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Chapter 4

Examples of Arakelov equality for semistable families of curves uniformized by the unit ball

4.1 Second fundamental form

Let $X$ be a complex manifold and let $E$ be a complex vector bundle on $X$ with a Hermitian metric. We recall that $k$-valued forms with values in $E$ on $X$ are sections of the bundle

$$\mathcal{A}^k(E) = \bigwedge^k T^*_C, X \otimes E,$$

where $T^*_C, X = T^*_X \otimes \mathbb{C}$ is the complexified cotangent bundle.

Using the decomposition $T^*_C, X = T^*_{X, 0} \oplus T^*_{X, 1}$, where $T^*_{X, 1}$ is isomorphic to the holomorphic cotangent bundle of $X$, we get the decomposition

$$\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E).$$

Hence, we can decompose any connection $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ on $E$ in two components:

$$\nabla^{1,0} : \mathcal{A}^0(E) \to \mathcal{A}^{1,0}(E) \text{ and } \nabla^{0,1} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$$

such that $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$.

**Proposition 4.1.** ([35], p.177) Let $E$ be a holomorphic vector bundle with a Hermitian metric on a complex manifold $X$. Then there is a unique connection $\nabla$ such that $\nabla^{0,1} = \overline{\partial}_F$. This connection is called the Chern connection on $E$.

Now, we suppose that we have a short exact sequence of holomorphic vector bundles on a complex manifold $X$:

$$0 \to E_1 \to E \to E_2 \to 0.$$
In general, such a sequence does not split holomorphically. However, the sequence of underlying smooth complex bundles always splits. Indeed, let $h$ be a Hermitian metric on the holomorphic vector bundle $E$. Then, $E_2$ is $C^\infty$-isomorphic to the orthogonal complement of $E_1$ with respect to the Hermitian metric $h$, hence one has a $C^\infty$-splitting:

$$E = E_1 \oplus E_2.$$ 

Let $\nabla$ be the Chern connection of the metric $h$ on $E$, i.e. the connection such that $\nabla^{0,1} = \overline{\partial}E$. On subbundles $E_1$ and $E_2$ one has the induced connections $\nabla_1$ and $\nabla_2$ defined by:

$$\nabla_i(s) := p_i(\nabla(s)),$$

where:

- $s$ is any section of $E_i$, and hence of $E$;
- $p_i$ is the projection $E_1 \oplus E_2 \to E_i$;

for $i \in \{1, 2\}$.

The connection $\nabla_1$ on $E_1$ satisfies $\nabla_1^{0,1} = \overline{\partial}E_1$, since $E_1$ is a holomorphic subbundle of $E$. We have the commutative diagram:

$$\begin{array}{ccc}
\mathcal{A}^0(E_1) & \xrightarrow{\nabla_1} & \mathcal{A}^1(E) \\
\downarrow{\nabla_1} & & \downarrow{pr} \\
\mathcal{A}^1(E_1) & & \mathcal{A}^1(E_1)
\end{array}$$

Now, we define the operator:

$$\beta = \nabla_{|\mathcal{A}^0(E_1)} - \nabla_1 : \mathcal{A}^0(E_1) \to \mathcal{A}^1(E_2).$$

**Definition 4.1.** The operator $\beta$ is called the second fundamental form of the subbundle $E_1 \subset E$. It is of type $(1, 0)$ and linear over $C^\infty$ functions, i.e.

$$\beta \in \mathcal{A}^{1,0}(\text{Hom}(E_1, E_2)).$$

By Theorem 14.3 from [17] the connection matrix of $\nabla$ is

$$\nabla = \begin{pmatrix}
\nabla_1 & -\beta^* \\
\beta & \nabla_2
\end{pmatrix},$$

where $\beta^* \in \mathcal{A}^{0,1}(\text{Hom}(E_2, E_1))$ is the adjoint of $\beta$ and $\overline{\partial}\beta^* = 0$. Hence $\beta^*$ defines a class

$$[\beta^*] \in H^1(X, \text{Hom}(E_2, E_1)).$$
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**Proposition 4.2.** ([17], Theorem 14.9) The correspondence $E \to [\beta^*]$ induces
a bijection from the set of isomorphism classes of extensions of $E_1$ by $E_2$ onto
the cohomology group $H^1(X, \text{Hom}(E_2, E_1))$. In particular $[\beta^*]$ vanishes if and
only if the exact sequence $0 \to E_1 \to E \to E_2 \to 0$ splits holomorphically.

**Definition 4.2.** If $i : X_0 \to X$ is an immersion of a complex manifold $X_0$
to a complex manifold $X$ with a Hermitian metric, then we say that $X_0$ is
totally geodesic if the second fundamental form $\beta_{X_0}$ of the tangent bundle $T_{X_0}$
vanishes identically on $X_0$.

If $X_0$ is a complex submanifold of a compact complex manifold $X$ with a
Hermitian metric, then the definition of the normal bundle $N_{X_0}$ is given by
the exact sequence:

$$0 \to T_{X_0} \to T_{X|X_0} \to N_{X_0} \to 0.$$ 

The second fundamental form of the tangent bundle $T_{X_0} \subset T_{X|X_0}$ will be
$\beta_{X_0} \in \mathcal{A}^{1,0}(X_0, \text{Hom}(T_{X_0}, N_{X_0}))$.

By Proposition 4.2 we get that for a totally geodesic submanifold $X_0$, the
previous exact sequence splits holomorphically, hence we have a holomorphic
isomorphism:

$$T_{X|X_0} \simeq T_{X_0} \oplus N_{X_0}.$$ 

### 4.2 Second projective fundamental form

In this section we will give a short review of Mok’s paper [53] and it can be
considered as the technical background for what follows. Here, we will explain
notions such as holomorphic projective connection, second projective funda-
mental form and tautological foliation on a projectivized tangent bundle. We
will see how on a space endowed with a holomorphic projective connection we
obtain the projective second fundamental form, using a holomorphic foliation
on the projectivized tangent bundle.

**Definition 4.3.** ([53] §2.1) Let $X$ be a $n$-dimensional complex manifold for
$n > 1$. A holomorphic projective connection $\Pi$ on $X$ consists of:

- an open covering $\{U_\alpha\}$, with holomorphic coordinates $\{z_1^\alpha, ..., z_n^\alpha\}$;
- holomorphic functions $\left(\alpha \Phi^k_{ij}\right)_{1 \leq i,j,k \leq n}$ on $U_\alpha$, symmetric in $i, j$
satisfying the trace condition $\sum_j \alpha \Phi^k_{ij} = 0$ for all $i$ and satisfying on $U_\alpha \cap U_\beta$
the transformation rule:

$$\beta \Phi^l_{pq} = \sum_{i,j,k} \alpha \Phi^k_{ij} \frac{\partial z_1^\alpha}{\partial z_p^\beta} \frac{\partial z_2^\alpha}{\partial z_q^\beta} \frac{\partial z_3^\alpha}{\partial z_k^\beta} + S(f_{\alpha \beta}),$$

where $S(f_{\alpha \beta})$ stands for the Schwarzian derivative of the holomorphic
transformation given by the change of variables $z^\alpha = f_{\alpha \beta} z^\beta$. 

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**Definition 4.4.** ([53] §2.2) Let \((\alpha^k_{ij})\) be Riemann-Christoffel symbols of any smooth connection on a complex manifold \(X\). Let \(\Pi\) be a projective connection on \(X\) defined as above. We can define a torsion free smooth connection \(\nabla\) associated to \(\Pi\), to be the connection with Riemann-Christoffel symbols:

\[
\alpha^k_{ij} = \Phi^k_{ij} + \frac{1}{n+1} \sum_l \delta^k_l \alpha^l_{ij} + \frac{1}{n+1} \sum_l \delta^k_l \alpha^l_{ji}.
\]

**Definition 4.5.** ([53] §2.2) Any two smooth connections \(\nabla\) and \(\nabla'\) on a complex manifold \(X\) are projectively equivalent if there exists a smooth \((1,0)\)-form \(\omega\) such that:

\[
\nabla_u v - \nabla'_u v = \omega(u)v + \omega(v)u,
\]

for any smooth \((1,0)\)-vector fields \(u\) and \(v\) on an open set of \(X\).

We assume now that a complex manifold \(X\) has a projective connection \(\Pi\). Let \(\nabla\) and \(\nabla'\) be two smooth torsion-free connections associated to the projective connection on the complex manifold \(X\). Then, \(\nabla\) and \(\nabla'\) are projectively equivalent. Moreover, for any submanifold \(X_0\) of \(X\) the second fundamental forms of the tangent bundle of \(X_0\), associated to \(\nabla\) and \(\nabla'\) are the same. As it can be found in [53], the second fundamental form with respect to a torsion-free smooth connection \(\nabla\) associated to the projective connection \(\Pi\) is independent of the choice of the background connection \((\alpha^k_{ij})\) and it is holomorphic.

**Remark 4.6.** When we refer to the second fundamental form of a submanifold \(X_0 \subset X\) we mean the second fundamental form of the holomorphic tangent bundle \(T_{X_0} \subset T_{X|X_0}\).

**Definition 4.7.** The second fundamental form \(\beta_{\Pi}\) of any submanifold \(X_0 \subset X\), with respect to a torsion-free smooth connection \(\nabla\) associated to a projective connection \(\Pi\) on \(X\) is called the projective second fundamental form of \(X_0\) with respect to \(\Pi\).

Now, it is plain to see that classes of the complex geodesics submanifolds of \(X\), with respect to smooth torsion-free connections associated to the projective connection \(\Pi\) on the complex manifold \(X\), will be the same.

**Definition 4.8.** The tautological lifting \(\hat{C}\) of some smooth holomorphic curve \(C \subset X\) is defined by lifting every point \(x \in C\) to the projectivization of the tangent line \([T_xC]\) in \(\mathbb{P}T_{X,x}\).

Here we will not give the definition of holomorphic foliation but we refer to [26] for details.

**Definition 4.9.** A holomorphic foliation of the projectivization of the tangent bundle whose leaves are tautological liftings of holomorphic curves is called a tautological foliation.
Definition 4.10. Complex geodesics on a complex manifold are totally geodesic submanifolds of dimension 1.

Let $X$ be a complex manifold equipped with a tautological foliation $\mathcal{F}$ of the projectivized holomorphic tangent bundle $\mathbb{P}T_X$. Let $\pi : \mathbb{P}T_X \to X$ be the canonical projection. By definition, the leaves of this foliation are 1-dimensional. Now, let $x \in X$ and let $\alpha \in T_xX$. We use the notation $T_{[\alpha]}\mathcal{F} = \mathcal{F}_{[\alpha]}$. Obviously, $d\pi(\mathcal{F}_{[\alpha]}) = \mathbb{C}\alpha$.

