COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

The expanding universe. Discussion of Lemaître’s solution of the equations of the inertial field, by W. de Sitter.

1. The differential equations.

In B. A. N. 185 it was pointed out that neither of the two possible static solutions of the differential equations

\[ G_{\mu\nu} - \frac{1}{3} \varepsilon_{\mu\nu} G + \lambda g_{\mu\nu} + \pi T_{\mu\nu} = 0 \]

The equations (1) then become, if we denote differential quotients \( \frac{d}{dt} \) by dots,

\[ \frac{2}{R} \dot{R} + \frac{R^2}{R^3} + \frac{1}{3} \left( \lambda + \pi \rho \right) = \frac{\lambda}{R^3} \]

and the equation of energy is

\[ \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \pi) = 0. \]

Lemaître puts

\[ \frac{\rho}{\dot{R}} = -\frac{\alpha}{\dot{R}}, \quad \pi = \frac{\beta}{R^3}. \]

The equation (4) can then be written

\[ R \dot{\alpha} + 3 \dot{\beta} = 0. \]

The three equations (3) and (4), or (3) and (4') are not independent of each other: (4') can easily be derived from (3). We will use the second of (3) and (4'). An assumption regarding \( \alpha \), or \( \beta \), must evidently be added in order to make a complete solution possible. The total volume of space is \( \pi^2 R_{3} \), consequently the total mass is, by (5), \( \pi^2 \alpha / R_{3} \). This Lemaître takes to be constant, and consequently by (4') \( \beta \) is also constant. Lemaître takes \( \beta = 0. \)

If we put

\[ y = R \sqrt{\lambda}, \quad \alpha + \delta = \alpha \sqrt{\lambda}, \quad \epsilon = 3 \beta \sqrt{\lambda}, \]

\( \lambda \) being a constant number, which must evidently be positive, then \( y, \delta, \epsilon \) are pure numbers, independent of the choice of units. The equations then become

\[ y' = \frac{1}{3} \lambda \left( y^2 - 3 y^2 + \alpha y + y^2 + \beta \right) \]

\[ y^2 + \epsilon = 0. \]

It will be shown in article 5 that there is observational...
evidence that \( \delta \) and \( \varepsilon \) are both very small, and \( \Lambda \) is supposed to be so chosen that also \( \delta \) itself is small. Lemaître takes \( \Lambda = 2 \) for reasons which will appear further on.

The constant \( \lambda \) is an absolute constant of the dimension \( L^{-3} \), and can be made equal to unity by an appropriate choice of the unit of length. If this is done, and if \( \delta \) and \( \varepsilon \) are neglected, and \( \Lambda = 2 \) is adopted, the equation for \( y \) becomes

\[
(7') \quad y^2 = \frac{3y^3 - 3y^2 + 2}{3y}.
\]

2. Integration.

Although (7) is probably the simplest form to which the equations can be reduced, we will use a slightly more general form, which is derived from the second equation of (3) by putting

\[
x = \frac{R}{R_0},
\]

\( R_0 \) being a certain initial value. The equation then becomes of the form

\[
x^3 = \frac{1}{3} \lambda \frac{Z}{x^3},
\]

where \( Z \) is of the fourth degree in \( x \), the term in \( x^3 \) being absent. By an appropriate choice of \( R_0 \) we can give the four roots of the equation \( Z = 0 \) the values \( 1 \pm \sqrt{1 - \hat{b}} \) and \( 1 \pm \sqrt{1 - \hat{b}} \). We then have

\[
(9) \quad Z = (x^3 - 2x + 1 + a)(x^3 + 2x + b)
\]

and

\[
(10) \quad \frac{R_0}{R} = \frac{6(1 + a - b)}{3 - a - b}, \quad \frac{R}{R_0} = \frac{(1 + a - b)}{3 - a - b}.
\]

We will assume provisionally that \( \alpha \) and \( \beta \), and consequently also \( a \) and \( b \), are constants.

