COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

On the flattening and the constitution of the earth, by W. de Sitter.

1. The outer potential of any body, which is symmetrical with reference to an equatorial plane and an axis perpendicular to this plane, may be developed in a series, which, to the order of accuracy here required is

\[ V = \frac{M_0}{r} \left[ 1 - \frac{J}{\frac{3}{2} r^3} P_2 (\sin \delta) + \frac{K}{r^4} P_4 (\sin \delta) \right], \]

where \( M_0 \) is the total mass, \( f \) the constant of gravitation, and \( b \) the equatorial radius. \( P_2 \) and \( P_4 \) are spherical harmonics of the declination \( \delta \) above the equatorial plane, and \( J \) and \( K \) are constants characteristic of the body. We have

\[ J = \frac{\frac{5}{3} C - A}{M_0 b^3}, \]

\[ C \text{ and } A \text{ being the moments of inertia with respect to the polar and an equatorial diameter respectively.} \]

If we introduce the ratio

\[ H = \frac{C - A}{C}, \]

we have

\[ J = qH \]

or

\[ q = \frac{\frac{5}{3} C}{M_0 b^3}. \]

If the surface of the body is a equipotential surface, its equation may be written as that of a spheroid

\[ r = \delta \left[ 1 - \varepsilon \sin^2 \varphi' - \left( \frac{1}{2} \varepsilon^2 + \chi \right) \sin^2 2\varphi' \right], \]

\( \varphi' \) being the geocentric latitude and \( \varepsilon \) the flattening, i.e. if \( c \) is the polar radius:

\[ \varepsilon = \frac{b - c}{b}. \]

This spheroid deviates from an ellipsoid by a depression \( -\frac{1}{2} \varepsilon^2 \sin^2 2\varphi', \) reaching its maximum at the latitude \( 45^\circ. \)

If the body rotates with a constant angular velocity \( \omega, \) so that at its surface the potential is \( V, \) being the value of \( V \) for the surface, \( \delta = \varphi' \)

\[ V = \frac{M_0}{r} + \frac{\frac{5}{3} \omega^2 r^2 \cos^2 \varphi' \varepsilon}{M_0 b^3}, \]

then the conditions that the surface shall be an equipotential surface are

\[ (5) \quad \varepsilon - \frac{\frac{5}{3} \rho \varepsilon - \frac{1}{2} x \varepsilon - \frac{1}{2} \varphi}{2} = 00017 \ 287 \]

\[ K = -\delta^2 \varepsilon + 3 \varepsilon^2 - \frac{1}{2} \varphi \varepsilon = 00000 \ 109, \]

and the acceleration of gravity at the surface is

\[ g = g_0 \left[ 1 + \beta \sin^2 \varphi' \varepsilon + \gamma \sin^2 2\varphi' \right] \]

with

\[ (6) \quad \beta + \varepsilon = \frac{\frac{5}{3} \rho \varepsilon + \frac{1}{2} \rho \varepsilon^2 - \frac{1}{2} \varphi \varepsilon + \frac{1}{2} x = 00086 \ 559 \]

\[ \gamma = \frac{1}{2} \varphi \varepsilon - \frac{1}{2} \varepsilon^2 - 3 \varphi = 00000 \ 104. \]

In these formulas \( \rho_1 \) is determined by

\[ \rho_1 = \frac{\omega^6 \rho^3}{f M_0}, \]

or

\[ \rho_1 + \frac{\frac{5}{3} \rho_1^2 \varepsilon^2}{\delta^4 r^4}, \]

\( r_1 \) being the mean radius, and \( g_1 \), the acceleration of gravity at the latitude of this mean radius. This gives

\[ \rho_1 = 000000 \ 4992 \pm 0000 \ 0002. \]

In the small terms in (5) and (6) I have omitted the index of \( \rho_1, \) and I have adopted

\[ \varepsilon = 1/206,7 = 0003373 \]

\[ x = 00000 \ 0050. \]

The value of \( x \), of course, entirely unknown. If the earth were homogeneous, it would be zero, on the other hand it cannot exceed \( \frac{1}{2} \varepsilon^2 = 00000 \ 0080, \) so that the adopted value appears plausible. It corresponds to 3.2 meters, and is thus entirely irrelevant.

The equations (5) and (6) are independent of any assumption regarding the distribution of mass inside the earth. For \( g \) on the other hand we have no rigo-
rous equation of this kind, but the theory of Clairaut on the constitution of the earth enables us to derive a very approximate value of \( q \). Then, by combining (2) and (5), we can derive \( \varepsilon \) from \( H \), and this determination of \( \varepsilon \) will be more accurate than by any other method.

If we wish to go beyond an accuracy of the order of a unit in the denominator of \( \varepsilon \), corresponding to the fifth decimal place in \( \varepsilon \) itself, it is necessary to include the second order, as has already been done above in the formulas (1) to (6). On the other hand, if the third order is neglected, all figures beyond the seventh decimal are meaningless.

The theory of Clairaut has been developed to the second order by Darwin \(^*\) and others. By taking as independent variable the mean, instead of the equatorial, radius, the formulas become somewhat simpler, and at the same time the range of uncertainty of \( q \) is considerably narrowed. As this change of independent variable does not affect the essential parts of the theory, which are well known, I will only state the principal steps and formulas very succinctly, without going into the details of their derivation. As we require only one term beyond the one of the lowest order in any equation, we can choose at random any one of the several definitions of the mean radius, which are equivalent to the first order. We may suppose it to be the radius of the sphere of equal volume.

2. On the theory of Clairaut the surfaces of equal density are equipotential surfaces. Let \( \beta \) be the mean radius of any such surface, expressed in that of the outer surface as unit, so that \( \beta \) ranges from 0 to 1, then the equation of this surface becomes

\[
(7) \quad r = \beta [1 - \frac{3}{8}(\varepsilon' + \frac{3}{8} \varepsilon)r_3^2 \sin \varphi + \frac{3}{8}(3\varepsilon^2 + 8x)P_4(\sin \varphi)],
\]

\( P_4 \) and \( P_4 \) being again the spherical harmonics, and

\[
(8) \quad \varepsilon' = \varepsilon - \frac{3}{8} \varepsilon^2 + \frac{3}{8} x.
\]

In the terms of the second order it is not necessary to distinguish between \( \varepsilon \) and \( \varepsilon' \), and the accent is dropped.