The existence of holomorphic projective connections on a complex manifold $X$ is equivalent to the existence of holomorphic foliations on the projectivized holomorphic tangent bundle, by tautological liftings of complex geodesics on $X$. This relation is described by Proposition 1 from [53]. Here, we will not give the proof of this fact, but we will explain one of its consequences, which is the most important part of this section, the construction of the projective second fundamental form of a submanifold of a complex manifold $X$, where $X$ is endowed with a tautological foliation, or equivalently with a projective holomorphic connection. This construction can be found in [53] §2.3:

Let $X_0$ be a complex submanifold of $X$ and let $N$ be the normal bundle of $X_0$. Let $x_0 \in X_0$ and let $\alpha \in T_{X_0,x_0}$ be a non-zero tangent vector. Let $C$ be a local holomorphic curve on $X$ passing through $x_0$ and such that $T_{C,x_0} = \mathbb{C}\alpha$, and such that its tautological lifting $\check{C}$ is a local leaf of the tautological foliation $\mathcal{F}$. Let $D$ be a holomorphic curve on $X_0$ passing through $x_0$ and such that $T_{D,x_0} = \mathbb{C}\alpha$. We denote the tautological lifting of $D$ to $\mathbb{P}T_X$ as $\check{D}$. It is clear that $[\alpha] \in \check{C} \cap \check{D}$.

Let $\pi : \mathbb{P}T_X \to X$ be the canonical projection, then we choose the unique vector

$$\eta \in T_{[\alpha]}(\check{C}) \subset T_{[\alpha]}\mathbb{P}T_{X,x_0},$$

such that

$$d\pi(\eta) = \alpha,$$

and the unique vector

$$\xi \in T_{[\alpha]}(\check{D}) \subset T_{[\alpha]}\mathbb{P}T_{X,x_0},$$

such that

$$d\pi(\xi) = \alpha.$$

Hence, we get $d\pi(\xi - \eta) = 0$, i.e.

$$\xi - \eta \in \text{Ker } d\pi_{[\alpha]}.$$

Let $T_\pi = \text{Ker } d\pi$ be the relative tangent bundle of the map $\pi : \mathbb{P}T_X \to X$, see [35] p.95. We define the previous assignment of the vector $\alpha$ to the vector $\xi - \eta$ as the map $A : T_{X,x_0} \to T_{\pi,[\alpha]}$, which assigns a vector $v \in T_{X_0,x}$ to the vector

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$r_v \in T_{\pi, [\alpha]}$ in the same way as $\alpha$ is assigned to $\xi - \eta$. Note that $A(\alpha) = \xi - \eta$ is independent of the choice of $D$. Now, we construct the diagram:

We have the exact sequence:

$$0 \to T_\pi \to T_{\mathcal{P}T_X} \to \pi^*T_X \to 0.$$

On the other side, using the Euler sequence ([35], p.95) we have:

$$0 \to L \to \pi^*T_X \to T_\pi \otimes L \to 0,$$

where $L$ is the relative tautological bundle on $\mathcal{P}T_X$. The previous exact sequence yields the isomorphism:

$$\phi : T_\pi \otimes L \simeq (\pi^*T_X/L).$$

One should note that:

$$(\pi^*T_X)_{[\alpha]} \cong T_{X,x_0},$$

and

$$L_{[\alpha]} \cong T_{D,x_0} \subseteq T_{X_0,x_0}.$$

Hence, we have the canonical projection:

$$\rho : (\pi^*T_X/L)_{[\alpha]} \to (T_X/T_{X_0})_{x_0} \cong (\pi^*N)_{[\alpha]}.$$ 

Now, we can define

$$B := \rho(\phi(A \otimes \text{id}_{L_{[\alpha]}})) : L_{[\alpha]} \otimes L_{[\alpha]} \to (\pi^*N)_{[\alpha]},$$

such that:

$$B(\alpha \otimes \alpha) = \rho\left(\phi\left(A \otimes \text{id}_{L_{[\alpha]}}(\alpha \otimes \alpha)\right)\right) = \rho(\phi((\xi - \eta) \otimes \alpha)).$$
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As we mentioned above $A(\alpha)$ is independent of choice of $D$, hence $B(\alpha \otimes \alpha)$ is also independent of the choice of $D$. Moreover, one can see by local considerations that $B(\alpha \otimes \alpha)$ varies holomorphically with $\alpha$. Hence, one gets an induced holomorphic section

$$\tilde{\sigma} \in H^0(\mathbb{P}T_{X_0}, L^{-2} \otimes \pi^*N),$$

which is defined on fibers by $B$, and also one has the corresponding section

$$\sigma \in H^0(X_0, S^2T_{X_0}^* \otimes N).$$

According to [53] §2.3, this section agrees with the projective second fundamental form of $X_0$ in $X$ with respect to the given holomorphic projective connection on $X$.

4.3 The complex unit ball

In this section we will define the complex hyperbolic $n$-space and the complex unit $n$-ball. We will show that these two Kähler varieties can be identified and then we will give a short review of their basic properties. Then, we will define the tautological foliation on the projectivization of the tangent bundle of the complex unit $n$-ball. We will end this part by recalling Mok’s result which states that the second fundamental form of a submanifold of the quotient of the complex unit $n$-ball by some discrete subgroup of $PU(n, 1)$, with respect to the Bergman metric, coincides with the projective second fundamental form induced by the tautological foliation.

**Definition 4.11.** Let $C^{n,1} = (\mathbb{C}^{n+1}, h)$ be $(n+1)$-dimensional complex space $\mathbb{C}^{n+1}$ with Hermitian form $h$ of signature $(n, 1)$, i.e.

$$h(z, w) = z_0\overline{w_0} + z_1\overline{w_1} + ... + z_{n-1}\overline{w_{n-1}} - z_n\overline{w_n}.$$  

We say that a vector $z \in \mathbb{C}^{n,1}$ is negative if $h(z, z) < 0$.

**Definition 4.12.** ([27]§3.1) The complex hyperbolic space is the subset of $\mathbb{P}^n$ consisting of negative lines of $C^{n,1}$. It is naturally biholomorphic to the complex unit $n$-ball $\mathbb{B}^n$:

$$\mathbb{B}^n = \{z \in \mathbb{C}^n| \langle z, z \rangle < 1\},$$

where $\langle ., . \rangle$ is the standard Hermitian product on $\mathbb{C}^n$.

**Definition 4.13.** The special unitary group $SU(n, 1)$ is the subgroup of the group $SL(n+1, \mathbb{C})$ of matrices which preserve the Hermitian form $h$, i.e.

$$SU(n, 1) = \{A \in SL(n+1, \mathbb{C})| h(Az, Az) = h(z, z)\}.$$  

The projectivization of $SU(n, 1)$ in $PGL(n+1, \mathbb{C})$ is the group $PU(n, 1)$. 

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We should note that $\text{PU}(n, 1)$ is the group of biholomorphisms of $\mathbb{B}^n$. Moreover, the group $\text{PU}(n, 1)$ acts transitively on $\mathbb{B}^n$, see Lemma 3.1.3 from [27], in other words for any $[z], [w] \in \mathbb{B}^n$ there is an element $A \in \text{PU}(n, 1)$, such that $A[z] = [w]$.

From now on we will consider that the complex unit ball (or the complex hyperbolic space) is endowed with the Bergman metric, sometimes called the Poincaré metric, whose sectional holomorphic curvature is constant.

Now, let us give the theorem which explains how we get the totally geodesic submanifolds in $\mathbb{B}^n$, which will help us to define the tautological foliation on $\mathbb{P}T_{\mathbb{B}^n}$.

**Theorem 4.3.** ([27], §3.1.10) Let $F \subset \mathbb{P}^n$ be a complex $m$-dimensional projective subspace which intersects $\mathbb{B}^n$. Then, $F \cap \mathbb{B}^n$ is a totally geodesic holomorphic submanifold with respect to the Bergman metric, biholomorphically isometric to $\mathbb{B}^m$.

Hence, by the previous theorem, the complex geodesics on $\mathbb{B}^n$ are obtained as intersections of $\mathbb{B}^n$ with projective lines. On the other side, projective lines are complex geodesics in $\mathbb{P}^n$, with respect to the Fubini-Study metric. We recall that given a point in $\mathbb{B}^n$ and a complex tangent line at this point, there is a unique complex geodesic trough that point tangent to the complex tangent line. Now we will define the tautological foliation of the tangent bundles of the projective space $\mathbb{P}^n$ and $\mathbb{B}^n$.

**Definition 4.14.** ([53]§2.3) The tautological foliation on the projectivization of the tangent bundle of the projective space $\mathbb{P}^n$ is defined by the tautological lifting of projective lines. The tautological foliation on the projectivization of the tangent bundle of $\mathbb{B}^n$ is defined by tautological liftings of restrictions of projective lines to the complex hyperbolic space $\mathbb{B}^n$.

**Definition 4.15.** A complex hyperbolic space form of dimension $n$ is a quotient of the complex hyperbolic $n$-space $\mathbb{B}^n$ by some torsion-free discrete subgroup $\Gamma$ of the group of holomorphic automorphisms $\text{PU}(n, 1)$. These quotients are also called ball quotients.

The group $\text{PGL}(\mathbb{C}, n+1)$ is the group of automorphisms of $\mathbb{P}^n$ and then the tautological foliation on $\mathbb{P}T_{\mathbb{B}^n}$ is invariant under the action of this group. One has to note that the tautological foliation on $\mathbb{P}T_{\mathbb{B}^n}$ descends also to the tautological foliation on the projectivization of the tangent bundle of any complex hyperbolic space form. This holds since $\text{PU}(n, 1)$, the group of holomorphic automorphisms of $\mathbb{B}^n$, is a subgroup of the projective linear group $\text{PGL}(\mathbb{C}, n+1)$.

**Definition 4.16.** The holomorphic projective connection which corresponds to the tautological foliation on $\mathbb{P}T_{\mathbb{B}^n}$, or to the tautological foliation on the projectivization of the tangent bundle of a complex hyperbolic space form is called
the canonical holomorphic projective connection. The projective second fundamental form of any holomorphic immersion, with respect to this connection is called the canonical projective second fundamental form.

Also, let us note that a complex hyperbolic space form is endowed with the Bergman metric, since the Bergman metric is $\text{PU}(n, 1)$-invariant, see Chapter 4 Proposition 2 in [54].

Let us finish this section by giving Lemma 1 and Lemma 2 from [53], which describe the second fundamental form of holomorphic immersions into a complex hyperbolic space form, with respect to the Bergman metric.

**Lemma 4.4.** Let $(X, g)$ be a complex hyperbolic space form endowed with the Bergman metric. The smooth connection of the Bergman metric $g$ is associated to the canonical holomorphic projective connection on $X$. Let $i : X_0 \to X$ be a holomorphic immersion, and denote by $\beta$ the $(1, 0)$-part of the second fundamental form of $X_0$, with respect to the Bergman metric $g$. Then $\beta$ is holomorphic. Moreover, the second fundamental form on $X_0$ with respect to the Bergman metric $g$ agrees with the canonical projective second fundamental form of $X_0$ in $X$.

### 4.4 Arakelov inequality and quotients of the complex 2-ball

In this section we will show that the direct image of the pluricanonical relative sheaf of a semistable family of curves, uniformized by the complex unit 2-ball, over a curve, contains an invertible subsheaf which satisfies the maximal case in Arakelov inequality, under the assumption that all singular fibers of the family are totally geodesic.

