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A method of deriving limits for the excentricity of the orbit and for the longitude of periastron of an eclipsing binary, by \( \mathcal{Z} \) Uitterdijk.

1. In the following pages a method will be developed by which, certain conditions being fulfilled, rather close limits for the excentricity of the orbit and for the longitude of periastron of an eclipsing binary may be found from the phases and the widths of both minima. The method has been illustrated by application to a newly discovered binary of high excentricity.

2. We define (see fig. 1).
\( R \) and \( R' \) real distances between the centres of the stars in the middle of the “first” and “second” eclipse respectively.

\[
R' = \mu R \quad \text{(I take} \mu > 1) \\
r_1 \text{ and } r_2 \ldots \ldots \text{radii of the stars.}
\]

(During the “first” minimum a part of the “first” star with radius \( r_1 \) is eclipsed).
\( \pi \sigma_1 \text{ and } \pi \sigma_2 \ldots \ldots \) eclipsed parts of the stars in the middle of the “first” and “second” minimum respectively.

\[ a \text{ and } a' \ldots \ldots \text{projected distances of the centres of the stars in the middle of the “first” and “second” minimum respectively. I take the sum of the radii } r_1 + r_2 \text{ as unit.} \]
\( \lambda_1 \text{ and } \lambda_2 \ldots \ldots \text{ratio between the light in the two minima respectively and the maximum light.} \)

If the eclipses are central, the ratio of the duration of the two minima \( t \) and \( t' \) is with close approximation, according to Kepler's law of areas,

\[
\frac{t}{t'} = \frac{R}{R'} = \frac{1}{\mu'}
\]

because the projected paths of the components over each other are equal. If the eclipses are not central, these paths will be \( 2\sqrt{(1-a^2)} \) and \( 2\sqrt{(1-a'^2)} \), the
sum of the radii \( r_1 + r_2 \) being taken as unit (see fig. 2). In this case the formula becomes

\[
\frac{t}{R} = \frac{1}{\mu} \sqrt{\frac{1 - \alpha^2}{1 - \lambda^2}} = C
\]

In this formula \( \alpha' \) is the larger of the two projected distances of the stars, since \( \alpha' = \alpha \mu \) (which follows from \( R' = \mu R \)).

We shall now try to get the limits of the quantity \( \mu \) from the limits of the quantity \( \alpha' \) (It can easily be seen, that \( \alpha' \) must have a maximum for given ranges, because, if \( \alpha' \) were very large, no eclipse would take place).

3. The minimum of \( \alpha' \) is zero, the maximum is still to be obtained.

We can derive the formula:

\[
(1 - \lambda_1) \frac{R_1^2}{\sigma_1} + (1 - \lambda_2) \frac{r_1^2}{\sigma_2} = 1
\]

The quantity \( \alpha \) is smaller than \( \alpha' \), thus the “first” minimum is the more central. Consider for a moment this first minimum due to a perfectly central eclipse. Then we can calculate for given range and for each adopted ratio of the radii the corresponding intensities of the two components. Turning now to the “second” minimum, we suppose the radii and intensities to be the same as those considered at the first minimum, then from the given range of the second minimum we may compute the value of \( \alpha' \) as defined above. The point is that for every ratio of the radii we get in this way a value for \( \alpha' \), and this value will be larger than the actual value of \( \alpha' \) if the same ratio in the radii is supposed. (The fact is, that the intensity of the “first” star, as calculated by means of the method mentioned, will be less than the actual intensity). It should be understood that this procedure only serves to get a maximum value of \( \alpha' \) and does not claim to represent a dynamically possible orbit, \( \alpha \) being considered = 0 and \( \alpha' \neq 0 \) (Turning in the next sections to the dynamical problem, we also return there to the condition \( \alpha = \alpha \mu \)).

We shall further prove that even an absolute maximum of \( \alpha' \) may be obtained, viz. by supposing \( r_1 = r_2 \), provided \( \lambda_1 \leq \frac{3}{4} \lambda_3 \) (range of 440 magnitude or more). This will be shown in the note (1).