The potential \( V \) at any point within the earth of which the coordinates are \( r \) and \( \theta \) is given by

\[
(9) \quad V = \frac{1}{r} \left[ \frac{3}{8} \beta^3 \frac{P_4(\sin \varphi)}{r^3} + \frac{1}{8} \beta^5 \frac{Q_5(\sin \varphi)}{r^5} \right] P_4(\sin \varphi),
\]

where

\[
\beta = \frac{\omega^2 b^3}{f M} = \frac{p}{D},
\]

\( W \) being the volume and \( M \) the mass within the surface of which the mean radius is \( \beta \), and

\[
D = \frac{3}{\beta^3} \int \frac{\beta}{\beta} d\beta,
\]

\[
S = \frac{1}{\beta^3} \int \beta \frac{d}{d\beta} [\beta^3(\varepsilon' + \frac{3}{8} \varepsilon)] d\beta,
\]

\[
T = \int \frac{d}{d\beta} [\varepsilon' + \frac{3}{8} \varepsilon] d\beta,
\]

\[
P = \beta \int \beta \frac{d}{d\beta} \left[ (e + \frac{3}{8} x) \right] d\beta,
\]

\[
Q = \beta^5 \int \frac{d}{d\beta} \left[ \frac{x}{\beta^3} \right] d\beta.
\]

In these formulas \( \beta \) is the density, expressed in the mean density as a unit, and consequently \( D \) is the mean density within the surface \( \beta \), expressed in the same unit. \(^{**}\) For the outer surface we have, of course:

\[
D_1 = 1,
S_1 = \frac{3}{2} J,
T_1 = 0,
P_1 = \frac{5}{2} K,
Q_1 = 0.
\]

\(^*\) Scientific Papers, III, p. 78 = M.N. 60, p. 82 (1900).

\(^{**}\) It should be remarked that in these units we have \( M_r = \frac{4}{3} \pi r^3, D_1 = \frac{3}{8} \pi \).
and differentiating again, we find the differential equation for \( \zeta' \):

\[
(14) \quad \beta \frac{d\zeta'}{d\beta} + \zeta' \zeta + 5 \zeta' - 2 \zeta (1 + \zeta') - \frac{3}{2} \zeta Q = 0,
\]

where *)

\[
(15) \quad Q = 7 \rho (1 + \zeta') - 3 \varepsilon (1 + \zeta')^2 - 4 \varepsilon.
\]

Omitting the term with \( Q \) (14) is the well known differential equation correct to the first order. The introduction of \( \varepsilon' \) for \( \varepsilon \) has removed from (14) a term of the form \( \varepsilon' \), thus having all small terms multiplied by \( \zeta \). DARWIN, using the equatorial instead of the mean radius, achieves the same purpose by introducing \( h = \varepsilon - \frac{3}{2} \varepsilon' + \frac{1}{2} x = \varepsilon' - \frac{3}{2} \varepsilon' \). His equation for \( \eta D = \frac{\partial}{\partial \beta} \) is of the same form as (14) with \( Q D = 7 \rho (1 + \eta) - 3 \frac{h}{x} \varepsilon (1 + \eta)^2 - 18 \varepsilon \).

It is evident that \( Q D \) is numerically larger than our \( Q \).

From (14) we derive, as was first done by RADAU:

\[
(16) \quad D \beta^3 \frac{V}{1 + \zeta'} = \frac{\beta}{5} \int D \beta^3 F(\zeta') d\beta,
\]

where

\[
(17) \quad F(\zeta') = \frac{1 + \frac{1}{2} \zeta' - \frac{1}{2} \zeta' \zeta + \frac{3}{2} \zeta Q}{V(1 + \zeta')}. \]

The value of \( F(\zeta') \) is always very near to unity. If by \( 1 + \lambda \) we denote a certain average value of it over the range of integration, we have

\[
(18) \quad \frac{\beta}{5} \int D \beta^3 d\beta = \frac{D}{V} \frac{\beta}{1 + \lambda}.
\]

Now we have, to the required order of accuracy

\[
C = \frac{\pi}{3} \frac{1}{5} \int D \beta^3 \left[ \beta^3 (1 + \frac{3}{2} \varepsilon) \right] d\beta = \frac{3}{5} \pi \int D \beta^3 \beta d\beta + \frac{3}{5} (C-A),
\]

and, since in our units

\[
M, b^3 = \frac{3}{5} \pi (1 + \frac{3}{2} \varepsilon),
\]

we have by (3)

\[
(19) \quad q = 3 (1 - \frac{3}{5} \varepsilon) \int D \beta^3 \beta d\beta + \frac{3}{5} J.
\]

Replacing in the integral \( \beta \) by its value from (12), then integrating by parts, and using (5), of which only the terms of the lowest order are here needed, we find

\[
(20) \quad q = 3 (1 - \frac{3}{5} \varepsilon) \int D \beta^3 \beta d\beta,
\]

or by (18)

\[
(21) \quad q = I - \frac{3}{5} \rho I - \frac{3}{5} \int (1 - \frac{3}{5} \varepsilon) \frac{V + \eta}{1 + \lambda}.
\]

From the equation (13) we find for the outer surface

\[
(22) \quad \varepsilon \eta' = \frac{3}{5} \rho - 2 \varepsilon + \frac{3}{5} \rho \varepsilon + \frac{3}{5} \varepsilon^2 - \frac{3}{5} \varepsilon \rho,
\]

from which, with \( \varepsilon^{-1} = 59650 + 5 \varepsilon^{-1} \), corresponding to \( \varepsilon' = 0033716 \), we find

\[
\eta' = \frac{55893 + 00863 \Delta \varepsilon^{-1}}{1 + \lambda}.
\]

Therefore

\[
(23) q = 50053 - 00140 \Delta \varepsilon^{-1} + 54983 + 0014 \Delta \varepsilon^{-1} \frac{\lambda}{1 + \lambda}.
\]

3. To derive a probable value for \( \lambda' \), and to ascertain the limits within which this value may be considered as trustworthy, we must discuss the function \( F(\zeta') \) given by (17). The first part of this function is the same as discussed by RADAU, POINCARE and others, viz:

\[
F_1(\zeta') = \frac{1 + \frac{1}{2} \eta - \frac{3}{2} \zeta \eta'}{V(1 + \eta)}.
\]

For convenience I drop the accent, which is of no importance here. The function \( F_1(\eta) - 1 \) is zero for \( \eta = 0 \), rises to a maximum of \(+00074\) for \( \eta = \frac{1}{2} \), and then decreases again, becoming zero for \( \eta = 0^28 \)

\[ (= \frac{5}{2} - 2 \sqrt{\frac{5}{2}}) \]. For the surface, \( \eta = 0^26 \), it is \(-00029\). It is thus positive for practically the whole range, and is larger than \(+00050\) from \( \eta = 0^19 \) to \( \eta = 0^44 \). Since \( \eta \) increases continually from zero at the centre to \( 5^6 \) at the surface, we can take as a plausible value

\[
(24) \quad F_1(\eta) = 100050 \pm 00015.
\]

The “probable error” attached to this value is meant to express the belief that there is an even chance that the true value of \( F_1(\eta) \) is included within a range of \( 00030 \), the adopted value \( 100050 \) being somewhere inside this range, and probably not far from the middle.