Comparing with (6) we have

\[
\lambda = \frac{2(1 + a - b)}{(1-a+b)/3} = \frac{(1+a)b}{(1-a+b)^3},
\]

thus \( \lambda = 2, \delta = \varepsilon = 0 \) corresponds to \( a = b = 0 \). Like \( \delta \) and \( \varepsilon \), \( a \) and \( b \) are pure numbers, independent of the choice of units.

If we introduce a new unit of time, by putting

\[
(11) \quad \tau = \sqrt[3]{\frac{\lambda}{3}} \cdot ct,
\]

the relation between \( x \) and \( \tau \), i.e. between the radius of the universe and the time, is given by

\[
(12) \quad \tau - \tau_0 = \int_0^{\tau} \frac{dx}{X \sqrt{x^2 + a}},
\]

\( \tau_0 \) being a constant. If we put further

\[
(13) \quad x^2 = x - 1, \quad X^2 = x^2 + 4x + B^2, \quad B^2 = 3 + b,
\]

the integral (12) becomes

\[
(12') \quad \tau - \tau_0 = \int_0^1 \frac{dx}{X \sqrt{x^2 + a}},
\]

which is an elliptic integral of the third kind. In the case \( a = 0 \) it can be integrated by logarithms, thus:

\[
(14) \quad \tau - \tau_0 = \ln (x + x + 2) + \ln \left( \frac{x + X - B}{x + X + B} \right).
\]

The first term becomes positively infinite for \( x = \infty \), the second term becomes negatively infinite for \( x = 0 \), i.e. \( x = 1 \), \( R = R_0 \). The radius of the universe thus increases from \( R_0 \) at \( t = -\infty \) to infinity at \( t = +\infty \), both the initial and the final value being reached asymptotically. This is the solution of Lemaître, who, however, only considers the case \( b = 0 \), \( B = \sqrt{3} \).

3. Special cases.

The condition \( a = 0 \) is the condition that the equation \( Z = 0 \) shall have a double root, which, by the choice of \( R_0 \), has the value \( x_0 = 1 \). If \( a \) is different from zero this double root separates into two separate ones \( x_0 = 1 \pm \sqrt{-a} \). If \( a \) is negative these are real, and the larger of the two, \( x_0 = 1 + \sqrt{-a} \), must be taken as the lower limit of the integral: the radius increases from \( R = R_0 \) to \( R = \infty \). If \( a \) is positive the two roots are imaginary and the lower limit of the integral must be taken zero: the radius increases from \( R = 0 \) to \( R = \infty \). In both cases the upper limit \( R = \infty \) is still reached asymptotically, but the time taken to reach any finite value \( R \) from the initial value \( R_0 \) or \( 0 \) is finite, and of the order of magnitude of the radius \( R_1 \) or \( R_0 \), itself, i.e. of the order of \( 10^9 \) years. This short time scale would tempt us to assume, as Lemaître does, that the true value of \( a \) in the actual universe is zero. It should be remarked, however, that this does not help us very much. The logarithmic infinity of \( -\tau \) does not begin to assert itself until very near the limiting value \( x = 1 \), and the time taken by the radius to increase from any initial value \( x_t \), \( t \) to a larger value \( x_t \), is practically the same for all values of \( a \) for which the value \( x_t \) can be reached at all. In consequence of (10) the value of \( a \) is limited to 

\[
-1 + \frac{1}{2} < a < \frac{3}{2} - b.
\]

The value of \( b \) has very little influence on the value of the integral. Evidently, by (10), \( b \) lies necessarily

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between the limits 0 and the smaller of the two quantities \(3 - a\) and \(1 + a\). In the actual universe it is probably very small, of the order of \(10^{-5}\), as will be shown below in art. 5.

In the accompanying diagram the relation between \(\tau\) and \(z\) has been plotted for some special values of \(a\) and \(b\). These are, \(\tau_{\infty}\) being \(\tau - \tau_0\) for \(x = z_0\).