Supposing the range of the first minimum larger than or equal to 440 magnitude, we put therefore:

\[
r_1 = r_2 = \frac{1}{2} \quad \text{and} \quad \alpha_1 = r_1^2
\]

because the first eclipse is total. Formula (2) becomes:

\[
o = \frac{1 - \lambda^2}{4 \lambda_i}
\]

\( o_2 \) is also geometrically given by the formula

\[
\pi o_2 = 2 r_1^2 (\varphi_i - \sin \varphi_i \cos \varphi_i)
\]

\[
a' = 2 r_1 \cos \varphi_i
\]

(see fig. 2; \( \varphi_2 = \varphi_1 \) as \( r_2 = r_1 \)).

Taking \( r_1 = \frac{1}{2} \) and eliminating \( \sigma_1 \) we get:

\[
\varphi_i - \sin \varphi_i \cos \varphi_i = \frac{\pi}{2} \frac{1 - \lambda_2}{\lambda_4}
\]

\[
\cos \varphi_i = \alpha'
\]

by which the maximum value (see above) of \( \alpha' \) is found.

These two formulae have the same aspect as the formulae (1) and (2) of the next paper (p. 248), and thus may be represented by the same table.

4. We shall now derive the limits of the quantity \( \mu \).

Taking \( a = a'/\mu \), we can derive from formula (1) (\( C \) being constant)

\[
\frac{\partial a'}{\partial \mu} = \frac{(1 - \alpha^2) (\mu^2 - 2 \alpha')}{\mu a' (\mu^2 - 1)}
\]

As \( \mu > 1 \) and \( a' < 1 \), \( \partial a'/\partial \mu \) will always be positive, provided \( a' < \frac{1}{\mu} \sqrt{2} \mu \) which will be true in practically all cases. In other words: the function \( \mu = f(a') \) is always increasing with \( a' \), thus the minimum and maximum of \( a' \) correspond to the minimum and maximum of \( \mu \).

So we get, taking \( a' = 0 \) in formula (1):

\[
\mu_{\text{min}} = \frac{1}{C}
\]

or in words: the ratio of the width of the two minima at once gives the minimum of \( \mu = R'/R \).

Taking in (1) the maximum value of \( \alpha' \), as derived in section 3, we obtain:

\[
\mu_{\text{max}} = \frac{1 \pm \sqrt{1 - 4 C^2 (1 - a''_{\text{max}}) a''_{\text{max}}}}{2 C^2 (1 - a''_{\text{max}})}
\]

Now \( \mu_{\text{max}} \) seems to have more than one value in consequence of:

1° the ± sign in formula (8),

2° the definition of the quantity \( C \) which is (see (1)):

\[
C = \frac{t}{\text{width of the “first” minimum}} = \frac{1}{t} \quad \text{width of the “second” minimum}
\]

As the “first” minimum is not identical with the “primary” minimum of the lightcurve, we may choose each of the two minima for the “first” minimum, each choice giving one value for \( a''_{\text{max}} \) and one value for \( C \). The product of the two values of \( C \) is 1.

Now we have always adopted the condition \( \mu > 1 \). This condition rules out some values of \( \mu_{\text{max}} \). The
discussion of formula (8) will be given in note (2).
Here we give only the results:

1°. Choosing the more narrow minimum for the "first" one \( (C < 1) \) we always find one value for \( \mu_{\text{max}} \) by

(2) taking the + sign in formula (8). Taking however the — sign, we never find a value of \( \mu \).

2°. Choosing the wider minimum for the "first" one

\( (C > 1) \), we never find a value of \( \mu \), provided

\( a^2 < \frac{1}{4} + \frac{1}{2} \sqrt{(1 - 1/C^2)}. \)

This condition will be fulfilled in practically all cases.

In practice we thus find one value of \( \mu_{\text{max}} \).

5. The limits of \( \mu \) give us the limits of \( \epsilon \) by means of the formulae of celestial mechanics:

\[
\begin{align*}
\sin E_0 & = -\frac{\mu - 1}{\sin E_0} \\
\epsilon & = \frac{1 - \mu \cos E_0 - \cos E_0}{\mu} \\
E_0' - E_0 & - \epsilon \sin E_0 + \epsilon \sin E_0 = 2 \pi D
\end{align*}
\]

The longitude of periastron is given by:

\[
\sin \omega = \frac{\cos E_0 - \epsilon}{1 - \epsilon \cos E_0}
\]

(9)

\( E_0 \) and \( E_0' \) are the eccentric anomalies at the two minima.

6. An application of this method will be given in section 4 of the next article.

Thus \( a_0 \) will be constant for given ranges. The part of the curves to the right of the line \( k = 1 \) will be lines of constant \( a_0 \).

