \(\omega\), we derive:

\[
\int (Pq_s - Qp_s) \frac{dt}{D} = \frac{1}{D} \int \frac{\partial H_s}{\partial x} dt - \frac{\partial u}{\partial x} \frac{\partial w}{\partial x}
\]

In these formulae \(H_s\) is considered as a function of \(x, u, t, \varepsilon, \tau, \omega\) and the additive constant in \(\omega\). An analogous formula can be derived for the integral:

\[
\int (-Pq_s + Qp_s) \frac{dt}{D} = - \frac{1}{D} \int \frac{\partial H_s}{\partial x} dt + \frac{\partial u}{\partial x} \frac{\partial w}{\partial x}
\]

Collecting results we get:

\[
\begin{align*}
\delta x &= \frac{\partial x}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt - \frac{\partial x}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt + \frac{1}{D} \left( \frac{\partial x}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} - \frac{\partial x}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} \right) \delta u, \\
\delta y &= \frac{\partial y}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt - \frac{\partial y}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt + \frac{1}{D} \left( \frac{\partial y}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} - \frac{\partial y}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} \right) \delta u. \tag{59}
\end{align*}
\]

19. A last transformation reduces equations (59) to the required form. We shall no longer consider \(H_s\) as a function of \(x, u, t, \varepsilon, \tau, \omega\). We distinguish the derivatives in the two cases by brackets. We thus get the relations:

\[
\delta x = \left[ \frac{\partial x}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt - \frac{\partial x}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt + \frac{1}{D} \left( \frac{\partial x}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} - \frac{\partial x}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} \right) \delta u, \right.
\]

\[
\delta y = \left[ \frac{\partial y}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt - \frac{\partial y}{\partial \varepsilon} \int \frac{\partial H_s}{\partial \varepsilon} dt + \frac{1}{D} \left( \frac{\partial y}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} - \frac{\partial y}{\partial \varepsilon} \frac{\partial w}{\partial \varepsilon} \right) \delta u \right].
\]

*) Cf. "Investigations in the theory of Hyperion" Ch. III.
Substitution of (60) in the second equation (59) reduces \( \delta y \) to:

\[
\delta y = \frac{\partial y}{\partial \tau} \frac{\partial H_1}{\partial \tau} \frac{dt}{D} - \frac{\partial y}{\partial \tau} \frac{\partial H_1}{\partial \tau} \frac{dt}{D} + \frac{\partial y}{\partial \tau} \frac{\partial H_1}{\partial \tau} \frac{dt}{D} - \frac{\partial y}{\partial \tau} \frac{\partial H_1}{\partial \tau} \frac{dt}{D},
\]

The terms in \( t \) can be eliminated by the use of the relations:

\[
\int t \frac{\partial H_1}{\partial \tau} dt = \int \frac{\partial H_1}{\partial \tau} dt = \int \frac{\partial H_1}{\partial \tau} dt = \int \frac{\partial H_1}{\partial \tau} dt = \int \frac{\partial H_1}{\partial \tau} dt = \int \frac{\partial H_1}{\partial \tau} dt,
\]

The resulting equation for \( \delta y \) is:

\[
\delta y = \frac{\partial y}{\partial \tau} \frac{\partial H_1}{\partial \tau} \frac{dt}{D} - \frac{\partial y}{\partial \tau} \frac{\partial H_1}{\partial \tau} \frac{dt}{D} + \frac{\partial y}{\partial \tau} \frac{\partial H_1}{\partial \tau} \frac{dt}{D} - \frac{\partial y}{\partial \tau} \frac{\partial H_1}{\partial \tau} \frac{dt}{D},
\]

The integral \( \int \frac{\partial H_1}{\partial \tau} dt \) occurring twice must have exactly the same values in both cases.

The denominator \( D \) is equal to:

\[
D = \frac{\partial x}{\partial \tau} \frac{\partial y}{\partial \tau} - \frac{\partial x}{\partial \tau} \frac{\partial y}{\partial \tau}.
\]

As \( D \) is independent from \( \tau \) we may compute its value by substituting a special value for \( \tau \).