Throughout this section we will suppose that $f : X \to Y$ is a projective family of curves such that:

- the genus of $Y$ is at least 2;
- the family is smooth over $U = Y \setminus S$, where $S$ is a finite set of points on $Y$;
- $X \simeq \mathbb{H}^2/\Gamma$ is a quotient of the complex 2-ball by a torsion-free discrete cocompact subgroup $\Gamma$ of $\text{PU}(2, 1)$, i.e. $X$ is a two dimensional complex hyperbolic space form.

We will denote by $g : \mathbb{H}^2 \to \mathbb{D}$, where $\mathbb{D} \cong \mathbb{H}$, a lift of the map $f$ at the level of the universal cover. For a local system of coordinates $z = (x, y)$ on $\mathbb{H}^2$ we will use the notations: $g_x = \frac{\partial g}{\partial x}, g_y = \frac{\partial g}{\partial y}$

Let $i : C \to X$ be a smooth fiber of the family:

$$C = f^{-1}(y), y \in U = Y \setminus S.$$
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The second fundamental form of the fiber $C$ with respect to the Bergman metric on $X$ is:

$$\beta_C \in A^{1,0}(C, \text{Hom}(T_C, N_C)).$$

Since $i: C \hookrightarrow X$ is a holomorphic embedding, by Lemma 4.4 $\beta_C$ is holomorphic and one gets:

$$\beta_C \in \Gamma(C, \omega_C^{\otimes 2} \otimes N_C).$$

Moreover, by Lemma 4.4 the second fundamental form $\beta_C$ coincides with the canonical projective second fundamental form of $C$ in $X$, induced by the tautological foliation on the projectivization of the tangent space of $\mathbb{P}^2$.

**Lemma 4.5.** For the canonical projective second fundamental form of $C$ in $X$, one has: $\beta_C \in \Gamma(C, \omega_C^{\otimes 2})$.

**Proof.** This is the consequence of the fact that the normal bundle of a smooth fiber in any family is trivial. Let us explain this. We have $C = f^{-1}(y), y \in U$. Then,

$$(f^*T_Y)|_C = C \times (T_Y)_y$$

is a trivial bundle. On the other hand, for every $x \in C$ the map $df_x: (T_X|_C)_x \to (f^*T_Y)_x$ is a surjective map, since $f$ is surjective and $C$ is smooth. The kernel of that map is $(T_C)_x$ and

$$(f^*T_Y)_x \cong (T_X|_C)_x/(T_C)_x.$$ 

Using the fact that

$$N_C \simeq T_X|_C/T_C,$$

we have $N_C \simeq (f^*T_Y)|_C$, hence $N_C$ is trivial. Finally, we get that for a smooth fiber $C$ from the family, the second fundamental form $\beta_C \in \Gamma(C, \omega_C^{\otimes 2})$. \qed

The second fundamental form $\beta_C$ gives rise to a holomorphic section since $f$ is holomorphic:

$$\beta \in H^0(X \setminus \text{Sing}(f), \omega_{X/Y}^{\otimes 2} \otimes f^*T_Y).$$

Since the set of singular points on fibers of the family has dimension 0, by Hartogs theorem $\beta$ can be extended to a section:

$$\beta \in H^0(X, \omega_{X/Y}^{\otimes 2} \otimes f^*T_Y). \quad (4.1)$$

**Definition 4.17.** The section $\beta \in H^0(X, \omega_{X/Y}^{\otimes 2} \otimes f^*T_Y)$ is called the second fundamental form of the family $f : X \to Y$. Those fibers along which $\beta$ vanishes are totally geodesic fibers.
**Definition 4.18.** Let \( D \) be a possibly singular curve in \( \mathbb{B}^2 \). The curve \( D \) is said to be totally geodesic in \( \mathbb{B}^2 \) if its irreducible components are smooth and totally geodesic in \( \mathbb{B}^2 \). Let \( \tau : \mathbb{B}^2 \to X \) be the universal covering map. Let \( X_s \) be a singular fiber in the family \( f : X \to Y \). The fiber \( X_s \) is totally geodesic in \( X \) if \( \tau^{-1}(X_s) \) is totally geodesic in \( \mathbb{B}^2 \).

The following lemma states that totally geodesic fibers in the family can only be singular fibers. The main technical tool for this lemma will be \( \$1 \) from [53].

**Lemma 4.6.** Totally geodesic fibers in the family \( f : X = \mathbb{B}^2 / \Gamma \to Y \) can’t be smooth fibers.

**Proof.** According to Hirzebruch’s proportionality theorem the Chern numbers of \( X = \mathbb{B}^2 / \Gamma \) are proportional with strictly positive proportionality factor to the Chern numbers of \( \mathbb{P}^2 \), since \( \mathbb{P}^2 \) is the compact dual of \( \mathbb{B}^2 \). The proportionality factor is equal to the volume of \( X \). The first Chern class of \( \mathbb{P}^2 \) is:

\[
[c_1(\mathbb{P}^2)] = 3[\xi],
\]

where \([\xi] \in H^2(\mathbb{P}^2, \mathbb{Z})\) is the Poincaré dual to the hyperplane divisor in \( \mathbb{P}^2 \).

On the other side, we have \( c_1(X) = -3\xi' \), where \( \xi' \) is the normalized metric form of the Bergman metric on \( X \), with constant negative sectional curvature \(-2\). By [9] \$1, for any submanifold \( C \) on \( X \), we have:

\[
c_1(C) = -(2\eta + \sigma),
\]

where \( \sigma = \text{tr} (i \beta \wedge \beta^*) \) is a \((1,1)\)-real form which vanishes if \( C \) is totally geodesic and \( \eta \) is the restriction of \( \xi' \) to \( C \), i.e. \( \eta = \xi'|_C \). Let \( C \) be a totally geodesic fiber of the family \( f : X \to Y \). Then, we have:

\[
c_1(C) = -2\eta.
\]

Moreover, if we suppose that \( C \) is a smooth fiber, then by the adjunction formula one has:

\[
\omega_C = (\omega_X \otimes \mathcal{O}_C(C))|_C = \omega_X|_C,
\]

since the normal bundle of the fiber \( N_C = \mathcal{O}_C(C) \) is trivial. This yields that:

\[
c_1(\omega_C) = c_1(\omega_X|_C),
\]

then for the dual bundles, the tangent bundles we get

\[
c_1(C) = c_1(X)|_C,
\]

and

\[-2\eta = -3\eta.
\]

Hence, \( \eta = 0 \). This is not possible since \( \eta \) is the pullback of the Kähler form on \( X \) of constant negative sectional curvature. Therefore, the assumption that \( C \) is a smooth fiber is false. So, totally geodesic fibers in the family, if any, can only be singular fibers. \( \square \)
4.4. Arakelov inequality and quotients of the complex 2-ball

Before we give the most important results of this section, let us recall several well known details about line bundles on the projective line:

- We recall the definition of the tautological bundle on $\mathbb{P}^1$. It is defined as:

$$L = \{([z], \xi) \in \mathbb{P}^1 \times \mathbb{C}^2 | \xi \in \mathbb{C} \cdot z \} \cong \mathcal{O}(-1),$$

with fibers

$$L_{[z]} = \mathbb{C} \cdot z, z \neq 0.$$

The transition functions of this line bundle are:

$$l_{01}(z) = \frac{z_0}{z_1}$$

and

$$l_{10}(z) = \frac{z_1}{z_0}.$$

- The dual bundle of $\mathcal{O}(-1)$ is denoted as $\mathcal{O}(1)$ with transition functions:

$$t_{01}(z) = \frac{z_1}{z_0}$$

and

$$t_{10}(z) = \frac{z_0}{z_1}$$

which are obtained as dual (transposed) maps of the transition functions of the line bundle $\mathcal{O}(-1)$. It is well known, that all line bundles on $\mathbb{P}^1$ are isomorphic to $\mathcal{O}(-m) = \mathcal{O}(-1)^{\otimes m}$ or to $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$, for $m$ a non-negative integer, with transition functions equal to the $m$-th power of the transition functions of $\mathcal{O}(-1)$ or $\mathcal{O}(1)$.

- An easy calculation shows that the transition functions of the tangent bundle on $\mathbb{P}^1$ are:

$$t_{01}(z) = -\left(\frac{z_1}{z_0}\right)^2$$

and

$$t_{10}(z) = -\left(\frac{z_0}{z_1}\right)^2.$$

It is well known that two holomorphic vector bundles of rank $r$ over some complex manifold are isomorphic if and only if there exists an open covering $\{U_\alpha\}$ on that manifold relative to which their cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ given by transition functions are equivalent, in the sense that there exist holomorphic maps:

$$\lambda_\alpha : U_\alpha \to \text{GL}(r, \mathbb{C})$$

such that

$$g'_{\alpha\beta} = \lambda_\alpha g_{\alpha\beta} \lambda^{-1}_\beta, \text{ in } U_\alpha \cap U_\beta.$$ 

If we define the maps, $\lambda_0 : U_0 \to \mathbb{C}$ as $\lambda_0(z) = 1$ and $\lambda_1 : U_1 \to \mathbb{C}$ as $\lambda_1(z) = -1$, then the transition functions of the tangent bundle of $\mathbb{P}^1$ become:

$$t_{01}(z) = \left(\frac{z_1}{z_0}\right)^2$$

and

$$t_{10}(z) = \left(\frac{z_0}{z_1}\right)^2, \text{ in } U_0 \cap U_1,$$

therefore the tangent bundle of $\mathbb{P}^1$ is isomorphic to the line bundle $\mathcal{O}(2)$.
4. Examples of Arakelov equality for semistable families of curves
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Lemma 4.7. One has an isomorphism:

\[ T_{P^1} \otimes L \simeq \left( \mathbb{P}^1 \times \mathbb{C}^2 \right) / L, \]

where \( T_{P^1} \) is the tangent bundle and \( L \) is the tautological bundle of \( \mathbb{P}^1 \).