I. \(b = 0\), \(a = 0\) \(\tau_0 = 0\) \(\tau_0 = -\infty\)

II. \(b = 0\), \(a = 0.01\) \(\tau_0 = 1.1\) \(-3.35\)

III. \(b = 0\), \(a = 0.1\) \(\tau_0 = 0\) \(-1.80\)

IV. \(b = 0\), \(a = 0.5\) \(\tau_0 = 1.3162\) \(-2.05\)

V. \(b = 0\), \(a = 1\) \(\tau_0 = 0\) \(-1.25\)

VI. \(b = 1\), \(a = 0\) \(\tau_0 = 2\) \(-0.49\)

VII. \(b = 1\), \(a = 0\) \(\tau_0 = 1\) \(-\infty\)

i.e. an entirely empty universe, without either matter or radiation. The integral in this case is

\[
\tau - \tau_0 = \frac{\log (\varepsilon + V z^2 - 4)}{\varepsilon + V z^2 - 4},
\]

which is equivalent to

\[
z = 2 \cosh (\tau - \tau_0),
\]

or

\[
R = R_0 \cosh (\tau - \tau_0), \quad R_0 = 2 R_\infty.
\]

This is identical to the solution (B), viz:

\[
ds^2 = -R_\infty^2 \left[ d\rho^2 + \sin^2 \rho (d\varphi^2 + \sin^2 \psi d\psi^2) \right] + R_\infty^2 \cos^2 \rho \, d\zeta^2.
\]

By the transformation

\[
\sin \rho = \cosh \tau, \quad \sin \chi = \tanh \vartheta = \tanh \tau, \quad \sec \chi = \sec \vartheta
\]

the line-element (17) is transformed into

\[
ds^2 = -R_\infty^2 \cosh^2 \tau \left[ 2 \vartheta^2 + \sin^2 \chi (d\vartheta^2 + \sin^2 \psi d\psi^2) \right] + R_\infty^2 \, d\tau^2.
\]

We have in this case, by (10), \(R_\infty^2 \lambda = \frac{1}{4}\), and consequently, by (11), \(R_\infty d\tau = c \, dt\).

4. Determination of the constants.

The radius vector is \(r = R\chi\). Consequently the radial velocity is

\[
V_c = R\chi + \chi R.
\]

If the coordinate \(\chi\) has no systematic motion, which will be shown below to be the case, the systematic radial velocity is proportional to the radius vector. Thus:

\[
\frac{V}{c} = \frac{R}{R} = \frac{z}{z}.
\]

We have found in B. A. N. 185 that the extragalactic nebulae do show a systematic radial velocity proportional the distance, and we have determined the ratio

\[
\frac{V}{c} = \frac{1}{R_B}, \quad R_B = 2000 ~ A.
\]

By (8) and (18) we have thus

\[
\frac{1}{R_B} = \frac{\lambda z^2}{3 \varepsilon^4},
\]

the suffix 1 denoting the values at the present moment.

In Einstein’s solution (A) we have, on the other hand

\[
\frac{2}{R_A^3} = z (\rho + 4 \beta),
\]

1) The solution (B) can thus be considered as a static or a non-static solution at will. This is possible due to the fact that in it the four-dimensional space is isotropic and of constant curvature.

2) See de Sitter, On Einstein’s theory of gravitation and its astronomical consequences, third paper, M. N., lxxxvii (1917), p. 21, footnote. It should be noted that the quantity called \(\rho_0\) in the formulas of that footnote is our \(\rho = \rho_0 + 3 \beta\).
which, with $\rho_o = 2.10^{-38}$ gr. cm$^{-3}$, $\rho = 0$, gives

$$R_a = 2300 \, A$$

and consequently, by (5) and (10),

$$\frac{1}{R_a} = \lambda \cdot \frac{3}{2} \frac{(1 + a - b)}{z_1} \frac{(1 + a) b}{3 z_1^2}$$