\( D \) is of the order \( n \sqrt{\mathcal{V}(m'|M)} \); \( H_1 \) is equal to a sum of terms of type:

\[
\text{coefficient} \times \cos \left( \omega \pm \tau \right), \quad s \in \text{integer number};
\]

the coefficient is of order \( m'e|M \). Integration of \( H_1 \)

\[
H_1 = \frac{m'}{M} \sqrt{M} \left[ e' e \cos (l' - l) | - x_t^2 + \frac{3}{2} V_3 \gamma + \sin (l' - l) \right] \left( t + P_{13} \right) + \frac{1}{2} V_3 \gamma + \sin (l' - l) \left( t + P_{13} \right);
\]

\( P_{13} \) and \( P_{12} \) are power-series in \( e^2 \), having no terms of degree zero. The connection between \( l' - l \) and the variables \( y \) and \( \omega \) results from (26):

\[
l' - l = - w - y - \frac{1}{3} \pi.
\]
Hence, $P_{14}$ and $P_{15}$ being power series in $e^t$ having no terms of degree zero:

\[
\begin{align*}
\cos(t' - t) &= \frac{1}{2} \left\{ \cos 3\nu - \frac{1}{4} \nu \dot{\nu} + \ldots \right\} \cos \nu + \frac{1}{2} \nu \dot{\nu} V^3 y + \frac{1}{4} \nu^2 V^3 y^2 + \ldots \sin \nu \\
\sin(t' - t) &= \frac{1}{2} \left\{ \cos 3\nu - \frac{1}{4} \nu \dot{\nu} V^3 y + \frac{1}{4} \nu^2 V^3 y^2 + \ldots \right\} \cos \nu + \frac{1}{2} \nu \dot{\nu} V^3 y^3 + \frac{1}{4} \nu^2 V^3 y^4 + \ldots \sin \nu
\end{align*}
\]

(69)

\[
H_i = \frac{m'}{M} \frac{\sqrt{M}}{a \frac{1}{2}} \left[ e e' \cos \nu \right] - \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots \left[ \frac{1}{2} \nu \dot{\nu} V^3 y^3 + \ldots \right] + e e' \sin \nu \left[ \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots \right] + \ldots
\]

The terms with argument $\omega$ in the periodic development of $H_i$ as a function of $t$ and $\omega$ result from this formula to the lowest order of $\sqrt{m'/(m|M)}$ and $u$, if we substitute for $y$ its constant part and $\frac{1}{2} x^2$ for $y^2$.

Hence

(70)

\[
H_i = \ldots + \frac{m'}{M} n' e e' \left[ \cos \nu + \frac{1}{2} \nu \dot{\nu} V^3 y + \ldots \right] \sin \nu + \ldots,
\]

as far as regards the terms in $\omega$, only including the lowest orders in $\sqrt{m'/(m|M)}$ and $u$; the factor $e$ is equal to $\sqrt{2} u$ with sufficient approximation.

21. The computation of the critical terms (66) can now be carried on. We shall state the results including only the lowest orders of the various small quantities $u, \nu, \sqrt{m'/(m|M)}$:

\[
\frac{\partial H_i}{\partial \nu} = \frac{m'}{M} n' e e' \left[ \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots \right] \sin \nu + \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots
\]

(71)

\[
\frac{\partial H_i}{\partial u} = \frac{m'}{M} n' e e' \left[ \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots \right] \cos \nu + \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots
\]

\[
\frac{\partial H_i}{\partial t} = \frac{m'}{M} n' e e' \left[ \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots \right] \cos \nu + \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots
\]

\[
\frac{\partial}{\partial t} = -e e' \left[ \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots \right] \cos \nu + \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots
\]

\[
D = \frac{\partial}{\partial t} \left( \frac{\partial H_i}{\partial \nu} \right) = \frac{1}{2} \nu \dot{\nu} V^3 x \sqrt{m'/(m|M)}
\]

Substituting these relations in (66) and including only the lowest orders we get:

(72) critical terms $\sin \frac{\partial y}{\partial t} \frac{\partial y}{\partial t} \left[ \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots \right] \cos \nu \left[ \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots \right] \sin \nu \left[ \frac{1}{2} \nu \dot{\nu} V^3 \nu y + \ldots \right] \sin \nu
\]

and reducing the products $\cos \tau \sin \omega$ and $\cos \tau \cos \omega$ to $\cos(\tau \pm \omega)$.