Proof. Let \( \{z_0, z_1\} \) be a system of local coordinates on \( \mathbb{C}^2 \), then on \( \mathbb{P}^1 \) we choose the standard open covering by sets:

\[ U_0 = \{[z] = [z_0 : z_1] | z_0 \neq 0\} \quad \text{and} \quad U_1 = \{[z] = [z_0 : z_1] | z_1 \neq 0\}. \]

Let \( u_i = \frac{z_j}{z_i}, i \neq j \) be the local coordinate on \( U_i \) and let us give the trivialisation maps for the bundles \( T_{P^1} \otimes L \) and \( \left( \mathbb{P}^1 \times \mathbb{C}^2 \right) / L \). We suppose that\[
[z] = [z_0 : z_1] \in U_i,
\]
then we have:

\[ (T_{P^1} \otimes L)_{[z]} = \mathbb{C}\{\frac{\partial}{\partial u_i} \otimes z\} \]

and the trivialisation map is given by

\[ \frac{\partial}{\partial u_i} \otimes z \mapsto z_i. \]

For the vector bundle \( \left( \mathbb{P}^1 \times \mathbb{C}^2 \right) / L \) we have:

\[ \left( \left( \mathbb{P}^1 \times \mathbb{C}^2 \right) / L \right)_{[z]} \simeq \mathbb{C}^2 / \mathbb{C} \cdot z, \]

then:

- if \( [z] \in U_0 \), representatives of the vectors from \( \left( \left( \mathbb{P}^1 \times \mathbb{C}^2 \right) / L \right)_{[z]} \) are of the form \( \begin{pmatrix} 0 \\ \mu \end{pmatrix} \) and the trivialisation map is:
  \[ \begin{pmatrix} 0 \\ \mu \end{pmatrix} \mapsto \mu; \]

- if \( [z] \in U_1 \), representatives of the vectors from \( \left( \left( \mathbb{P}^1 \times \mathbb{C}^2 \right) / L \right)_{[z]} \) are of the form \( \begin{pmatrix} \nu \\ 0 \end{pmatrix} \) and the trivialisation map is:
  \[ \begin{pmatrix} \nu \\ 0 \end{pmatrix} \mapsto \nu. \]
Let $\phi : T_{\mathbb{P}^1} \otimes L \to (\mathbb{P}^1 \times \mathbb{C}^2)/L$ be the map defined by:

$$
\phi|_{U_0} \left( \frac{\partial}{\partial u_0} \otimes z \right) = \begin{pmatrix} 0 \\ z_0 \end{pmatrix}, [z] \in U_0;
$$

$$
\phi|_{U_1} \left( \frac{\partial}{\partial u_1} \otimes z \right) = \begin{pmatrix} z_1 \\ 0 \end{pmatrix}, [z] \in U_1.
$$

We will show that the map $\phi$ is well defined. Let us suppose that $[z] \in U_0 \cap U_1$, then a tangent vector $r = \frac{\partial}{\partial u_0}$ at the point $[z]$ can be seen as:

$$
r = -\left( \frac{z_0}{z_1} \right)^2 \frac{\partial}{\partial u_1},
$$

using the transition maps of the tangent bundle $T_{\mathbb{P}^1}$, defined as above. Then, for $[z] \in U_0 \cap U_1$ one has:

$$
v_0 = \phi|_{U_0}(r \otimes z) = \begin{pmatrix} 0 \\ z_0 \end{pmatrix};
$$

$$
v_1 = \phi|_{U_1}(r \otimes z) = \phi|_{U_1} \left( -\left( \frac{z_0}{z_1} \right)^2 \frac{\partial}{\partial u_1} \otimes z \right) = \begin{pmatrix} -\left( \frac{z_0}{z_1} \right)^2 z_1 \\ 0 \end{pmatrix} = \left( -\frac{z_0^2}{z_1^2} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

Let us show that the vectors $v_0$ and $v_1$ are equal in the fiber $((\mathbb{P}^1 \times \mathbb{C}^2)/L)[z]$. By the following computation:

$$
v_1 = \begin{pmatrix} -\frac{z_0^2}{z_1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ z_0 \end{pmatrix} - \frac{z_0}{z_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$

we get that vectors $v_0$ and $v_1$ are equal in the quotient space $((\mathbb{P}^1 \times \mathbb{C}^2)/L)[z]$. This implies that the map $\phi$ is well defined. Besides that, it is an injective and surjective map, so we have an explicitly described isomorphism between the bundles $T_{\mathbb{P}^1} \otimes L$ and $(\mathbb{P}^1 \times \mathbb{C}^2)/L$.

In section 4.2, following the result of Mok we explained that it is possible to define the second fundamental form of a holomorphic immersion into $X = \mathbb{B}^2/\Gamma$ only using the tautological foliation of $\mathbb{P}T_X$, without any references to the affine connection. Here, following that idea which was presented at the end of section 4.2, we will explicitly describe the second fundamental form of the projective family

$$
f : X = \mathbb{B}^2/\Gamma \to Y,
$$

at the level of the universal cover $\mathbb{B}^2$. 

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**Lemma 4.8.** Let $g : \mathbb{B}^2 \to \mathbb{D}$ be a lift of the map $f : X = \mathbb{B}^2/\Gamma \to Y$ at the level of the universal covering. The second fundamental form of fibers of the family $g : \mathbb{B}^2 \to \mathbb{D}$ can be identified with the function:

$$F(z) = g_y^2(z)g_{xx}(z) - 2g_x(z)g_y(z)g_{xy}(z) + g_x^2(z)g_{yy}(z),$$

in the system of coordinates $z = (x, y)$ on $\mathbb{B}^2$.

More precisely, let $\tilde{\beta} \in H^0(\mathbb{B}^2, \omega_{\mathbb{B}^2/\mathbb{D}} \otimes g^*T_D)$ be the second fundamental form of the family $g : \mathbb{B}^2 \to \mathbb{D}$. In the trivialization of the relative tangent bundle $T_{\mathbb{B}^2/\mathbb{D}}$ given by

$$z \mapsto s(z) = \begin{pmatrix} g_y(z) \\ -g_x(z) \end{pmatrix},$$

the function $F$ is defined by

$$F(z) = \tilde{\beta}_z \left( \begin{pmatrix} g_y(z) \\ -g_x(z) \end{pmatrix} \otimes \begin{pmatrix} g_y(z) \\ -g_x(z) \end{pmatrix} \right) \in (g^*T_D)_z \cong \mathbb{C}.$$

**Proof.** (A) The tangent bundle of $\mathbb{B}^2$ is trivial, i.e. $T_{\mathbb{B}^2} = \mathbb{B}^2 \times \mathbb{C}^2$. The projectivization of the tangent bundle is $\mathbb{P}T_{\mathbb{B}^2} = \mathbb{B}^2 \times \mathbb{P}^1$. The map $\pi : \mathbb{P}T_{\mathbb{B}^2} \to \mathbb{B}^2$ is the canonical projection. Let $T_{\pi} = \text{Ker} \, d\pi$. It is plain to see that: $T_{\pi} \cong T_{\pi^1}$.

Let $D = g^{-1}(c), c \in \mathbb{D}$. Let $z_0 = (x_0, y_0) \in D$ be a smooth point in the fiber $D$. We suppose that for the points $z$ around $z_0$, i.e. for $z \in D_{z_0} = B(z_0, \epsilon) \cap D$ for $\epsilon$ small enough, one has $g_y(z) \neq 0$. Then, a trivialization of $T_{D_{z_0}}$ is given by the section:

$$z \mapsto s(z) = \begin{pmatrix} g_y(z) \\ -g_x(z) \end{pmatrix}.$$

The implicit function theorem provides a holomorphic parametrization of the curve $D$ near the fixed point $z_0$, given by:

$$\lambda \mapsto (x(\lambda), y(\lambda)), \text{ where } g(x(\lambda), y(\lambda)) = c,$$

and such that $x(0) = x_0, y(0) = y_0, \dot{x}(0) = g_y(z_0)$ and $\dot{y}(0) = -g_x(z_0)$.
The tautological lifting of $D_{z_0}$ to the space $\mathbb{P}T_{\mathbb{B}^2}$ is defined by the lifting of a point $z \in D_{z_0}$ to the point $[s(z)] \in \mathbb{P}T_{\mathbb{B}^2,z} \cong \mathbb{P}^1$. More precisely, the tautological lifting of $D_{z_0}$ is defined as the curve:

$$\hat{D}_{z_0} = \{(z, [s(z)]), z \in D_{z_0}\} \subset D_{z_0} \times \mathbb{P}^1 \subset \mathbb{P}T_{\mathbb{B}^2},$$

i.e. $\hat{D}_{z_0}$ as the graph of the function:

$$z \in D_{z_0} \mapsto [s(z)] = [g_y(z) : -g_x(z)],$$

and since we supposed that $g_y(z) \neq 0$ then

$$z \in D_{z_0} \mapsto [s(z)] = \left[1 : -\frac{g_x(z)}{g_y(z)}\right] \in U_0,$$

where $\{U_0, U_1\}$ is the standard open covering on $\mathbb{P}^1$. So, we get the local description of the curve $\hat{D}_{z_0}$ by the parametrization:

$$\lambda \mapsto \left(x(\lambda), y(\lambda), \left[1 : -\frac{g_x(x(\lambda), y(\lambda))}{g_y(x(\lambda), y(\lambda))}\right]\right),$$

or to simplify notation, the parametrization of $\hat{D}_{z_0}$ is given as:

$$\lambda \mapsto \left(x(\lambda), y(\lambda), \frac{g_x(x(\lambda), y(\lambda))}{g_y(x(\lambda), y(\lambda))}\right).$$

(B) Let $w = s(z_0) = \left(\frac{g_y(z_0)}{-g_x(z_0)}\right)$ be a tangent vector of $D_{z_0}$ at the point $z_0$. Let $C$ be a geodesic on $\mathbb{B}^2$ intersecting $D_{z_0}$ at the point $z_0$, such that $w$ is a common tangent vector for both curves. Being a geodesic, $C$ is a line on the ball in the direction of the vector $w$. We have the following parametrization for $C$:

$$\lambda \mapsto (\lambda g_y(z_0), -\lambda g_x(z_0)).$$

The tautological lifting $\hat{C}$ of the curve $C$ is a leaf of tautological foliation on $\mathbb{P}T_{\mathbb{B}^2}$ and it intersects $\hat{D}_{z_0}$ at the point $[w]$. We can describe the curve $\hat{C}$ by the parametrization:

$$\lambda \mapsto \left(\lambda g_y(z_0), -\lambda g_x(z_0), \left[1 : -\frac{g_x(z_0)}{g_y(z_0)}\right]\right),$$

or

$$\lambda \mapsto \left(\lambda g_y(z_0), -\lambda g_x(z_0), \frac{g_x(z_0)}{g_y(z_0)}\right).$$
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Using the parametrization introduced above, a tangent vector of $\hat{D}_{z_0}$ at the point $[w] \in \mathbb{P}T_{\mathbb{B}^2}$ is:

$$\xi = \left(\dot{x}(\lambda), \dot{y}(\lambda), \frac{\partial}{\partial \lambda} \left( -\frac{g_x(x(\lambda), y(\lambda))}{g_y(x(\lambda), y(\lambda))} \right) \right)_{|\lambda=0} \in T_{[w]}(\hat{D}_{z_0}) \subseteq T_{[w]}\mathbb{P}T_{\mathbb{B}^2}.$$  

Using that:

- $\dot{x}(0) = g_y(z_0)$, $\dot{y}(0) = -g_x(z_0)$,
- $\frac{\partial}{\partial \lambda} (g_x(x(\lambda), y(\lambda))) = g_{xx}\dot{x} + g_{xy}\dot{y}$,
- $\frac{\partial}{\partial \lambda} (g_y(x(\lambda), y(\lambda))) = g_{yy}\dot{y} + g_{xy}\dot{x}$,

we get:

$$\left. \frac{\partial}{\partial \lambda} \left( -\frac{g_x(x(\lambda), y(\lambda))}{g_y(x(\lambda), y(\lambda))} \right) \right|_{\lambda=0} = \left. \left( -\frac{g_{xx}\dot{x} + g_{xy}\dot{y}}{g_y^2} \right) - \frac{g_x(z_0)}{g_y(z_0)}^2 g_{yy}(z_0) \right|_{\lambda=0}$$

and hence

$$\xi = \left( g_y(z_0), -g_x(z_0), -g_{xx}(z_0) + 2\frac{g_x(z_0)}{g_y(z_0)}g_{xy}(z_0) - \left( \frac{g_x(z_0)}{g_y(z_0)} \right)^2 g_{yy}(z_0) \right),$$

whilst a tangent vector of $\hat{C}$ at the point $[w]$ is

$$\eta = (g_y(z_0), -g_x(z_0), 0) \in T_{[w]}(\hat{C}) \subseteq T_{[w]}\mathbb{P}T_{\mathbb{B}^2}.$$  