Eliminating $\lambda$ from (19) and (20) we find for $z_1$ the equation of the fourth degree

$$(21) \quad z_1^4 - 2z_1^3 + 1 + a (z_1^2 + 2z_1 + b) = \frac{R_a^2}{R_b^2} [3 (1 + a - b)^2 + 2 (1 + a) b]$$

If we put $b = 0$, the equation becomes the third degree, viz:

$$(21') \quad z_1^3 - 2z_1 - 2 + a (z_1 - 2) = 0,$$

and substituting the value of $R_a/R_b = 1.15$, or very nearly

$$\frac{R_a^2}{R_b^2} = \frac{4}{3}$$

we find

$$(21'') \quad z_1 = 2,$$

independent of the value of $a$. This is an accidental circumstance. If we had taken any other value of $R_a/R_b$, the value of $z_1$ would, of course, depend on $a$.

Then from (19) or (20) we can find $\lambda$, and then $R_o$ from the first of (10). If we take $b = 0$ we have

$$\lambda = \frac{1}{1 + a} \cdot \frac{z_1^2}{R_a^2}$$

and

$$R_o = \frac{3}{3 - a} \cdot \frac{R_a^2}{z_1^2}$$

Now, if $R_a^2/R_b^2$ is not too small, the preponderating terms in (21') are

$$z_1^3 \approx \frac{3}{2} \frac{R_A^2}{R_B^2} (1 + a)$$

or

$$\frac{R_a^2}{z_1^2} \approx \frac{R_b^2}{3 (1 + a)}$$

and consequently

$$\lambda \approx \frac{3}{R_B^2}, \quad R_o \approx \frac{R_b^2}{3 - a}$$

The values of $\lambda$ and $R_o$ thus depend almost entirely on $R_b$, and not on $R_A$, as might be suggested by (23). It has been already remarked in B. A. N. 185 that the uncertainty of the adopted value of $R_b$ is probably not more than corresponds to a probable error of a fourth or a fifth of the amount, while $R_A$, which

depends on $V_{\rho_o}$ may be uncertain by a factor of 10, or more. The product $R_b^2 \lambda$ is, of course, independent of both $R_A$ and $R_b$.

Consequently $R_o$ and $\lambda$ are known within narrow limits of uncertainty. The present radius $R_1 = R_o z_1$, depends on $R_A$, and is much less certain.

Taking $R_a^2/R_b^2 = 4/3$ we find for some of the special cases considered in the preceding article

$$\begin{array}{cccc}
1: & R_o \lambda = 1, & z_1 = 2, & R_0 = 816A, \lambda = 1.15 \\
II: & 0.997 & 2 & 811 \\
III: & 1.003 & 2 & 819 \\
VII: & 1.5 & 1.623 & 877 \\
\end{array}$$

For the case VI, $b = 0, a = -1$, we have $\rho_o + 4\eta = 0$, and the formulas of the present article are not applicable. They would lead to $\lambda = \infty$, $R_o = 0$, which is devoid of meaning.

If we take another value for $\rho_o$, and consequently for $R_a^2/R_b^2$, the values of $z_1$, $R_o$ and $\lambda$ are changed, $R_b^2 \lambda$ remaining the same. For the case I ($a = 0, b = 0$) taking $\rho_o = 2.10^{38}$, $R_a^2/R_b^2 = 400/3$, we find

$$(24') \quad z_1 = 7.49, \quad R_0 = 1126A, \quad \lambda = 0.787 \times 10^{-6} A^{-2}$$

By (23) the maximum value of $R_o$ (for very small $\rho_o$) is $\lambda R_b V_3 = 1155 A$.