To discuss the value of \( Q \) we put

\[
(25) \quad \varepsilon = \rho (1 - \eta).
\]

We then have

\[
Q = Q_1 + Q_2,
\]

\[
Q_1 = (\eta - 3\eta')\rho,
\]

\[
Q_2 = \rho (7 + 6\eta + \eta^2)
\]

Consider first the first part \( Q_1 \). This varies between the limits \(+08 \rho \) for \( \eta = \frac{1}{2} \), and \(-36 \rho \) at the surface \( (\eta = 5^6) \). At the centre it is zero. The value of \( \rho \) is always smaller than \( 1 \); \( \sqrt{1 + \eta} \) is always larger than \( 1 \), and it is probable that \( \xi \) never considerably exceeds its surface value \( \xi_1 = 0^25 \). We can thus take

\[
\frac{3}{5} \frac{\xi}{\sqrt{1 + \eta}} < \frac{3}{5} \frac{\xi}{\sqrt{1 + \eta}} \rho_1 = 00010.
\]
Consequently the value of \( \epsilon Q_\tilde{\phi} \) is certainly comprised between the limits \( +0.000008 \) and \( -0.00036 \). We can take as a plausible average:

\[
F_\tilde{\phi}(\nu) = -0.000014 \pm 0.000010.
\]

To discuss the function \( Q_\phi \) we must consider the possible values of \( \nu \). Differentiating (25) we find

\[
\frac{\beta}{1 - \nu} \frac{d\nu}{d\beta} = \zeta - \eta
\]

from which, in consequence of the well known inequalities

\[
o \leq \eta \leq \zeta \leq 3
\]

it follows that \( \nu \) increases continuously from the centre outwards, or

\[
\nu_0 < \nu < \nu_1,
\]

\( \nu_0 \) and \( \nu_1 \), being the values for the centre and the surface respectively. For the earth the surface value is \( \nu_1 = +0.02 \). The value of \( \nu_0 \) can be included in narrow limits. We have

\[
\frac{\nu_0 \rho_0}{\rho_1} = 1 - (1 - \nu_1) \frac{\epsilon_0}{\epsilon_1}
\]

For \( \epsilon_0/\epsilon_1 \) we find upper and lower limits from the formula (42) given by Tisserand, *Méc. Cél.* II, p. 227, viz:

\[
\frac{3 \delta_0 + 2 + \eta_1}{\delta_0} > \frac{\epsilon_0}{\epsilon_1} > \frac{3 \delta_1 + 2 + \eta_1}{3 \delta_1 + 2 \delta_0}.
\]

Taking \( \delta_1 = 4.95 \) this gives the following limits:

<table>
<thead>
<tr>
<th>( \delta_0 )</th>
<th>Limits of ( \frac{\epsilon_0}{\epsilon_1} )</th>
<th>Limits of ( \frac{\nu_0 \rho_0}{\rho_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.28</td>
<td>1.00</td>
<td>(-.25)</td>
</tr>
<tr>
<td>1.5</td>
<td>.94</td>
<td>(-.26)</td>
</tr>
<tr>
<td>2.0</td>
<td>.86</td>
<td>(-.23)</td>
</tr>
<tr>
<td>2.5</td>
<td>.80</td>
<td>(-.39)</td>
</tr>
<tr>
<td>3.0</td>
<td>.77</td>
<td>(-.42)</td>
</tr>
</tbody>
</table>

We can thus take with certainty

\[
-0.40 < \frac{\nu_0 \rho_0}{\rho_1} < -0.21.
\]

Now put for abbreviation

\[
\frac{Q_\phi}{V_1 + \eta} = \nu \frac{7 + 6 \eta + \eta^2}{V_1 + \eta} = \omega \rho_1.
\]

For the surface we have

\[
\omega_0 = \frac{7 \nu_0 \rho_0}{\rho_1},
\]

or

\[
-2.8 < \omega_0 < -1.5.
\]

Of the two factors of which \( \omega \) is made up, \( \nu \rho_0/\rho_1 \) increases continually from the centre, where it is included between the limits (27), to the surface, where it is \( +0.02 \). The factor \( (7 + 6 \eta + \eta^2)/(1 + \eta) \) increases continually from \( 7 \) at the centre to \( 8.54 \) at the surface. The only factor of which we are not certain is \( \zeta \). For the centre it is zero, and for the surface its value is \( \zeta_1 = 1.515 \), but it is not excluded that at intermediate depths it may be larger than \( \zeta_1 \), though it can never exceed the value \( 3 \). Still it is probable that the product \( \zeta \omega_0 \), as we proceed from the surface to the centre, will with some rough approximation follow the diagonal of the following table, starting from the left hand top corner, and reaching the bottom somewhere between the last two columns.