We get the vector:

$$\eta - \xi = \left( 0, 0, g_{xx}(z_0) - 2\frac{g_x(z_0)}{g_y(z_0)}g_{xy}(z_0) + \left( \frac{g_x(z_0)}{g_y(z_0)} \right)^2 g_{yy}(z_0) \right).$$

Therefore, we get $d\pi(\eta - \xi) = 0$, i.e.

$$\eta - \xi \in T_{\pi,[w]}.$$  

As we explained in the end of Section 4.2, in order to define the second fundamental form of $D_{z_0}$ which is a submanifold of dimension 1 on the complex unit 2-ball, which is equipped with the projective connection, we have to assign to the vector $w$, the unique vector, $r_w = A(w)$ from the kernel of the map $\pi$. Summarising what we have seen so far we get:

$$r_w = \eta - \xi.$$
As \( \eta - \xi \in T_{\pi,[u]} \) and \( [w] = \begin{bmatrix} 1 : - \frac{g_x(0)}{g_y(0)} \end{bmatrix} \in U_0 \), then in the local coordinates on \( \mathbb{P}(T_{\mathbb{B}^2,z_0}) \cong \mathbb{P}^1 \) we can see the vector \( \eta - \xi \) as the vector:

\[
r_w = \left( g_{xx}(z_0) - 2 \frac{g_x(z_0)}{g_y(z_0)} g_{xy}(z_0) + \left( \frac{g_x(z_0)}{g_y(z_0)} \right)^2 g_{yy}(z_0) \right) \frac{\partial}{\partial u_0},
\]

where \( u_i = \frac{v_i}{v_0}, i \neq j \), is a local coordinate on the standard open set \( U_i \) of \( \mathbb{P}(T_{\mathbb{B}^2,z_0}) \) and \( \{v_0, v_1\} \) are local coordinates on \( T_{\mathbb{B}^2,z_0} \).

(C) Let us make a short review of what we saw in the end of Section 4.2 but here applied to the case of the complex unit 2-ball. Following the notations from Section 4.2 we have the map:

\[
A : T_{D_{z_0}} \to T_{\pi,[v]},
\]

or using the isomorphism \( T_{D_{z_0},z} \cong L_{[v]} \) we have the map

\[
A : L_{[v]} \to T_{\pi,[v]},
\]

\[
A(v) = r_v,
\]

where \( T_{D_{z_0},z} = \mathbb{C}v \), at a point \( z \in D_{z_0} \). This map gives rise to the map \( \tilde{\sigma}_{[v]} \) which will induce the second fundamental form of \( D_{z_0} \):

\[
\tilde{\sigma}_{[v]} : L_{[v]} \otimes L_{[v]} \to T_{\pi,[v]} \otimes L_{[v]} \to (\pi^*T_{\mathbb{B}^2}/L)_{[v]},
\]

which is defined as:

\[
\tilde{\sigma}_{[v]} = \phi(A \otimes \text{id}_{L_{[v]}}).
\]

In the case of the complex unit 2-ball, using the isomorphism \( \phi \) constructed in Lemma 4.7, \( \tilde{\sigma} \) will be defined as:

\[
\tilde{\sigma}_{[v]}(v,v) = \phi(A \otimes \text{id}_{L}(v,v)) = \phi(r_v \otimes v),
\]

\[
\tilde{\sigma}_{[v]}(v,v) = \begin{cases} 
0, & \text{if } [v] \in U_0 \text{ and } r_v = \alpha \frac{\partial}{\partial u_0}; \\
v_0 \alpha, & \text{if } [v] \in U_1 \text{ and } r_v = \beta \frac{\partial}{\partial u_1}.
\end{cases}
\]

The morphism \( \tilde{\sigma}_{[v]} \) gives rise to the holomorphic section:

\[
\tilde{\sigma} \in H^0(\mathbb{P}T_{D_{z_0}}, L^{-2} \otimes \pi^*T_{\mathbb{B}^2}/L).
\]

Note that the isomorphism \( L_{[v]} \cong T_{D_{z_0},z} \) implies:

\[
\pi^*T_{\mathbb{B}^2}/L \cong \pi^*N_{D_{z_0}}.
\]
Hence,
\[ \tilde{\sigma} \in H^0(\mathbb{P}T_{D_{z_0}}, L^{-2} \otimes \pi^* N_{D_{z_0}}). \]

At the end we will apply the isomorphism \( dg : N_{D_{z_0}, z} \cong T_{D, g(z)} \cong \mathbb{C} \), given by the linear operator \( dg(z) \left( \begin{array}{c} \lambda \\ \gamma \end{array} \right) = (g_x, g_y)(z) \left( \begin{array}{c} \lambda \\ \gamma \end{array} \right) = g_x(z)\lambda + g_y(z)\gamma \) and we get the holomorphic section:
\[ \sigma \in H^0(D_{z_0}, T_{D_{z_0}}^{-2} \otimes g^* T_{\mathbb{B}}) \]
which is the second fundamental form of \( D_{z_0} \). This section on fibers is given by:
\[ \sigma_z = dg(z)(\tilde{\sigma}_v) : T_{D_{z_0}, z} \cong \mathbb{C}, \]
such that:
\[ \sigma_z(v, v) = \begin{cases} v_0 \alpha g_y(z) & \text{if } [v] \in U_0 \text{ and } r_v = \alpha \frac{\partial}{\partial u_0}; \\ v_1 \beta g_x(z) & \text{if } [v] \in U_1 \text{ and } r_v = \beta \frac{\partial}{\partial u_1}. \end{cases} \]

Now, applying what we have seen so far, to the vector \( w \) at point \( z_0 \) on \( D_{z_0} \), with
\[ r_w = \left( g_{xx}(z_0) - 2g_x(z_0)g_{xy}(z_0) + \left( \frac{g_x(z_0)}{g_y(z_0)} \right)^2 g_{yy}(z_0) \right) \frac{\partial}{\partial u_0}, \]
we get:
\[ \sigma_{z_0}(w, w) = g_y(z_0) \left( g_{xx}(z_0) - 2g_x(z_0)g_{xy}(z_0) + \left( \frac{g_x(z_0)}{g_y(z_0)} \right)^2 g_{yy}(z_0) \right) g_y(z_0) \]
\[ = g_y^2(z_0)g_{xx}(z_0) - 2g_x(z_0)g_y(z_0)g_{xy}(z_0) + g_x^2(z_0)g_{yy}(z_0), \]
which is the expression for the second fundamental form of \( D_{z_0} \) in the local trivialization, i.e. \( \sigma_z = \tilde{\sigma}_z \). Since the last expression is symmetric in \( g_x \) and \( g_y \), the condition \( g_y(z) \neq 0 \) did not play any role.

Moreover, the last expression is well defined at every point of the fiber \( D \), therefore the extension of the second fundamental form of the fiber \( D \) can be identified with the function
\[ F : \mathbb{B}^2 \rightarrow \mathbb{C}, \]
where
\[ F(z) = g_y^2(z)g_{xx}(z) - 2g_x(z)g_y(z)g_{xy}(z) + g_x^2(z)g_{yy}(z). \]
Lemma 4.9. In the previous notations, let the family $f : X \to Y$ be a semistable family of curves, then the function $F : \mathbb{B}^2 \to \mathbb{C}$ vanishes at order exactly one on the totally geodesic fibers, if any, in the family $g : \mathbb{B}^2 \to \mathbb{D}$.

Proof. The family $f : X \to Y$ is a semistable family of curves, hence the singular fibers of the family $g : \mathbb{B}^2 \to \mathbb{D}$ are reduced normal crossing divisors.

By the previous lemma, the second fundamental form of a fiber at a point $z$ in the family $g : \mathbb{B}^2 \to \mathbb{D}$ is:

$$F(z) = g_y^2(z)g_{xx}(z) - 2g_x(z)g_y(z)g_{xy}(z) + g_x^2(z)g_{yy}(z).$$

Note that the map $F$ is well defined everywhere on $\mathbb{B}^2$, including singular points of the family $\text{Sing}(g)$. Let $D$ be a singular and totally geodesic fiber in the family $g : \mathbb{B}^2 \to \mathbb{D}$. Since $\text{PU}(2,1)$ acts transitively on $\mathbb{B}^2$, we can suppose, after some change of coordinates on the ball, that:

- $D = g^{-1}(0)$;
- $(0,0) \in D$ and it is a singular point for the map $g$.

Since any singular point of the fiber $D$ is a normal crossing of two branches and $D$ is a geodesic on the ball, we can suppose that the branches of $D$ which intersect at the point $(0,0)$ are the branches $\{y = 0\}$ and $\{x - cy = 0\}, c \in \mathbb{C}$. Recall that the second fundamental form is invariant under the action of $\text{PU}(2,1)$.

One has $g(cy, y) = 0$ and $g(x, 0) = 0$. Since $D$ is totally geodesic, on the branch $\{y = 0\}$ we have:

$$F(x, 0) = 0.$$ 

We write the function $g$ as:

$$g(x, y) = \sum_{i,j} a_{ij}x^i y^j,$$

where $a_{ij}$ are Taylor’s coefficients in the neighborhood of the point $(0,0)$. The equation $g(x, 0) = 0$ gives:

$$a_{k0} = 0, \text{ for all } k.$$ 

The fact that $(0,0)$ is a singular point yields:

$$a_{01} = 0.$$ 

On the other side, for the function $g$ in the neighborhood of the point $(0,0)$ one has:

$$g(x, y) = ay(x - cy) + \sum_{i+j \geq 3} a_{i,j}x^i y^j,$$

with $a \neq 0$ since $D$ is nodal at $(0,0)$. 