As a compromise it is perhaps convenient to adopt an intermediate value, which gives $R_o = 1000 A$, $\lambda = 10^{-6} A^{-2}$ for case I. This value is

$$\frac{R_a^2}{R_b^2} = 7.2, \quad \rho_o = 3.73 \times 10^{-39}.$$
away energy, by which their mass is diminished. According to the formulas (15) and (16) of B. A. N. 185 the absolute magnitude (visual) of the spiral nebulae is \(-15.56\) and of the elliptical nebulae \(-16.06\). The latter number is too uncertain to merit any confidence. If we adopt the former, the difference in absolute magnitude with the sun is \(-20.4\), and the radiation from one spiral consequently is \(1.5 \times 10^8\) times that of the sun, or \(2.10^{48}\) grams per year. The assumed mass was \(2.10^{44}\) grams. Thus, if we chose the unit of time so as to make \(c=1\) corresponding to our unit of distance of \(1\ A=10^{24}\) cm, i.e. approximately a million years, the rate of conversion of matter into energy would be:

\[
\frac{\dot{\alpha}}{\alpha} = -10^{-10}.
\]

(27)

The radiation may have been underestimated, as we have neglected the reduction from visual to bolometric magnitude for the spirals, but, so far as the stellar radiation from the spirals is involved, this can hardly amount to more than one or two tenths of a magnitude, and also other radiations — penetrating radiation, etc. — cannot contribute much. The uncertainty on this account does probably not amount to more than a factor of 2. On the other hand the mass may have been overestimated. Perhaps on the whole \(10^{-9}\) may be thought more probable than the value (27). This would correspond closely to the rate of generation of energy by a dwarf of somewhat later type than the sun.

We can measure the rate of conversion of matter into radiant energy against the rate of expansion of the universe, by putting

\[
\frac{\dot{\alpha}}{\alpha} = -\gamma \frac{\dot{R}}{R},
\]

\(\gamma\) being positive. The value (27) of \(\dot{\alpha}/\alpha\) gives for the value of \(\gamma\) at the present moment \(1\) \(\gamma = 2.10^{-7}\). It is not probable that \(\gamma\) will be a constant throughout the evolution of the universe, but as its true value for the distant past and future is unknown, the best we can do is to treat it as a constant. Consequently

\[
\alpha = \alpha_0 R^{-\gamma},
\]

and consequently

\[
\nu R = \frac{\alpha_0 \gamma}{3(1-\gamma)} R^\gamma.
\]

(29)

Then (49) gives

\[
\beta = \beta_0 + \frac{\alpha_0 \gamma}{3(1-\gamma)} R^{1-\gamma},
\]

and consequently

\[
\nu \rho = \frac{\beta_0}{R^1} + \frac{\alpha_0 \gamma}{3(1-\gamma)} \cdot \frac{1}{R^2 + \gamma}.
\]

(30)

The second term is the pressure of the radiation produced by the radiating matter. If we suppose that there is no radiation in the universe not emanated from matter, then the first term represents the kinematical pressure, corresponding to the random motions of the extragalactic nebulae, considered as the molecules of a gas. According to HUBBLE, and to our own finding in B. A. N. 185, these random motions are of the order of \(150\) km/sec. or \(0.5 \times 10^{-3}\). The ratio \(\rho/\rho_0\) between the kinematical pressure and the density is equal to the square of this, or \(1.10^{-6}\). Consequently, if we neglect the second term of (30), we have by (10) for the present moment:

\[
\frac{(1+a) \beta}{6(1+a-b) \alpha} \frac{1}{a^{-1} - \gamma} = \frac{1}{4} \times 10^{-6}.
\]

It follows that \(a\) is of the order of magnitude of \(10^{-6}\) or \(10^{-5}\).

Neglecting the second term of (30), i.e. treating \(a\) and \(\beta\) as constants, we have

\[
\frac{\rho}{\rho} = \text{const},
\]

and consequently the random motions should decrease as \(1/\sqrt{R}\).