<table>
<thead>
<tr>
<th>( \zeta )</th>
<th>+0.17</th>
<th>0</th>
<th>-0.1</th>
<th>-1.0</th>
<th>-1.5</th>
<th>-2.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>+0.25</td>
<td>0</td>
<td>-0.75</td>
<td>-1.25</td>
<td>-1.5</td>
<td>-2.10</td>
</tr>
<tr>
<td>2.25</td>
<td>+0.21</td>
<td>0</td>
<td>-0.50</td>
<td>-1.00</td>
<td>-1.5</td>
<td>-2.10</td>
</tr>
<tr>
<td>1.00</td>
<td>+0.17</td>
<td>0</td>
<td>-0.38</td>
<td>-0.75</td>
<td>-1.12</td>
<td>-2.10</td>
</tr>
<tr>
<td>0.75</td>
<td>+0.04</td>
<td>0</td>
<td>-0.25</td>
<td>-0.75</td>
<td>-1.40</td>
<td>-2.10</td>
</tr>
<tr>
<td>0.50</td>
<td>+0.01</td>
<td>0</td>
<td>-0.25</td>
<td>-0.75</td>
<td>-1.40</td>
<td>-2.10</td>
</tr>
<tr>
<td>0.25</td>
<td>+0.01</td>
<td>0</td>
<td>-0.25</td>
<td>-0.75</td>
<td>-1.40</td>
<td>-2.10</td>
</tr>
<tr>
<td>0</td>
<td>+0.00</td>
<td>0</td>
<td>-0.25</td>
<td>-0.75</td>
<td>-1.40</td>
<td>-2.10</td>
</tr>
</tbody>
</table>

It thus appears probable that the average value of this product over the range of integration is included between comparatively narrow limits, say between \(-0.20\) and \(-1.20\). We can take as a plausible value

\[
\bar{\zeta} \omega_0 = -0.70 \pm 0.30,
\]

the probable error being taken in its true meaning to indicate that there is an even chance that the true average is between the limits \(-40\) and \(-100\). We then find

\[
(28) \quad F_\tilde{\phi}(\nu) = \epsilon_0 \rho_0 \omega_1 = -0.000046 \pm 0.000020.
\]

Adding the three partial values (24), (26) and (28) together we find:

\[
(29) \quad \lambda_1 = +0.00044 \pm 0.00015.
\]

It will be seen that the effect of the second order terms (\( \xi Q \)) is only about one tenth of the whole. Substituting this value of \( \lambda_1 \) in (23) we find:

\[
(30) \quad e = 50075 \pm 0.00008 - 0.0140 \Delta t^{-1} + 499 \Delta \lambda_1.
\]
4. We must now consider the question whether these results, which have been derived from the theory of Clairaut, are applicable to the actual earth.

The actual surface of the earth is, of course, neither a surface of equal density nor an equipotential surface. We can however safely assume that up to a certain distance from the centre the material out of which the earth is made up is — so far as secular forces are concerned — in hydrostatic equilibrium, and consequently satisfies the conditions of the theory of Clairaut. The last surface for which this theory is applicable is called the isostatic surface, and will be denoted by $S_e$. Above this there exist, of course, further equipotential surfaces, but these are not as a rule surfaces of equal density, and not necessarily spheroids. The actual land surface is, of course, not such a surface, but the undisturbed surfaces of the different oceans can be assumed to form parts of one and the same equipotential surface, which is called the geoid. This geoid is determined from geodetic measures on the continents, and from determinations of the intensity of gravity on the continents and on the oceans. It is found that it deviates only very little from an ellipsoid of revolution. The ellipsoid of revolution, or rather the spheroid, best fitting the geoid is called the normal surface, and is denoted by $S$. The differences between the geoid and the normal surface never amount to more than a few tens of meters. This fact has led to the well known theory of isostasy, which asserts that within any cylinder erected over a (not too small) surface element $\omega$ of the isostatic surface there is the same mass as there would be with a certain ideal distribution, which we can take to be in accordance with the theory of Clairaut. The upper surface, that would result if the conditions of this theory were satisfied throughout, will be called the ideal surface, and will be denoted by $S_i$. To this surface $S$, the equations (5) and (2), with the value (21) of $q$, are applicable.

The normal surface $S$ on the other hand is not an equipotential surface, but it is the spheroid best fitting the geoid, which is an equipotential surface. For the condition of this "best fitting" we can take that in the developments of both surfaces in series of spherical harmonics the coefficients of the harmonics of the orders zero, two and four are the same. Then the equation (5) is applicable if for $\varepsilon$ we take the compression of the normal surface, and for $\mathcal{F}$ its actual value for the real earth. We can, of course, again write down the equation (2), taking for $H$ also its actual value, but now $q$ is determined by (3) and not by (21).

We have thus for the ideal surface:

$$\mathcal{F}_i = q_i H_i, \quad q_i \text{ from (21)},$$

and for the actual earth:

$$\mathcal{F} = q H, \quad q \text{ from (3)},$$

and the problem before us is to find, the difference between $q$ and $q_i$.

If the earth were entirely constituted according to the theory of Clairaut, it would be covered by an ocean of an average depth of about 2.5 km, of which the upper surface would be the ideal surface $S_i$, and the bottom would also be an equipotential surface, which we will call $S_b$. The true distribution of mass differs from this ideal one on the one hand by an excess of mass in the continents, and the shallow seas, and a defect in the deep oceans, and on the other hand by the isostatic compensations of these excesses and defects.

This compensation is assumed to be equally distributed over the layer between the surfaces $S_i$ and $S_b$ or, in the case of the deep oceans, between $S_i$ and the bottom of the ocean.

The formulas have been worked out by me in 1915 *).

In that publication the layer between $S_i$ and $S_b$ was however treated very summarily, and a corresponding correction was applied to the final result. If we compute the effect of this layer for each surface element $\omega$ separately, we find, for the continents:

$$\Delta (m r^2) = 2.70 \omega (Z + h) [r - 0.5 (Z - h)] + 0.70 B (1.70 Z + 2.70 h) [r - 0.5 (Z + B)],$$

for the shallow seas, of depth not exceeding $B$:

$$\Delta (m r^2) = 1.70 \omega (B - d) (Z - d) [r - 0.5 (Z + B + d)],$$

and for the deep oceans:

$$\Delta (m r^2) = -1.70 \omega (d - B) (Z - d) [r - 0.5 (Z + B + d)].$$

In these formulas $r$ is the radius of the ideal surface at the latitude considered, $Z$ is the depth of the isostatic surface, and $B$ that of the surface $S_i$, below $S_i$, while $h$ and $d$ are the height of the land surface, and the depth of the sea bottom, also referred to the ideal surface. **)


** Strictly speaking $d$ and $h$ are not known, but the depth $d_i$ and height $h_i$ referred to the geoid are used. These can be assumed practically to be referred to the normal surface. The correction to reduce from the normal to the ideal surface is given in the quoted paper, formula (7) p. 1503, but it is entirely negligible. In the formulas as given above the densities are referred to that of water as unit, and the depths and heights are supposed to be expressed in kilometers.
Then we have
\[ \Delta H = \Sigma \Delta (m r^2), \]
\[ \frac{\Delta C}{C} = \Sigma \Delta (m r^2) \cdot \cos^2 \phi', \]
We find \(^*\) in units of the seventh decimal place:

<table>
<thead>
<tr>
<th>Parts of the world</th>
<th>surface</th>
<th>( \Delta H )</th>
<th>( \frac{\Delta C}{C} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. North Polar regions</td>
<td>'0277</td>
<td>+ 1.7</td>
<td>+ 2.2</td>
</tr>
<tr>
<td>2. Europe</td>
<td>'0260</td>
<td>- 1.1</td>
<td>+ 2.4</td>
</tr>
<tr>
<td>3. Asia</td>
<td>'0876</td>
<td>+ 8.8</td>
<td>+ 14.5</td>
</tr>
<tr>
<td>4. North America</td>
<td>'0485</td>
<td>- 1.0</td>
<td>+ 5.2</td>
</tr>
<tr>
<td>5. Northern Atlantic Ocean</td>
<td>'0781</td>
<td>- 1.2</td>
<td>- 4.3</td>
</tr>
<tr>
<td>6. South America</td>
<td>'0408</td>
<td>+ 2.3</td>
<td>+ 9.4</td>
</tr>
<tr>
<td>7. Southern Atlantic Ocean</td>
<td>'0840</td>
<td>+ 1.2</td>
<td>- 10.7</td>
</tr>
<tr>
<td>8. Africa</td>
<td>'0662</td>
<td>+ 2.9</td>
<td>+ 14.3</td>
</tr>
<tr>
<td>9. Indian Ocean</td>
<td>'1434</td>
<td>+ 1.2</td>
<td>- 9.2</td>
</tr>
<tr>
<td>10. Indian Archipelago</td>
<td>'0215</td>
<td>- 7.0</td>
<td>+ 2.0</td>
</tr>
<tr>
<td>11. Australia and N. Guinea</td>
<td>'0284</td>
<td>+ 5.0</td>
<td>+ 3.7</td>
</tr>
<tr>
<td>12. Northern Pacific Ocean</td>
<td>'1475</td>
<td>- 7.4</td>
<td>- 17.7</td>
</tr>
<tr>
<td>13. Southern Pacific Ocean</td>
<td>'1700</td>
<td>- 2.1</td>
<td>- 13.8</td>
</tr>
<tr>
<td>14. South Polar regions</td>
<td>'0294</td>
<td>- 6.6</td>
<td>+ 9.0</td>
</tr>
<tr>
<td>Total</td>
<td>'0000</td>
<td>- 8.5</td>
<td>- 3.4</td>
</tr>
</tbody>
</table>

If there were no isostatic compensation, these numbers would be increased roughly in the ratio of \( r/Z \), i.e. about 55 times. The approximation of the computations is such that each of the partial numbers is correct to a few percents of its amount. The sums may thus easily be a unit or more in error.

We have thus as the result of this computation:
\[ \Delta C = \frac{C - C_1}{C_1} = -0.0000 0034, \]
\[ \Delta H = H - H_1 = -0.0000 0085. \]

Now we have from (5) and (2)
\[ \varepsilon = \mp 0.017 287 + q H, \]
from which
\[ \Delta \varepsilon = q \Delta H + H \Delta q, \]
and from (3)
\[ \Delta q = \frac{\Delta C}{C} - 2 \frac{\Delta b}{b}. \]

We have
\[ \frac{\Delta b}{b} = \frac{1}{2} \Delta \varepsilon. \]

Therefore
\[ \Delta q \left( \frac{1}{q} + \frac{3}{2} \frac{\Delta H}{H} \right) = \frac{\Delta C}{C} - \frac{3}{2} q \Delta H, \]
or
\[ \Delta q = -0.0000 0003 \]
\[ \Delta \varepsilon = -0.0000 0043. \]

The correction to \( q \) would still be negligible if there were no isostatic compensation. The difference in the compression between the ideal and the normal surfaces gives in the denominators
\[ \varepsilon^{-1} - \varepsilon_1^{-1} = +0.04. \]

This is of course entirely negligible. It means that the polar semidiameter of the ideal surface is 1.8 meters shorter, and the equatorial radius 0.9 meters longer, than of the normal surface. If there were no isostatic compensation, however, the difference between \( \varepsilon \) and \( \varepsilon_1 \) would be of the order of two units in the denominator.

5. It has thus been proved that we can in the formula (31) for the actual earth use the value (21) or (30) of \( q \) derived from the theory of Clairaut.

The value of \( H \) can be derived with great accuracy from the constant of precession. Adopting for the reciprocal of the mass of the moon
\[ \mu^{-1} = 81.50 \pm 0.07 + \Delta \mu^{-1} \]
I find
\[ H = 0.0032 774 + 0.0000 270 \Delta \mu^{-1}. \]

The probable error of \( H \) is made up of \( \pm 19 \) in the seventh decimal place due to the uncertainty of \( \mu \), and \( \pm 0.6 \) due to the constant of precession.

Now taking
\[ \varepsilon = 0.0033 727 + \Delta \varepsilon, \]
and substituting (30) into (31), we find, since
\[ \Delta \varepsilon^{-1} = 87 900 \Delta \varepsilon, \]
(33)
\[ 597 \Delta \varepsilon = -0.0000 028 + 0.0000 135 \Delta \mu^{-1} + 0.00163 \Delta \lambda_1, \]
from which
\[ \Delta \varepsilon = -0.0000 048 \]
\[ \varepsilon = 0.0033 679, \]
or
(34)
\[ \varepsilon^{-1} = 296.92 \pm 136.00 \Delta \mu^{-1} - 152.00 \Delta \lambda_1. \]

The probable error of \( \varepsilon^{-1} \) is made up as follows:
from the precessional constant \( \pm 0.04 \)
from \( \mu \) \( \pm 0.132 \)
from \( \lambda_1 \) \( \pm 0.035 \)
The remaining uncertainty of $\varepsilon$ is thus due almost entirely to that of the mass of the moon.

In my paper of 1915 I derived from the same value of $H$ the value 296.6 of $\varepsilon^{-1}$. This was based on DARWIN’s computations. There is, however, at this point a numerical mistake in DARWIN’s work. On p. 113 (Sci. Pa., III) the value of $D_0$ corresponding to his adopted data is $0.4979173$, instead of $0.4991436$ as printed. This can be verified by interpolation from the table on p. 112, which gives, for $e = 1/299$, $C / 3 \text{ Ma} = 0.49793$, the last figure being, of course, uncertain. This mistake, of course, makes his formula for $\delta_0$, which was used by me in my former work, erroneous. Moreover DARWIN’s value of $q$ i.e. of $\lambda_0$, was derived from ROCHE’s hypothesis, which, as will be shown below, is not in agreement with the observed facts. The formula (33) must therefore be used, which does not depend on any hypothesis, except so far as the adopted probable error of $\lambda_0$ is concerned.