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So, we get:

$$a_{11} = \frac{\partial^2 g}{\partial x \partial y}(0,0) = a \neq 0. \quad (4.2)$$

We have:

$$F(x,y) = \left( \sum_{j \geq 1} ja_{ij}x^i y^{j-1} \right)^2 \left( \sum_{i \geq 2} i(i-1)a_{ij}x^{i-2}y^j \right) - 2\left( \sum_{i \geq 1} ia_{ij}x^{i-1}y^j \right) \left( \sum_{j \geq 1} ja_{ij}x^i y^{j-1} \right) \left( \sum_{j \geq 1} ij a_{ij}x^{i-1}y^{j-1} \right) + \left( \sum_{i \geq 1} ia_{ij}x^{i-1}y^j \right)^2 \left( \sum_{j \geq 2} j(j-1)a_{ij}x^{i-2}y^j \right) = \sum_{k,m} b_{km} x^k y^m.$$ 

Since $F(x,0) = 0$, we have:

$$b_{k0} = 0.$$ 

Also, using that $a_{k0} = 0$ for $k \geq 0$, an easy calculation gives:

$$b_{k1} = \sum_{m+n+p=k} (p+2)(p+1)a_m a_n a_{p+2,1} - 2 \sum_{m+n+p=k} (m+1)(p+1)a_m a_n a_{p+1,1}.$$ 

We suppose that the function $F(x,y)$ vanishes at order bigger than 1 along the singular totally geodesic fiber $D$. In the neighborhood of the point $(0,0)$ the fiber $D$ is given as the intersection of branches $y = 0$ and $x - cy = 0$, hence our assumption implies that

$$F(x,y) = y^2 H(x,y),$$

for some analytic function $H(x,y)$. This implies that all

$$b_{k1} = 0.$$ 

In particular, from $b_{11} = 0$ using that $a_{01} = 0$, we get:

$$a_{11} = 0.$$ 

Therefore, we have a contradiction with (4.2) and our assumption that $F(x,y)$ vanishes at order bigger than 1 along a singular geodesic fiber is false, hence the second fundamental form on totally geodesic fibers vanishes of order 1. 

We should underline the fact that the second fundamental form induced by the canonical projective connection on the ball $\mathbb{B}^2$ is invariant under the action of the group $\text{PU}(2,1) \subset \text{PGL}(3)$, as we explained in Section 4.3. Hence, it descends to the ball quotient $X = \mathbb{B}^2/\Gamma$. Moreover, by Lemma 4.9 on the geodesic fibers of the semistable family $f : X \to Y$, the second fundamental form will vanish at order exactly 1.

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**Theorem 4.10.** If all singular fibers in the family $f: X \to Y$ are totally geodesic, then there is an invertible subsheaf of $f_*\omega_{X/Y}^{\otimes 2}$ which satisfies the maximal case in the Arakelov inequality.

**Proof.** The second fundamental form $\beta \in H^0(X, \omega_{X/Y}^{\otimes 2} \otimes f^*T_Y)$, of the fibration $f: X \to Y$ induces a section:

$$\beta_Y \in H^0(Y, f_*\omega_{X/Y}^{\otimes 2} \otimes T_Y).$$

By Lemma 4.6, $\beta$ can vanish only on singular fibers, i.e. fibers over the set $S$. This yields that $\beta_Y$ can vanish only at points of the set $S$.

The section $\beta_Y \in H^0(Y, f_*\omega_{X/Y}^{\otimes 2} \otimes T_Y)$ provides an invertible subsheaf

$$\mathcal{F} \subset f_*\omega_{X/Y}^{\otimes 2} \otimes T_Y,$$

and one has:

$$\deg \mathcal{F} = \sum \text{multiplicities of zeros of } \beta_Y.$$ 

By Lemma 4.9, $\beta$ vanishes at order 1 along the totally geodesic fibers, so the multiplicities of zeros of the section $\beta_Y$ will be 1. Hence, we get:

$$\deg \mathcal{F} = \# \text{Zeros}(\beta_Y) \leq \# S.$$

If we suppose that all singular fibers are totally geodesics then $\beta_Y$ vanishes at every point of $S$ and:

$$\deg \mathcal{F} = \# S.$$

Then, the invertible subsheaf $\mathcal{H} = \mathcal{F} \otimes \omega_Y \subseteq f_*\omega_{X/Y}^{\otimes 2}$, satisfies

$$\deg \mathcal{H} = \deg \mathcal{F} + \deg \omega_Y = \deg \omega_Y(S).$$

The invertible subsheaf $\mathcal{H} \subseteq f_*\omega_{X/Y}^{\otimes 2}$ reaches the bound in the Arakelov inequality. \hfill \Box

The semistable families of curves $f: X \to Y$, where $X$ is a quotient of the complex 2-ball by a torsion-free discrete cocompact subgroup of $PU(2,1)$, all whose singular fibers are totally geodesic are examples of families whose bicanonical relative sheaf $f_*\omega_{X/Y}^{\otimes 2}$ contains an invertible subsheaf which satisfies the maximal case in the Arakelov inequality.

### 4.5 Example

In this section we will give examples of semistable families whose geometric variation contains a subvariation whose Higgs field is an isomorphism. In other words, by Theorem 2.22, the base curves in these families are Teichmüller curves. These examples are families $g: W_{\Gamma(N)} \to Y_{\Gamma(N)}$, where the map $g$ is
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the composition of a $\nu$-cyclic covering over the elliptic modular surface $X_{\Gamma(N)}$ of level $N$ and the elliptic fibration $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$, where $Y_{\Gamma(N)}$ is a modular curve attached to the principal congruence subgroup of level $N$, for $N \geq 3$. The degree $\nu$ of the cyclic covering $W_{\Gamma(N)}$ will depend on $N$.

Moreover, Livné shows in his thesis [48] that for $\nu = \frac{N}{N-6}$, i.e., for $N \in \{7, 8, 9, 12\}$, the surfaces $W_{\Gamma(N)}$ are of general type with $c_1^2 = 3c_2$. By Yau’s result, these surfaces are uniformized by the complex unit 2-ball.

In the previous section we proved that in the case when all singular fibers of a family, uniformized by the complex unit 2-ball, are totally geodesic there is an invertible subsheaf of the direct image of the relative bicanonical sheaf of the family which satisfies the maximal case in the Arakelov inequality. Here, we prove that the pluricanonical relative sheaf of the families $g : W_{\Gamma(N)} \to Y_{\Gamma(N)}$ contains an invertible subsheaf which satisfies the maximal case in the Arakelov inequality. Later, we will prove that in the case when $N \in \{7, 8, 9, 12\}$ the singular fibers are totally geodesic, as expected.

4.5.1 Preliminaries

Let $\Gamma$ be a torsion free subgroup of finite index of the group $\text{SL}_2(\mathbb{Z})$. The group $\Gamma$ acts discretely on $\mathbb{H}$. The quotient $\mathbb{H}/\Gamma$ is a non-compact Riemann surface. Adding a finite number of cusps to $\mathbb{H}/\Gamma$ we get a compact Riemann surface $Y_{\Gamma}$, called the modular curve attached to $\Gamma$.

**Definition 4.19.** ([64] §1.4 or [65] §4) A point $y \in Y_{\Gamma}$ is called a cusp of width $b$ if one of its representing points $z \in \mathbb{Q} \cup \{\infty\}$ has the stabilizer generated by an element which is conjugate in $\text{SL}_2(\mathbb{Z})$ to either \( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \) or \( \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \), for $b > 0$. Respectively, $y$ is called a cusp of the first or of the second kind.

We say that a point $y \in Y_{\Gamma}$ is an elliptic point if $y$ is not a cusp and its representing point $z \in \mathbb{H}$ has the stabilizer $\Gamma_z$ in $\Gamma$, generated by an element of order 3, which is conjugate in $\text{SL}_2(\mathbb{Z})$ to either \( \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

We will use the notations:
\[ \text{o} \quad \mu \text{ is the index of the projectivization of } \Gamma \text{ in } \text{PSL}(2, \mathbb{Z}); \]
\[ \text{o} \quad t_1 \text{ is the number of cusps of the first kind in } Y_{\Gamma}; \]
\[ \text{o} \quad t_2 \text{ is the number of cusps of the second kind in } Y_{\Gamma}; \]
\[ \text{o} \quad t' = t_1 + t_2 \text{ is the number of cusps in } Y_{\Gamma}; \]
\[ \text{o} \quad s \text{ is the number of elliptic points in } Y_{\Gamma}; \]
\[ \text{o} \quad t = t' + s. \]

**Lemma 4.11.** ([§4][65]) The genus $g$ of $Y_{\Gamma}$ satisfies:
\[ 2g - 2 + t' + \frac{2}{3}s = \frac{1}{6}\mu. \]
**Definition 4.20.** Let \( f : X \to Y \) be a projective family of curves, smooth over \( U = Y \setminus S \). The surface \( X \) is called an elliptic fibration or an elliptic surface if:

1. all smooth fibers \( f^{-1}(y), y \in U \) are elliptic curves, and singular fibers contain no \((-1)\)-curves;
2. there exists a holomorphic map \( \varphi_0 : Y \to X \) such that \( f \circ \varphi_0 = id_Y \), called a zero section of \( f \).

Following Chapter 12 from [44] and Chapter 2 from [65], we give several facts about elliptic fibrations over a modular curve.

**\( \text{A}_1 \)** The quotient of \( \mathbb{H} \times \mathbb{C} \) by automorphisms of the form:

\[
(\tau, z) \to (\gamma(\tau), \frac{z + m\tau + n}{c\tau + d}),
\]

where \( \gamma = \left( \begin{matrix} a & b \\ c & d \end{matrix} \right) \in \Gamma, (m, n) \in \mathbb{Z}^2 \), defines a surface equipped with a morphism to the modular curve \( Y_\Gamma \). The fiber over the image in \( Y_\Gamma \) of a general point \( \tau \in \mathbb{H} \) is the elliptic curve corresponding to the lattice \( \mathbb{Z} \oplus \mathbb{Z}\tau \). The surface obtained in this way can be extended to an elliptic surface \( X_\Gamma \) over the modular curve \( Y_\Gamma \). In this fibration singular fibers lie over cusps and elliptic points of the modular curve \( Y_\Gamma \). The surface \( X_\Gamma \) is called the elliptic modular surface attached to the group \( \Gamma \).

**\( \text{A}_2 \)** Let us denote by \( \varphi_\Gamma : Y_\Gamma \to X_\Gamma \) a section of the elliptic modular fibration \( f : X_\Gamma \to Y_\Gamma \). By Theorem 6.8 from [62] one has:

\[
\omega_{X_\Gamma} = f^*(\omega_{Y_\Gamma} \otimes L^{-1}),
\]

for some line bundle \( L \) such that \( \deg L^{-1} = \chi(\mathcal{O}_{X_\Gamma}) \). Moreover, one gets that \( K_{X_\Gamma} \) is a vertical divisor, i.e. it is linearly equivalent to a smooth fiber of the family and one has:

\[
K_{X_\Gamma} \approx (2g - 2 + \chi(\mathcal{O}_{X_\Gamma}))C,
\]

where \( C \) is a smooth fiber. Hence, one gets:

\[
K_{X_\Gamma} \cdot K_{X_\Gamma} = 0,
\]

or

\[
c_1^2(X_\Gamma) = 0.
\]

On the other side, the Noether formula:

\[
12\chi(\mathcal{O}_{X_\Gamma}) = c_1^2(X_\Gamma) + c_2(X_\Gamma)
\]

yields

\[
12(1 - q + p_g) = 12 \deg L^{-1} = \deg c_2(X_\Gamma) = \chi(X_\Gamma),
\]

(4.4)
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where \( q \) is the irregularity of \( X_\Gamma \) and \( p_g \) is geometric genus of \( X_\Gamma \).