We know:

1°. Taking \( k \) constant and increasing \( \alpha' \), the value of \( a_0 \) will diminish. In words: taking the same radii and increasing the projected distance of the stars, the eclipsed fraction will diminish.

2°. Taking \( \alpha' \) constant and increasing \( k \), the value of \( a_0 \) will diminish. For the width of the eclipsed part of star 2, \( A_1 A_2 \) (see fig. 2) will be \( r_1 + r_2 - a' = 1 - a' = \text{constant} \), whilst the radius of this star \( r_4 \) is increasing.

According to 1° and 2° \( a_0 \) can only remain constant for increasing \( k \), if we decrease the value of \( \alpha' \). Thus the line of constant \( a_0 \) will ascend in diagram 3 for decreasing \( k \).

Therefore the maximum value of \( \alpha' \) in the region \( k > 1 \) will be reached, taking exactly \( k = 1 \).

We shall now discuss the part of the curve to the left of the line \( k = 1 \).

\( a_0 \) is geometrically given by (see fig. 2):

\[
\begin{align*}
\pi a_0 & = p_1 r_1 + p_2 r_2 - r_3 \sin p_1 \cos p_2 - r_4 \sin p_3 \cos p_4 \\
r_3 \sin p_1 & = r_3 \sin p_2 \\
r_4 \cos p_1 + r_5 \cos p_2 & = a'
\end{align*}
\]

or inserting \( k \) and \( a_0 \):

\[
\begin{align*}
(11) \begin{cases}
k^2 p_0 = p_1 + k^2 p_2 - \sin p_1 \cos p_2 - k^2 \sin p_2 \cos p_3 \\
\sin p_3 = k \sin p_2 \\
\cos p_2 + k \cos p_3 = a' (1 + k)
\end{cases}
\end{align*}
\]

\( k \) being less than 1, the first eclipse will be annular. Thus we must take

\[
\begin{align*}
\frac{r_5}{r_4} & = 1 \\
\text{in formula (2), which becomes:}
\end{align*}
\]

\[
\begin{align*}
a_0 & = \frac{(1 - \lambda_2) k^2}{k^2 - (1 - \lambda_2)}
\end{align*}
\]
Differentiating (11) and (12) with respect to \( \phi \), we get, taking \( \lambda_1 \) and \( \lambda_2 \) constant:

\[
2k \pi a_s + k^2 \pi \frac{\partial a_s}{\partial k} = \frac{\partial \pi}{\partial k} + k^2 \frac{\partial \pi}{\partial k} + 2k \frac{\partial a_s}{\partial k} - \\
- \cos 2 \pi \frac{\partial \pi}{\partial k} - k \cos 2 \pi \frac{\partial \pi}{\partial k} + 2k \sin \pi \cos \pi.
\]

(13 I)

\[
(13 \text{II}) \cos \pi \frac{\pi}{\partial k} = k \cos 2 \pi \frac{\partial a_s}{\partial k} + \sin \pi \frac{\partial a_s}{\partial k} - \\
- \sin \pi \frac{\pi}{\partial k} - k \sin \pi \frac{\partial a_s}{\partial k} + \cos \pi \frac{\partial a_s}{\partial k} = (1 + k) \frac{\partial \pi}{\partial k} + \\
+ a' = (1 + k) \frac{\partial \pi}{\partial k} + \cos \pi \frac{\partial a_s}{\partial k} + \frac{k \cos \pi}{1 + k}.
\]

(13III)

(In 13III \( a' \) is eliminated by means of (11)).

\[
\frac{\partial a_s}{\partial k} = -\frac{2k (1 - \lambda_2)}{[k^2 - (1 - \lambda_2)]^2}
\]

The purpose is, to get the quantity \( \partial a'/\partial k \) in an explicit form in the case of \( k = 1 \).