Turning now to the second term of (30), we find that the total amount of radiation pressure in the universe, so far as it originates from the radiation of the stars and other matter, is \(\pi^2 R^2 \beta\), or

\[
\pi^2 \frac{\alpha_0 \gamma}{3(1-\gamma)} R^{-\gamma},
\]

and consequently, since \(\gamma\) is positive, the total amount of radiation decreases as a consequence of the conversion of matter into radiation. The explanation of this paradox is simple. By the adiabatic expansion of the universe the pressure is diminished, and this more than counterbalances the increase by the conversion of matter into radiation. This can easily be verified by making up the account of loss and gain of energy. If we denote the total amount of radiative energy in the universe by \(E\), and the total material mass by \(M\), the change of energy is

\[
\dot{E} = \dot{E}_1 + \dot{E}_2,
\]

\(1\) In a paper (Proc. Nat. Acad. of Sci., Washington, April 1930), which comes to hand while the present note is being prepared for the press, R. C. TOLMAN gives an approximate solution of the equations (3), which leads to the value \(\gamma = 3\) (ibid. p. 331). This seems inadmissible. See also the last article of the present paper.

\(2\) Strictly speaking we should use the ratio \(\beta/(\rho_0 + 4 \beta)\) (see EDDINGTON, Mathematical theory of relativity, p. 122).
where $\dot{E}_t$ is the gain by the conversion of matter, thus

$$\dot{E}_t = -\dot{M} = \gamma \frac{R}{R} \dot{R},$$

and $\dot{E}_s$ is the loss by degradation consequent on the increase of wavelength corresponding to the receding velocities by Doppler's principle. Thus, by Planck's equation $E = h \nu$, we have

$$\frac{\dot{E}_s}{E} = \dot{\nu} = -\gamma \frac{\dot{R}}{R},$$

as will be shown in the next article. Consequently

$$\dot{E} = \frac{\dot{R}}{R} (\gamma M - E),$$

or, since $M = \pi^* R^2 \rho_0/\kappa$, by (29)

$$ERK + R\dot{E} = \gamma M \dot{R} = \pi^* \kappa_0 \gamma R^{-\gamma} \dot{R},$$

from which

$$(32) \quad E = \frac{\pi^* \kappa_0 \gamma}{\kappa} R^{-\gamma},$$

which, by $\pi^* R^3 \rho/\kappa = \frac{1}{2} E$, is identical with the second term of (30).

The new theory thus incidentally gives an answer to the old question what becomes of the energy that is continually being poured out into space by the stars. It is used up, and more than used up, by the work done in expanding the universe. Nevertheless it would not be correct to say that the universe is expanded by the radiation pressure. It would expand just the same if $\gamma$ were zero, i.e. if no radiation was emitted by matter. The expansion is due to the constant $\lambda$.

In the integration of the differential equation for $ER$ we have omitted to add a constant of integration. This would in (32) give an additional term

$$\frac{E_0}{R},$$

which represents the initial energy of radiation, if any, not emanated from matter, and forms part of the first term of (30). This, like the kinematical pressure, diminishes proportional to $1/R$ by the adiabatic expansion of the universe.

It should be kept in mind that the approximation $\gamma = \text{constant}$ is not supported (nor contradicted) by any observational evidence. The formulae derived in the present article can thus only be considered as describing the state and rate of change of the universe at the present moment, and must not be used for extrapolation into the distant past or future.


We have seen in the preceding article that $\dot{\nu}$ is very small. We can thus, in discussing the phenomena in the actual universe at the present moment, as a good approximation treat $\alpha$ and $\beta$, and consequently also $a$ and $\delta$, as constants, as was already done above. Also we can, with sufficient approximation, take $b = 0$.

For a ray of light $ds = 0$, and therefore $d\sigma = c dt/\dot{R}$. The equation of a ray travelling between two points characterised by the coordinates $\sigma$ and $\sigma_t$ is thus

$$(33) \quad \sigma_t - \sigma = c \int_{\sigma}^{\sigma_t} \frac{dt}{\dot{R}} = V \frac{3}{R_o \lambda} \int_{z}^{z_t} \frac{dz}{Z},$$

where $z_t$ and $z$ are the radii of the universe, in $R_o$ as unit, at the times when the light travelled through the points $\sigma_t$ and $\sigma$ respectively. In the case $I$ of art. 3 ($a = o$, $b = o$) we have thus

$$(34) \quad \sigma_t - \sigma = \frac{1}{\sqrt{R_o \lambda}} \left| \log \frac{x + X - V}{x + X + V} \right|.$$
from (33) with the value of \( z \) derived from (35).