The best values of $\varepsilon^{-1}$ derived by other methods are:

- From geodetic measures in the United States by HAYFORD\textsuperscript{*)}:
  
  \[ \varepsilon^{-1} = 297.0 \pm 5 \]

- From the intensity of gravity by HELMERT\textsuperscript{**)}:
  
  \[ \varepsilon^{-1} = 296.7 \pm 4. \]

Both agree with the value (34) within their probable errors, but both are very much less accurate. I have stated the probable errors as given by the authors (CONVERTING HELMERT’s mean error into a probable error), though these appear to me to be underestimated in both cases.

By (32) we have for the ideal surface

\[ \varepsilon = 0.033683 \]

and consequently

\[ \varepsilon' = 0.033672. \]

6. The actual distribution of density within the earth is unknown. We can make hypotheses regarding this distribution, and by their aid compute the different integrals occurring in the theory of CLAIRAUT, and thus arrive at values of the different quantities, which can be determined by observation, such as $\varepsilon$, $\beta$, $J$, $H$ and the surface density $\delta_s$. The agreement or non-agreement of these computed values with the observed ones then is a test of the hypothesis, in this sense that, if there is no agreement, the hypothesis is condemned, but if there is agreement, this does not prove the hypothesis to be true, since it may be very well possible to arrive at the same final values by different hypotheses.

The equations (3) and (6) are independent of the inner constitution. Consequently any hypothesis, which will reproduce any one of the three quantities $\varepsilon$, $\beta$, $J$, will also give the correct value for the other two. $H$ and $\delta_s$ however are independent data, so that there are practically three conditions to be satisfied by any hypothesis on the distribution of mass.

ROCHE’s hypothesis

\[ \delta = \delta_0 \left( 1 - k \beta^2 \right) \]

contains only two constants,\textsuperscript{*)} and can thus not be expected to satisfy the conditions. This expectation is confirmed by DARWIN’s computations. which are based on this hypothesis. The table on p. 112 of his paper (Sci. Pa., III) gives corresponding values of $1/\delta_s$, $1/s$ and $q$. The observed values are, by (34) and (30):

\[ \delta_s = 0.495, \quad s = 1/296.92, \quad q = 50016. \]

If we interpolate in DARWIN’s table for these values we find the following sets of corresponding values

\[
\begin{array}{lll}
\delta_s & 0.495 & 0.4226 & 0.4186 \\
\varepsilon^{-1} & 0.2881 & 0.2969 & 0.2974 \\
q & 0.51323 & 0.50082 & 0.50016 \\
\end{array}
\]

This shows that, as was to be expected, it is not possible to bring ROCHE’s hypothesis into agreement with the observed facts.

WIECHERT’s hypothesis, according to which the earth consists of a core and a crust, each of constant density, separated by a surface of discontinuity at which the density changes abruptly, contains three parameters, viz. the two densities and the radius of the surface of discontinuity. It is thus theoretically possible by this hypothesis to satisfy the three conditions, but it remains to be seen whether the values of the parameters by which this is effected, are otherwise acceptable.

In order to test this and other hypotheses we must, as has been already said, compute the different integrals occurring in the theory of CLAIRAUT. I begin by deriving the parts contributed to these integrals by the mass above the isostatic surface, as these must be the same for any hypothesis. We must in this computation start from the ideal surface $S_s$, and the ideal distribution of mass described above in article 4, p. 101. It has been proved in that article that the results derived from this ideal distribution are applicable to the actual earth. The radius of the isostatic surface $S_s$ is denoted by $s$, and I put

\[ \delta_s = i(1 - \frac{3}{2} \varepsilon). \]

\textsuperscript{*)} Or rather only one constant, since the condition $D_s = 1$ gives $\delta_s = i(1 - \frac{3}{2} \varepsilon)$.

\textsuperscript{**)} U. S. Coast and Geodetic Survey: Supplementary investigation in 1909 of the figure of the earth and isostasy, 1910. ***) Berlin Sitzungsberichte: Neue Formeln für den Verlauf der Schwerkraft in Meeresniveau beim Festlande, 1915, XLI, p. 676.
\[ \tau = 3 \int \beta \beta^2 d \beta. \]

Since
\[ D_1 = 3 \int \beta \beta^2 d \beta = s^3 D_1 + 3 \int \beta \beta^2 d \beta, \]
we have
\[ D_1 = (1 - \tau)s^3. \]

Further
\[ \sigma = \int \beta \frac{d}{d \beta} [\epsilon + \frac{1}{2} \frac{\epsilon^2}{\beta}] d \beta, \]
\[ \chi = 5 \int \beta \beta^4 d \beta, \]
\[ \psi = 5 \int D \beta^5 d \beta. \]

We find easily
\[ \psi = \frac{1}{2} \left( 1 - s^2 (1 - \tau) \right) - \frac{3}{2} \chi. \]

The values of \( \tau \) and \( \chi \), and therefore of \( \psi \), are, of course, independent of \( \epsilon \). I denote the density of the upper shell, between the surfaces \( S_b \) and \( S_b \), by \( \delta \), and between \( S_b \) and \( S_b \), I assume the density to increase proportionally to \( b - \beta \) from \( \delta_b \) at \( S_b \) to a certain value \( \delta_s \) at \( S_b \). I make \( \delta_s \) equal to \( m. \)

From \( b = 99962, \quad \delta_b = 0.186, \quad s = 98200, \quad \delta = 0.495. \)

Then we find
\[ \tau = \frac{b^3 - s^3}{b - s} \left( \delta_b - s \delta_b \right) - \frac{b^3 - s^3}{b - s} \left( \delta_b - s \delta_b \right) + \delta' (1 - b^3), \]
\[ \chi = \frac{b^5 - s^5}{b - s} \left( \delta_b - s \delta_b \right) - \frac{b^5 - s^5}{b - s} \left( \delta_b - s \delta_b \right) + \delta' (1 - b^3). \]