To form a judgment about the magnitude of the terms in question it is advantageous to combine the terms in $\sin \omega$ and $\cos \omega$ in (72):

(73)

\[
\frac{\partial y}{\partial t} = \nu \cos \tau
\]

and the term $\cos \tau \sin \omega$ and $\cos \tau \cos \omega$ to $\sin(\omega \pm \omega)$.

22. The formula (72) exhibits the possibility of interpreting the critical terms as a perturbation of the libration-argument.

We may pass on to the goniometrical form by substituting:

(74)

\[
\nu \dot{\nu} V^3 \nu y - \nu \dot{\nu} \cos \nu = 2.2 \sin(\nu - 68^\circ).
\]

The coefficient of the perturbation in $\tau$ becomes:

(75)

\[
2.2 \frac{ee'}{\sqrt{m/(m|M)}}
\]

Substituting numerical values we get:

(76)

\[
3.4 e.
\]

As $e$ may be a quantity of the order 0.1, the coefficient of the perturbation of $\tau$ may reach the value:

(77)

\[
0.34 = 19^\circ.
\]

The period of this perturbation is:

(78)

\[
2 \pi \frac{8}{27} \sqrt{\frac{M}{m'}} = 37 \text{ centuries}.
\]

If we do not adhere to the goniometrical form for all perturbations, we may develop $\sin \omega$ and $\cos \omega$ in powers of $t$. In that case we simply get an addition to the mean motion of the argument $\tau$, proportional to $ee' \sqrt{m/(m|M)}$.

23. Though the computation of the terms in $\partial x, \partial y, \partial u$ and $\partial w$ with argument $\omega$ does not belong to our subject, we shall briefly derive their values as these terms are of special importance in another respect.

It is easily seen that the term in $\omega$ in $\partial y$ is at least of the order $ee'$. If we neglect $x, \partial x$ is of order $m/(m|M)$. The term in $\partial w$ of lowest order results from the integration of:

(79)

\[
\frac{\partial w}{\partial t} = \frac{\partial H_i}{\partial u} = \frac{m'}{M} n' e e' \left[ \frac{1}{2} \nu \dot{\nu} \cos \omega \right] + \frac{1}{2} \nu \dot{\nu} \nu y \sin \omega.
\]

Hence, including lowest orders only:

(80)

\[
\frac{\partial w}{\partial t} = \frac{e}{e} \left[ \frac{1}{2} \nu \dot{\nu} \nu y \cos \omega \right] = -\frac{e}{e} \sin(\omega + 60^\circ).
\]
The value of $\delta u$ from (71) is:

\begin{equation}
\delta u = \epsilon \dot{e} \cos (\omega + 60^\circ).
\end{equation}

The influence of these terms on the coordinates $r$ and $\psi$ can be computed from the formulae:

\begin{equation}
\frac{\delta r}{a} = - \cos l \dot{e} + \epsilon \sin l \dot{w},
\end{equation}
\begin{equation}
\frac{\delta \psi}{a} = 2 \sin l \dot{e} + 2 \epsilon \cos l \dot{w}.
\end{equation}

As:

\begin{equation}
\delta u = \epsilon \dot{e},
\end{equation}

we get:

\begin{equation}
\frac{\delta r}{a} = - \epsilon \dot{e} \cos (l - \omega - 60^\circ),
\end{equation}
\begin{equation}
\frac{\delta \psi}{a} = 2 \epsilon \sin (l - \omega - 60^\circ).
\end{equation}

If $\epsilon = 0$:

\begin{equation}
l - \ell = 60^\circ + \omega;
\end{equation}

hence:

\begin{equation}
\frac{\delta r}{a} = - \epsilon \dot{e} \cos \ell, \quad \frac{\delta \psi}{a} = 2 \epsilon \sin \ell,
\end{equation}

which is the elliptical triangular solution.