For the case I \((a = 0)\) of art. 3 we have the following comparison between \( \sigma' = r / R \) computed by (18) and 

\[ -\sigma \] 

by (35) and (34). For \( \sigma \), we have taken \( \sigma = 3 \) and to convert \( \tau \) into years we have used the corresponding factor 1850, \( \tau \) itself being computed by (14).

<table>
<thead>
<tr>
<th>( z )</th>
<th>( \frac{V}{c} )</th>
<th>( \Omega )</th>
<th>( \frac{d \lambda}{\lambda} )</th>
<th>( \frac{\sigma}{\sigma'} )</th>
<th>( \pi )</th>
<th>( \tau )</th>
<th>( t ) 10^6 years</th>
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<tr>
<td>1.05</td>
<td>1.342</td>
<td>1</td>
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<td>2.971</td>
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<td>+0.31</td>
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</table>

When the distance \( -\sigma \) given in the fourth column exceeds \( \pi = \pi 141 \), the light has travelled entirely round the world before reaching us. It will be seen that light emitted after the epoch corresponding to the radius \( z \), which makes \( \pi - \sigma = \pi \) in (33) with \( z = \infty \), can never complete the circuit of the world, the expansion being too quick for light to overtake it by a complete circuit.

For the case I this value is \( z = 1.0728 \). For the case II light which is emitted at the origin of time \((z = 1.1)\) can never travel a further distance than \( 0.68 \pi (z = \infty) \), which is reached asymptotically for \( t = \infty \).

It is important to note that the velocity of light is independent of the space-coordinates. In the coordinates \( \chi, \psi, \theta \) the rays of light are geodesics described with constant velocity. It follows that on triangles formed by rays of light the ordinary spherical trigonometry is applicable. Consequently the parallax is given by

\[ \varphi = \frac{a}{\tan \chi} \]

and the apparent diameter by

\[ d = \frac{d}{\sin \chi} \]

The intensity of light is given by

\[ I = \frac{I_0}{\sin^2 \chi} \]

Consequently the diameter and the magnitude still follow the same law, log \( \sin \chi \) taking the place of \( \log r \) in the formulas of B.A.N. 185, as well for \( \log r_d \) as \( \log r_m \).

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1) This corresponds to \( \phi = 4.10^{-9} \), \( R_A / R_B = 20/3 \), instead of (26).

---

7. Motion of a Material Particle.

The equations of motion of a material point under the influence of inertia alone in the expanding world can easily be derived. The Christoffel symbols for the coordinates \( \chi, \psi, \theta, \phi \) are:

\[
\begin{align*}
(22) & = -\sin \chi \cos \chi, & (33) & = -\sin \chi \cos \chi \sin^2 \psi \\
(12) & = \cos \chi, & (33) & = -\sin \psi \cos \psi, \\
(23) & = \cot \psi, & (33) & = R \sin^2 \chi \sin^2 \psi.
\end{align*}
\]

We can, as always, suppose \( \psi = \frac{1}{2} \pi, d\psi/ds = 0 \), and consider the coordinates \( \chi, \theta \) only. The equations for the geodesic then become

\[
\begin{align*}
\frac{d\varphi}{ds} &= -2 \frac{R}{R} \frac{d\varphi}{ds} \\
\frac{d\varphi}{ds} + 2 \cot \chi \frac{d\chi}{ds} &= -2 \frac{R}{R} \frac{d\varphi}{ds}.
\end{align*}
\]