If we take \( \delta_s = 0.009 + 0 \) we find
\[ \tau = 0.0262 + 0.025729 \]
\[ \chi = 0.0429 + 0.041969 \]
and consequently
\[ \psi = 0.08049 - 0.00769 \]
\[ D_1 = 0.02852 - 0.027249. \]

As to \( \sigma \) we have
\[ \sigma = \int \beta \epsilon \epsilon' (1 + \frac{1}{2} \frac{\epsilon^2}{\beta}) d \beta. \]

We split up the integral in two parts, from \( s \) to \( b \), and from \( b \) to \( 1 \). For the second part it is sufficiently accurate to take
\[ [\sigma] = \frac{b}{\beta} (1 - b) \epsilon', \quad \epsilon' = 0.0000012. \]

For the first part we first derive values of \( \epsilon' \) and \( \epsilon' \) for the surfaces \( S_b \) and \( S_b \). Then we take
\[ \epsilon' - \epsilon' = - (1 - b) \epsilon', \epsilon', \quad \epsilon' - \epsilon' = - (1 - b) \left( \frac{d \epsilon'}{d \beta} \right). \]

The differential coefficient
\[ \left( \frac{d \epsilon'}{d \beta} \right) \]

is derived from (14), using \( \xi = 3 (1 - b) = 2.422. \)

With this value of \( \epsilon' \) we find again \( \left( \frac{d \epsilon'}{d \beta} \right) \) from (14), now using \( \xi = 3 (1 - b) = 2.515, \) and this gives a first approximation to \( \epsilon'. \) Then, with this value of \( \epsilon' \) and \( \xi = 3 (1 - b) = 2.515, \) we find \( \left( \frac{d \epsilon'}{d \beta} \right) \) from (14), and then a new approximation, which can be taken as final, for \( \epsilon' \), by \( \epsilon' - \epsilon' = - \frac{b - s}{b + s} (\epsilon' \epsilon' + \epsilon' \epsilon'), \)

Then we take first \( \epsilon' - \epsilon' = - \frac{b - s}{b + s} (\epsilon' \epsilon' + \epsilon' \epsilon'), \)

and then \( \epsilon' - \epsilon' = - \frac{b - s}{b + s} (\epsilon' \epsilon' + \epsilon' \epsilon'). \)

I find thus, adding the previously found value of \( \epsilon' - \epsilon' = - 0.0000329 \)
giving \( \epsilon' = 0.0033343. \)

Using these values of \( \epsilon' \) and \( \epsilon' \), we find finally \([\sigma] \) by the formula already stated. Adding the value of \([\sigma] \) we have
\[ \sigma = 0.0001621 + 0.000157. \]

This value of \( \sigma \) was computed from the originally adopted value of \( \epsilon = 1/296'5, \) but the effect of a small change in \( \epsilon \) is entirely negligible in \( \sigma \).)

We are now in a position to write down the three conditions mentioned above which must be satisfied by any hypothesis regarding the constitution of the earth. To begin with we have as already stated above
\[ (35) \quad D_1 = \frac{3}{s} \int s d \beta \beta^2 d \beta = 0.02852 - 0.027249. \]

The equation (10) for the isostatic surface gives
\[ \frac{1}{s} \int \beta \beta \beta' \left[ \beta \beta' \right] d \beta = \frac{3}{D_1} \epsilon', \quad \epsilon' - \frac{1}{s} \epsilon' - \frac{3}{D_1} \epsilon'. \]

where we have put for abbreviation

\[ *) \quad \text{In my paper of 1914 I found in the same way, working with somewhat less precise data, and to one decimal place less,} \quad \epsilon' = - 0.000033. \]

\[ **) \quad \text{As a matter of fact the value of} \ \sigma \ \text{as given above, in consequence of the dropping of the last decimal place, happens to correspond even more exactly to the final value} \ \epsilon = 1/296'5 \ \text{than to} \ 1/296'5. \]
of which the value for $S_0$ is
$$
\varepsilon' = \varepsilon' + \frac{s}{2} \varepsilon' ,
$$
of which should be noted that $S_0$ is
$$\varepsilon' = 0033376 = 1/299'62.
$$
We find, using also (35)

$$
\int_0^{\beta} \frac{d\beta}{\beta} \left[ \beta^2 \varepsilon' \right] d\beta = 8.4685 - 0.5008 \theta.
$$

Finally we have from (18)
$$
\int_0^{\beta} \frac{d\beta}{\beta} = \frac{\varepsilon}{\beta} - \frac{\varepsilon}{\beta} = \frac{1 + \frac{\sigma_1}{\pi}}{1 + \lambda_i} - \psi
$$
or the equivalent formula
$$
\int_0^{\beta} \frac{d\beta}{\beta} = \frac{\varepsilon}{\beta} - \frac{\varepsilon}{\beta} = \frac{1 + \frac{\sigma_1}{\pi}}{1 + \lambda_i} - \chi,
$$
or numerically, since for the ideal surface $\sigma_1' = 56204,

$$
\int_0^{\beta} \frac{d\beta}{\beta} = 1.16179 + 0.0007669 - 1.258 \frac{\lambda_i}{1 + \lambda_i}
$$
or

$$
\int_0^{\beta} \frac{d\beta}{\beta} = 0.9084 - 0.041969 + 0.833 \frac{\lambda_i}{1 + \lambda_i}
$$

It need hardly be mentioned that in all these formulas the value of the last decimal place given is entirely fictitious owing to the uncertainty of the data used. I have computed the effect on the fundamental quantities $\tau$, $\chi$, and $\sigma$ of changes in $\sigma_1$, $\delta_1$, $s$ and $\delta$, which may be taken as about the largest that can still be considered as not improbable. I assumed these to be 1 percent of $\sigma_1$, 2% of $\delta_1$, 10% of $1-s$ and 5% of $1-\delta$. These changes applied in the sense that the depths of both the isostatic surface $S_0$ and the surface $S_0$, and also the densities of the sea-water and of the material of the solid crust are all increased (i.e., the mean density of the earth diminished) have the following effects, which are expressed in units of the fifth decimal place for $\tau$ and $\chi$ and of the seventh decimal place for $\sigma$, i.e., in all cases in units of the last decimal place given above.