Using Lemma VI.4 from [4] which describes the Euler characteristic of a fibration of a surface one gets:

\[
\chi(X_\Gamma) = \chi(Y_\Gamma)\chi(C) + \sum_{t \in T}(\chi(X_{\Gamma,t}) - \chi(C)),
\]

where \( C \) is a smooth fiber in the family, the set \( T \) is the discriminant locus of the family and \( X_{\Gamma,t} \) is the singular fiber corresponding to the point \( t \in T \). The Euler characteristic of an elliptic curve vanishes and one gets:

\[
e_2(X_\Gamma) = \sum_{t \in T}\chi(X_{\Gamma,t}). \tag{4.5}
\]

By Theorem 12.2 from [44] and Proposition 4.2 from [65] we get:

\[
e_2(X_\Gamma) = \sum_{t \in T}\chi(X_{\Gamma,t}) = \mu + 6t_2 + 8s. \tag{4.6}
\]

Therefore, the Euler characteristic of the modular elliptic surface \( X_\Gamma \) is positive and by (4.4) one gets that \( \chi(O_{X_\Gamma}) \) is positive. In particular, the degree of the line bundle \( L^{-1} \) is positive, hence it is an ample divisor on \( Y_\Gamma \).

One should note that:

\[
p_g = h^0(X, \omega_{X_\Gamma}) = h^0(X, f^*(\omega_{Y_\Gamma} \otimes L^{-1})) = h^0(Y, \omega_{Y_\Gamma} \otimes L^{-1}).
\]

Using the Riemann-Roch formula, the Serre duality and the fact that \( h^0(Y, L) = 0 \) (since \( L^{-1} \) is ample on \( Y \)) one gets:

\[
h^0(Y, \omega_{Y_\Gamma} \otimes L^{-1}) = \deg(\omega_{Y_\Gamma} \otimes L^{-1}) - g + 1, \tag{4.7}
\]

i.e.

\[
p_g = \deg(\omega_{Y_\Gamma} \otimes L^{-1}) - g + 1 = g - 1 - \deg L, \tag{4.8}
\]

where \( g \) is the genus of \( Y_\Gamma \).

By (4.4) and (4.8) we get:

\[
q = p_g + 1 - \frac{e_2(X_\Gamma)}{12} = g - 1 - \deg L + 1 + \deg L = g.
\]

Hence, the irregularity of the elliptic modular surface is equal to the genus of the modular curve \( Y_\Gamma \).
By Lemma (4.11) one has:

$$\mu = 12g - 12 + 6t_1 + 6t_2 + 4s.$$  

As we saw before by (4.6) one has:

$$c_2(X_\Gamma) = 12 \deg L^{-1} = \mu + 6t_2 + 8s,$$

hence

$$\deg L^{-1} = g - 1 + \frac{1}{2}t_1 + t_2 + s.$$  

Then by (4.8) we get:

$$p_g = 2g - 2 + \frac{1}{2}t_1 + t_2 + s. \quad (4.9)$$

**Definition 4.21.** The principal congruence subgroup of $SL_2(\mathbb{Z})$ of level $N$ is defined to be:

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | a, d \equiv 1 (\text{mod} \ N); b, c \equiv 0 (\text{mod} \ N) \right\}.$$  

The curve $Y_{\Gamma(N)}$ is called the modular curve of level $N$. The elliptic modular surface $X_{\Gamma(N)}$ attached to the group $\Gamma(N)$ is called the elliptic modular surface of level $N$.

From now on, we will suppose that $N \geq 3$. Following results from [37] and the previous results about elliptic modular surfaces, we will list here several properties of the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$.

**(B1)** The group $\Gamma(N)$ is torsion free, so $s(N) = 0$.

**(B2)** All cusps are of the first kind, hence $t_2(N) = 0$ and $t(N) = t_1(N)$. The set of cusps will be denoted by $T$. This set is the discriminant locus of the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$.

**(B3)** $t(N) = \frac{\mu(N)}{N}$, where $\mu(N) = \frac{1}{2}N^3 \prod_{p \mid N}(1 - \frac{1}{p^2})$, where the product is taken over prime numbers $p$.

**(B4)** From point (A3), the genus of the curve $Y_{\Gamma(N)}$ is given by:

$$g(N) = 1 + \frac{(N - 6)\mu(N)}{12N}.$$  

It is equal to the irregularity $q(N)$ (see (A2)). The geometric genus of the surface $X_{\Gamma(N)}$ is given by:

$$p_g(N) = \frac{N - 3}{6N} \mu(N).$$

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(B5) The elliptic modular surface $X_{\Gamma(N)}$ of level $N$ has $N^2$ disjoint sections over the base curve $Y_{\Gamma(N)}$. These sections will be denoted by $D_i$ and the sum of these sections is $D = \sum_{i=1}^{N^2} D_i$. For $N \geq 4$, the divisor $D = \sum_{i=1}^{N^2} D_i$ is divisible in Pic($X_{\Gamma(N)}$) by $N$ if $N$ is odd or by $\frac{N}{2}$ otherwise.

(B6) From point (A2), the canonical bundle of $X_{\Gamma(N)}$ is given as

$$\omega_{X_{\Gamma(N)}} = f^*(\omega_{Y_{\Gamma(N)}} \otimes L^{-1}),$$

where $L$ is a line bundle on $Y_{\Gamma(N)}$ such that

$$\deg L^{-1} = p_g(N) - q(N) + 1 = \frac{\mu(N)}{12}.$$

(B7) Non-singular fibers of the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$ are non-singular elliptic curves. The divisor $D$ intersects a smooth fiber in $N^2$ points. Singular fibers of the family are above the cusps of $Y_{\Gamma(N)}$. A singular fiber $F$ of the family is of the form $F = \sum_{i=0}^{N-1} F_i$, where $F_i$ are non-singular rational curves with $F_i^2 = -2$. A curve $F_i$ intersects transversely curves $F_{i-1}$ and $F_{i+1}$. Hence, singularities of the divisor $F$ are nodes. The $N$–cycle of these curves is such that every $F_i$ intersects the divisor $D$ in $N$ points, i.e. every curve $F_i$ intersects exactly $N$ sections. The points of intersections of divisors $F$ and $D$ are not the nodes of divisor $F$. It is obvious that the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$ is semistable. Then by Lemma 1.9 one gets:

$$\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} = \Omega^1_{X_{\Gamma(N)}/Y_{\Gamma(N)}}(\log f^*(T)).$$

4.5.2 The construction

We will suppose that $N \geq 4$. Let $\nu \geq 2$ be an integer such that $\nu$ divides $N$ if $N$ is odd or $\nu$ divides $\frac{N}{2}$ if $N$ is even. Hence, by (B5), there exists a line bundle $\mathcal{M}$ in Pic($X_{\Gamma(N)}$) such that:

$$\mathcal{O}_{X_{\Gamma(N)}}(D) = \mathcal{M}^\nu.$$

Now, we can construct a cyclic covering $W_{\Gamma(N)}$ over $X_{\Gamma(N)}$ of degree $\nu$, ramified along the divisor $D$. Since all components of $D$ are disjoint, $D$ is a smooth divisor. Therefore, the cyclic covering $\tau : W_{\Gamma(N)} \to Y_{\Gamma(N)}$ ramified along $D$ is smooth, see Lemma 1.6. The induced family $g = f \circ \tau : W_{\Gamma(N)} \to Y_{\Gamma(N)}$, has singular fibers over the set of cusps $T$ of the curve $Y_{\Gamma(N)}$, which is also the discriminant locus for the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$. There is no fibers of the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$ in the branched locus $D$ of the covering, so
all fibers of the family \( g : W_{\Gamma(N)} \to Y_{\Gamma(N)} \) are reduced, on the other side the branch divisor \( D \) does not contain singular points (nodes) of singular fibers of the family \( f : X_{\Gamma(N)} \to Y_{\Gamma(N)} \) (see (B6)), hence \( \tau^*(D) + g^*(T) \) is a normal crossing divisor, see Remark 1.14. We have that all fibers of \( g : W_{\Gamma(N)} \to Y_{\Gamma(N)} \) are reduced, singular fibers are normal crossing divisors and they do not contain exceptional curves of first kind, hence the family \( g : W_{\Gamma(N)} \to Y_{\Gamma(N)} \) is semistable.

The semistability of the family \( g : W_{\Gamma(N)} \to Y_{\Gamma(N)} \) and Lemma 1.9 yield that:

\[
\omega_{W_{\Gamma(N)}/Y_{\Gamma(N)}} = \Omega^1_{W_{\Gamma(N)}/Y_{\Gamma(N)}}(\log(g^*(T))).
\]

By Lemma 1.16 one has:

\[
\tau_*\omega_{W_{\Gamma(N)}/Y_{\Gamma(N)}} = \omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} \oplus \bigoplus_{i=1}^{\nu-1} \Omega^1_{X_{\Gamma(N)}/Y_{\Gamma(N)}}(\log(f^{-1}(T) + \Gamma_i)) \otimes \mathcal{M}^{-i},
\]

where \( \Gamma_i \) is the sum of components of \( D \) whose multiplicities multiplied by \( i \) are not divisible by \( \nu \). Then it is plain to see that \( \Gamma_i = D \), for all \( i = 1, ..., \nu - 1 \), since \( D \) is a reduced divisor. We get:

\[
f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} \subset \tau_*\omega_{W_{\Gamma(N)}/Y_{\Gamma(N)}}, \quad \text{or}
\]

\[
f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} \subset g_*\omega_{W_{\Gamma(N)}/Y_{\Gamma(N)}}. \quad (4.10)
\]

**Lemma 4.12.** The sheaf \( f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} \) is an invertible sheaf and for any positive integer \( n \) the sheaf

\[
\mathcal{H} = (f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}})^{\otimes n}
\]

is a subsheaf of \( g_*\omega_{W_{\Gamma(N)}/Y_{\Gamma(N)}}^{\otimes n} \) and satisfies the maximal case in the Arakelov inequality for the semistable family \( g : W_{\Gamma(N)} \to Y_{\Gamma(N)} \), i.e.

\[
\deg \mathcal{H} = \frac{n}{2} \deg \omega_{Y_{\Gamma(N)}}(T).
\]

**Proof.** As \( f : X_{\Gamma(N)} \to Y_{\Gamma(N)} \) is an elliptic fibration, the canonical bundle of a generic fiber \( C \) is trivial, i.e. \( \omega_C = \mathcal{O}_C \) which yields \( h^0(C; \omega_C) = 1 \). This means that the rank of the sheaf \( f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} \) is 1, so it is an invertible sheaf.