Putting

\[
\varphi = \left(\frac{d\varphi}{ds}\right)^2 = \left(\frac{d\chi}{ds}\right)^2 + \sin^2 \chi \left(\frac{d\theta}{ds}\right)^2,
\]

\[ \gamma = \sin^2 \chi \frac{d\theta}{ds}, \]

we derive the two equations

\[
\begin{align*}
\frac{d\varphi}{ds} &= -4 \frac{R}{R} \frac{d\varphi}{ds} \\
\frac{d\gamma}{ds} &= -2 \frac{R}{R} \gamma \frac{d\varphi}{ds},
\end{align*}
\]

from which we find easily

\[
\varphi = \frac{\varphi_0}{R^2}, \quad \gamma = \frac{\gamma_0}{R^2},
\]

corresponding to the integrals of energy and areas in the stationary universe.

Eliminating \( ds \), we have for the differential equation of the track:

\[
\frac{\varphi^2}{\Gamma^2} = \frac{d\varphi^2 + \sin^2 \chi d\theta^2}{\sin^2 \chi d\theta^2} = \text{const.} = \sec^2 \chi \varphi^2,
\]

giving

\[
(36) \quad \tan \chi \cos (\theta - \theta_0) = \tan \chi \varphi,
\]

which is the equation of a geodesic (great circle) in three-dimensional space. To find the velocity, we must introduce the time instead of the interval \( ds \). We have

\[
\left(\frac{d(\varphi)}{ds}\right)^2 = 1 + \frac{R^2}{R^2} \left(\frac{d\varphi}{ds}\right)^2 = 1 + R^2 \varphi^2.
\]
Thus, putting
\[ \varphi = \left( \frac{d\chi}{d(\sigma t)} \right) + \sin^2 \chi \left( \frac{d\theta}{d(\sigma t)} \right)^2 = \left( \frac{d\sigma}{d(\sigma t)} \right)^2, \]
\[ G = \sin^2 \chi \frac{d\theta}{d(\sigma t)}, \]
we have
\[ \varphi = v \frac{d(\sigma t)}{ds}, \quad \Gamma = G \frac{d(\sigma t)}{ds}. \]

The velocity \( v \) is thus given by
\[(\frac{R}{v})^2 = -\frac{\varphi_o^2}{\varphi + \varphi_o^2}. \tag{37}\]

The random velocities are thus decreasing, and unless \( \varphi_o \) is very large, are practically proportional to \( t/\bar{R} \). This is in contradiction with the result found in art. 5 (formula (31), page 215) that the random velocities decrease as \( t/\sqrt{R} \). The formula (31), of course, supposes \( \alpha \) and \( \beta \) to be constant, of which assumption the present result is independent. But this can hardly be the complete explanation of the paradox. The question must be left open for the present.

For a ray of light we have, of course, \( \varphi_o = \infty \), and (37) gives \( v = 1/\bar{R} \), as has been found already (art. 6).

It should be noted that the results of this article are entirely independent of the integration of the differential equation for \( R \), and of the assumption that \( \alpha \) and \( \beta \) are constant.

8. Miscellaneous remarks.

In an interesting paper, 1) which was published while the present communication was being prepared for the press, Professor R. C. TOLMAN independently derives the same equations that are used by LEMAITRE. If in his equations (34) \( R.e^{ir} \) is called \( \bar{R} \), they become identical with our equations (3). Using the notation of the present paper, TOLMAN’s solution is \( R = R_o e^{kt} \).

If to (3) we add the condition \( k \) = constant, or
\[ \frac{d^2}{dt^2} (\log R) = \frac{\dot{R}}{R} - \frac{\dot{R}^2}{R^2} = 0, \]
they can be completely integrated. We find easily (with the help of (4'), if desirable)
\[ \alpha = 6 R - 4 \beta_o R^3, \]
\[ \dot{\beta} = \beta_o R^4 - R^2, \]
\[ R = R_o e^{k(t - \omega)}, \]
\[ 3 k^2 + \beta_o = \lambda. \tag{38}\]

If we add the further condition, as TOLMAN does,