Therefore we take \( k = 1 \) (and thus \( \pi = \pi_2 \); see fig. 2) in

(11I), (13I), (13III), (12) and (14) respectively:

\[
\pi a_s = 2 \pi_1 - 2 \sin \pi_1 \cos \pi_1
\]

\[
2 \pi a_s + \pi \frac{\partial a_s}{\partial k} = \left[ \frac{\partial \pi}{\partial k} + \frac{\partial \pi_1}{\partial k} \right] (1 - \cos 2 \pi_1) + \\
+ 2 \pi_1 - 2 \sin \pi_1 \cos \pi_1
\]

\[
- \sin \pi_1 \left( \frac{\partial \pi_1}{\partial k} + \frac{\partial \pi_2}{\partial k} \right) + \cos \pi_1 = 2 \frac{\partial a'}{\partial k} + \cos \pi_1.
\]

\[
a_2 = \frac{1 - \lambda_2}{\lambda_2}
\]

\[
\frac{\partial a_s}{\partial k} = -\frac{2 (1 - \lambda_2)}{\lambda_2^2}
\]

Finally we get, eliminating the expressions \( \partial \pi_1/\partial k + \partial \pi_2/\partial k \) and \( 2 \pi_1 - 2 \sin \pi_1 \cos \pi_1 \):

\[
\frac{\partial a'}{\partial k} = \frac{\pi}{4 \sin \pi_1} \frac{(2 - 3 \lambda_1) (1 - \lambda_2)}{\lambda_2^2}
\]

This gives us the following results:

I. \( \lambda_1 < \lambda_2 \). Thus \( \partial a'/\partial k > 0 \). The curve shows a (sharp) maximum at \( k = 1 \).

II. \( \lambda_1 = \lambda_2 \). (Critical range of '440 magnitude). Thus \( \partial a'/\partial k = 0 \). The same case as I, but now the tangent is horizontal.

III. \( \lambda_2 > \lambda_1 \). Thus \( \partial a'/\partial k < 0 \). The maximum will not be at \( k = 1 \).

The results are in agreement with diagram 3. Nevertheless it should be remembered that we have only given a discussion for \( k = 1 \). It is not possible to calculate the quantity \( \partial a'/\partial k \) in an explicit form taking \( k \neq 1 \). But it has been numerically proved, that the curve is always descending in the cases I and II if we decrease the value of \( k \). So there will be always one maximum.

Summary. The problem of getting a maximum value of \( a' \) has been solved. In the cases I and II the solution is given by formulae (4) and (5). In case III \( a'_{\text{max}} \) is found by constructing the curve as has been done in the diagram.

(2) A condition is given by the fact, that the root in formula (8) must be real. Distinguishing the two cases \( C > 1 \) and \( C < 1 \), we get:

First condition:

1°. \( C < 1 \). \( a' \) arbitrary.

2°. \( C > 1 \). \( \quad a' > \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{C^2}} \)

or:

\[
a' > \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{C^2}}
\]

We shall first give the discussion of (8), taking the + sign.

\[
\mu > 1 \quad \text{thus:}
\]

\[
1 + \sqrt{1 - 4 C^2 (1 - a') a'^2} > 2 C^2 (1 - a')
\]

or:

\[
\sqrt{1 - 4 C^2 (1 - a') a'^2} > 2 C^2 (1 - a') - 1
\]

Thus:

\[
2 C^2 (1 - a') - 1 < 0
\]

or:

\[
a' > 1 - \frac{1}{2 C^2}
\]

will be sufficient (not necessary).

If this relation has not been fulfilled, both terms of the inequality (15) will be positive. Therefore the squares of these terms form the following inequality:

\[
1 - 4 C^2 (1 - a') a'^2 > 4 C^2 (1 - a') - 4 C^2 (1 - a' + 1)
\]

or:

\[
(C^2 - 1)(1 - a') < 0
\]

or: \( C < 1 \). Therefore \( C > 1 \) will be impossible, unless the relation (16) is fulfilled. Thus we have obtained:

Second condition:

1°. \( C < 1 \). \( a' \) arbitrary.

2°. \( C > 1 \). \( a' > \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{C^2}} \)

Turning to the first condition in the beginning of this note, we see, that for \( C > 1 \) only the condition

\[
a' > \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{C^2}}
\]

remains (because

\[
\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{C^2}} > 1 - \frac{1}{2 C^2}
\]

which easily can be proved).

Summary:

1°. \( C < 1 \). \( a' \) arbitrary.

2°. \( C > 1 \). \( a' > \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{C^2}} \)

We shall now give the discussion of (8) taking the — sign. Then we must have:

\[
1 - \sqrt{1 - 4 C^2 (1 - a') a'^2} > 2 C^2 (1 - a')
\]

or:

\[
\sqrt{1 - 4 C^2 (1 - a') a'^2} < 1 - 2 C^2 (1 - a')
\]