Also, by (B6) one has:

\[
\deg f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} = \deg f_*(\omega_{X_{\Gamma(N)}} \otimes f^*\omega_{Y_{\Gamma(N)}}^{-1}) = \deg(f_*\omega_{X_{\Gamma(N)}} \otimes \omega_{Y_{\Gamma(N)}}^{-1})
\]

\[
= \deg(\omega_{Y_{\Gamma(N)}} \otimes \mathcal{L}^{-1} \otimes \omega_{Y_{\Gamma(N)}}^{-1}) = \deg \mathcal{L}^{-1} = \mu(N) \frac{12}{12}.
\]

On the other hand, one has:

\[
\deg \omega_{Y_{\Gamma(N)}}(T) = 2g(N) - 2 + t = \mu(N) \frac{6}{6}.
\]

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By (4.10) we have:
\[ H = (f_*\omega_{X_{\Gamma(N)}}/Y_{\Gamma(N)})^{\otimes n} \subset (g_*\omega_{W_{\Gamma(N)}}/Y_{\Gamma(N)})^{\otimes n} \subseteq g_*\omega_{W_{\Gamma(N)}}^{\otimes n} \]
and
\[ \deg H = nf_*\omega_{X_{\Gamma(N)}}/Y_{\Gamma(N)} = n\mu(N)/12 = n/2 \deg \omega_{Y_{\Gamma(N)}}(T). \]

So, for the semistable family \( g : W_{\Gamma(N)} \rightarrow Y_{\Gamma(N)} \), with discriminant locus the set of cusps \( T \), the invertible sheaf \( H = (f_*\omega_{X_{\Gamma(N)}}/Y_{\Gamma(N)})^{\otimes n} \subseteq g_*\omega_{W_{\Gamma(N)}}^{\otimes n} \)
satisfies the case of the equality in the Arakelov inequality.

**Proposition 4.13.** For the family \( g : W_{\Gamma(N)} \rightarrow Y_{\Gamma(N)} \), the curve \( Y_{\Gamma(N)} \setminus T \) is a Teichmüller curve.

**Proof.** This is a consequence of Lemma 3.13 for \( n = 1 \).

Livné showed in his thesis [48] §1.6 that the surfaces \( W_{\Gamma(N)} \) with \( \nu = \frac{N}{N-6} \), i.e. \( N = 7, 8, 9, 12 \), satisfy \( c_1^2 = 3c_2 \). Due to well known Yau’s result [84], the surfaces with \( c_1^2 = 3c_2 \) are quotients of the complex unit 2-ball by a discrete, co-compact, torsion-free subgroup of \( \text{PU(2,1)} \).

In order to prove that all singular fibers in the semistable families:
\[ g : W_{\Gamma(N)} \rightarrow Y_{\Gamma(N)} \text{ for } N \in \{7, 8, 9, 12\}, \]
are totally geodesic let us state and prove one auxiliary lemma which gives an approach for detecting the totally geodesic curves on a smooth complex 2-ball quotient.

**Lemma 4.14.** Suppose \( D \) is a reduced (not necessarily irreducible) curve on a smooth complex two-ball quotient \( X \) self-intersecting only at \( k \) distinct points with simple multiplicities given by \( (b_1, \ldots, b_k) \) and let us denote by \( D_i \) (\( i = 1, 2, \ldots, n \)) its irreducible components, \( \hat{D}_i \) their normalization. Let \( \alpha : D = \bigcup_i \hat{D}_i \rightarrow D \) be the normalization of \( D \). Then \( D \) is totally geodesic if and only if
\[ K_X \cdot D = 3 \sum_{i=1}^{n} (g(\hat{D}_i) - 1). \]

**Proof.** The direction when \( D \) is totally geodesic is Lemma 6 in [11]. Here, we will prove the other direction. We suppose that:
\[ K_X \cdot D = 3 \sum_{i=1}^{n} (g(\hat{D}_i) - 1). \]

Note that we also have:
\[ K_X \cdot D = \int_D c_1(K_X) = \int_D \alpha^* c_1(K_X) = \int_D c_1(\alpha^* K_X), \]

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and
\[ 3 \sum_{i=1}^{n} (g(D_i) - 1) = \frac{3}{2} \sum_{i=1}^{n} \deg K_{D_i} = \frac{3}{2} \sum_{i=1}^{n} \int_{D_i} c_1(K_{D_i}). \]

This yields:
\[ \int_{D} c_1(\alpha^* K_X) = \frac{3}{2} \int_{D} c_1(K_D). \]  

(4.11)

By [9] §1 or Lemma 4.6 we have:
\[ c_1(\alpha^* K_X) = 3\alpha^* \xi, \]
\[ c_1(K_D) = 2\alpha^* \xi + \alpha^* \sigma, \]

where \( \xi \) is the Kähler form of the metric on \( X \) and \( \sigma \) is (1,1)-form, non negative definite at every point of \( D \). It is explained in §1 of [9] that if \( \sigma \) vanishes identically on \( D \), then the second fundamental form vanishes on \( D \), or equivalently \( D \) is totally geodesic. Using that \( \sigma \) is non-negative and equality 4.11, one gets:
\[ \int_{D} 3\alpha^* \xi = \frac{3}{2} \int_{D} (2\alpha^* \xi + \alpha^* \sigma), \]

hence \( \alpha^* \sigma = 0 \) on \( \bar{D} \). As a consequence one has that the irreducible components of \( D \) are totally geodesic curves. \( \square \)

**Lemma 4.15.** In the notations from the beginning of the section, for \( \nu = \frac{N}{N-6} \), the singular fibers in the family
\[ g : W_{\Gamma(N)} \rightarrow Y_{\Gamma(N)} \]
are totally geodesic.

**Proof.** The singular fibers of the fibration \( g : W_{\Gamma(N)} \rightarrow Y_{\Gamma(N)} \) are the fibers over cusps of \( Y_{\Gamma(N)} \) and they are \( \nu \)-cyclic coverings of singular fibers of the family \( f : X_{\Gamma(N)} \rightarrow Y_{\Gamma(N)} \). Recall that the map \( \tau : W_{\Gamma(N)} \rightarrow X_{\Gamma(N)} \) is the \( \nu \)-cyclic covering ramified along the divisor \( D = \sum_{i=1}^{N^2} D_i \).

For a singular fiber of the family \( f : X_{\Gamma(N)} \rightarrow Y_{\Gamma(N)} \), one has:
\[ F = \sum_{i=0}^{N-1} F_i, \]

where \( F_i \cong \mathbb{P}^1 \). Then \( \tau^*(F) = C = \sum_{i=0}^{N-1} C_i \), where the \( C_i \)'s are \( \nu \)-cyclic coverings of \( F_i \cong \mathbb{P}^1 \). The cyclic coverings \( C_i \)'s of \( \mathbb{P}^1 \) are ramified over \( N \) points of \( F_i \), which are not nodes of \( F \). This holds by (B7). Hence, by the Riemann-Hurwitz formula one gets:
\[ \deg K_{C_i} = \nu \deg K_{\mathbb{P}^1} + N(\nu - 1) = -2\nu + N(\nu - 1), \]
and \[ g(C_i) = \frac{(\nu - 1)(N - 2)}{2}. \]

In Livné’s thesis §1.5 we can find that the canonical divisor of the surface \( W_{\Gamma(N)} \) is given by:

\[ K_{W_{\Gamma(N)}} = (\nu - 1) \sum_{i=1}^{N^2} \hat{D}_i + \frac{N - 4}{4N} \mu(N) \hat{\Phi}, \]

where \( \hat{\Phi} = \tau^*(\Phi) \), for \( \Phi \) a smooth fiber in the family \( f : X_{\Gamma(N)} \to Y_{\Gamma(N)} \) and \( \hat{D}_i = \tau^*D_i \).

Using the fact that fibers in a family do not intersect, we have \( F \cdot \hat{\Phi} = 0 \), then \( \tau^*F \cdot \tau^*\Phi = 0 \), i.e.

\[ C \cdot \hat{\Phi} = 0. \]

Again by (B_7) and the fact that \( C_i \) is ramified at \( N \) points one gets

\[ C_i \cdot \sum_{j=1}^{N^2} \hat{D}_j = N. \]

Bringing together all these facts we have:

\[ K_{W_{\Gamma(N)}} \cdot C_i = \left( (\nu - 1) \sum_{i=1}^{N^2} \hat{D}_i + \frac{N - 4}{4N} \mu(N) \hat{\Phi} \right) \cdot C_i = (\nu - 1)N, \]

and

\[ 3 \sum_{i=1}^{N} (g(C_i) - 1) = 3N \frac{N\nu - 2\nu - N}{2}. \]

Now, for \( \nu = \frac{N}{N-6} \) we get:

\[ K_{W_{\Gamma(N)}} \cdot C = N(K_{W_{\Gamma(N)}} \cdot C_i) = 6 \frac{N^2}{N - 6}, \]

and

\[ 3 \sum_{i=1}^{N} (g(C_i) - 1) = 6 \frac{N^2}{N - 6}, \]

which yields that \( C \) is totally geodesic by the previous lemma.

The elliptic modular surface \( X_{\Gamma(12)} \) has 144 sections which form the divisor \( D \). This divisor is divisible in the group \( \text{Pic}(X_{\Gamma(12)}) \) by 6. Hence, we can construct the cyclic covering \( W_{\Gamma(12)} \) over \( X_{\Gamma(12)} \) ramified over the divisor \( D \) of
degree $\nu = \frac{12}{12 - 6} = 2$. Therefore, $W_{\Gamma(12)}$ is a quotient of a complex 2-ball by a torsion free cocompact discrete subgroup of $PU(2, 1)$. All singular fibers of the family $g : W_{\Gamma(12)} \to Y_{\Gamma(12)}$, i.e. the fibers over the set of cusps $T$ of $Y_{\Gamma(12)}$ are totally geodesic. Let us calculate the genus of smooth fibers $W_y(N)$ in the family. By Hurwitz-Riemann one gets:

$$g(W_y(N)) = \frac{144 + 2}{2} = 73,$$

since the divisor $D$ intersect a smooth fiber of the family $f : X_{\Gamma(12)} \to Y_{\Gamma(12)}$ (an elliptic curve) in $N^2 = 144$ points. The invertible sheaf $f_*\omega_{X_{\Gamma(12)}/Y_{\Gamma(12)}}$ is a subsheaf of $g_*\omega_{W_{\Gamma(12)}/Y_{\Gamma(12)}}$ and it satisfies the maximal case in the Arakelov inequality. The curve $Y_{\Gamma(12)} \setminus T$ is a Teichmüller curve in $M_{73}$, the moduli space of curves of genus 73, for the family $g : W_{\Gamma(12)} \to Y_{\Gamma(12)}$.

In the case when $N \in \{7, 8, 9\}$ we get $\nu \in \{7, 4, 3\}$. The invertible sheaves which satisfy the maximal case in the Arakelov inequalities are $f_*\omega_{X_{\Gamma(7)}/Y_{\Gamma(7)}}$, $f_*\omega_{X_{\Gamma(8)}/Y_{\Gamma(8)}}$ and $f_*\omega_{X_{\Gamma(9)}/Y_{\Gamma(9)}}$. The curves $Y_{\Gamma(7)} \setminus T$, $Y_{\Gamma(8)} \setminus T$ and $Y_{\Gamma(9)} \setminus T$ are Teichmüller curves in $M_{148}$, $M_{97}$ and $M_{82}$, for the families $g : W_{\Gamma(7)} \to Y_{\Gamma(7)}$, $g : W_{\Gamma(8)} \to Y_{\Gamma(8)}$ and $g : W_{\Gamma(9)} \to Y_{\Gamma(9)}$, respectively.