<table>
<thead>
<tr>
<th>$\Delta \tau$</th>
<th>$\Delta \chi$</th>
<th>$\Delta \sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \delta_1$</td>
<td>$-2.58$</td>
<td>$+42.5$</td>
</tr>
<tr>
<td>$\Delta s$</td>
<td>$-1.3$</td>
<td>$+263$</td>
</tr>
<tr>
<td>$\Delta \beta$</td>
<td>$+2.0$</td>
<td>$-36$</td>
</tr>
<tr>
<td>$\Delta \delta$</td>
<td>$+260$</td>
<td>$+5218$</td>
</tr>
</tbody>
</table>

The effect of $\Delta s$ has been split up into two parts: (i) due to variation of $s$ under the integral signs, and (ii) due to variation of the lower limit of $s$ of the integrals. It is evident that this last variation is of no importance for the final result of the testing of the theory, for by a change of depth of the isostatic surface exactly as much as is added to the integral relating to the shell above this surface, is subtracted from the integral over the volume below it. Since $\delta$ is presumably not larger than $\delta_1$, it will be seen that the effect of all the other causes is not very serious.

On the other hand the probable errors of the right-hand members of the equations (35), (37) and (37') corresponding to the adopted probable error of $1/\varepsilon$ are:

$$
(35): \pm 0.0039
$$

$$
(37): \pm 0.0047
$$

$$
(37'): \pm 0.0031.
$$

The right-hand member of (35) of course is independent of $\varepsilon$, owing to the use of the mean radius instead of the equatorial one as the independent variable.

7. As an example I have applied these formulas to the theory of WIECHERT. We will first investigate this theory in its simplest form, with only one surface of discontinuity and constant densities below it and between it and $S_0$, and then if necessary modify it so that between the centre and the surface of discontinuity the density follows ROCHE's law. The mean radius of the surface of discontinuity I call $\beta_2$, the sudden increase of density at this surface is called $\Delta$ and the further increase between this surface and the centre is $\Delta'$. We have thus

$$
\begin{align*}
&\text{for } 0 < \beta < \beta_2: \quad \delta = \delta_0 (1 - k \beta^2) \\
&\text{for } \beta_2 < \beta < s: \quad \delta = \delta_2 = 500 + \theta
\end{align*}
$$

with

$$
\frac{\delta_0}{\delta_2} = \frac{\delta_1 + \Delta + \Delta'}{\beta_2}.
$$

Consequently we have

$$
\begin{align*}
&\text{for } 0 < \beta < \beta_2: \quad D = \delta_0 (1 - \frac{s}{2} k \beta^2) \\
&\text{for } \beta_2 < \beta < s: \quad D = \delta_2 + \left(\frac{\beta_2}{\beta}\right)^3 (\Delta + \frac{s}{2} \Delta')
\end{align*}
$$

We put

$$
\Delta' = x \Delta,
$$

and we will first investigate the simple case $x = 0$. We will however derive the formulas at once for the more general case.

Putting

$$
\beta_2 = \rho s
$$

we find at once

$$
D = \delta_2 + \rho^3 (\Delta + \frac{s}{2} \Delta')
$$

and the condition (35) consequently gives
\( \Delta \rho^3 (1 + \frac{\alpha}{3} x) = A = \frac{52852}{-102724} \)

Now call the value of \( e^* = e^* + \frac{3}{2} e^* \) at the surface of discontinuity \( e^* \) and put

\[ e^* = e^* (1 - \xi). \]

Then

\[
\int_\beta \frac{d}{d\beta} \left[ \frac{\beta^6 \beta^*}{\beta^*} \right] d\beta = \beta^6 \beta^* \xi + 2 \int \beta \beta^* d\beta.
\]

For the integral from the centre to the surface of discontinuity we have

\[
\int \frac{d}{d\beta} \left[ \frac{\beta^6 \beta^*}{\beta^*} \right] d\beta = \beta^6 \beta^* \xi + 2 \int \beta \beta^* d\beta.
\]

The last integral can be found, since, when \( \beta \), and therefore \( \xi \), is given as a function of \( \beta \), \( e^* \) can be found by (14) and then \( e^* \) and consequently \( e^* \) as a function of \( \beta \). We will however not attempt to effect this integration, but we will put

\[
\begin{equation}
\int \beta \beta^* d\beta = \frac{1}{7} \beta^6 \beta^* e^*,
\end{equation}
\]

where \( \alpha \) is necessarily a small positive quantity. Having regard to the limits of \( e^*/e^* \) stated above, it appears that \( \alpha \) can certainly not exceed the value \( \frac{1}{3} \) and is probably much smaller.

We then find

\[
\Delta (1 - \xi) (1 + \frac{\alpha}{3} x + \frac{3}{2} \alpha x) = \frac{1}{7} \Delta \beta^6 \beta^* \xi + \frac{3}{2} \Delta \beta^6 \beta^* \xi = \frac{1}{7} \Delta \rho^3 (1 + \frac{\alpha}{3} x) - \frac{3}{2} \Delta \rho^3 (1 - \xi) (1 + \frac{\alpha}{3} x - \frac{3}{2} \alpha x),
\]

or numerically, using (38) and (39)

\[
\Delta (1 - \xi) (1 + \frac{\alpha}{3} x + \frac{3}{2} \alpha x) = [1 + \frac{3}{2} \alpha x + \frac{3}{2} \alpha x] \xi + \frac{1}{7} \Delta \beta^6 \beta^* \xi = \frac{1}{7} \Delta \rho^3 (1 + \frac{\alpha}{3} x) - \frac{3}{2} \Delta \rho^3 (1 - \xi) (1 + \frac{\alpha}{3} x - \frac{3}{2} \alpha x),
\]

Denoting the right hand members of (38), (39), (40), (41) by \( A, B, C, D \) respectively and the coefficients of \( \xi \) and \( \xi^2 \) in (41) by \( E \) and \( F \), we have for every value of \( \theta \) two determinations of \( \xi \), one by (39) and (40), viz:

\[
\xi = \frac{1}{C} \left( 1 + M \right)
\]

and one by (38), (39), (41), which leads to an equation in \( \xi \) of the form

\[
\frac{1}{2} (G + E) \xi - (\frac{1}{2} G + F) \xi^2 + \frac{1}{2} G \xi^2 = G - D,
\]

where the powers of \( \xi \) higher than the third have been neglected, and we have put

\[
G = A^2, \quad B = \frac{2}{7} \left( 1 + N \right).
\]

The corrections \( M \) and \( N \) are defined as follows

\[
M = \frac{2 \alpha x}{7 + \alpha x},
\]

\[
1 + N = \left[ 1 + \frac{3}{2} x + \frac{3}{2} \alpha x \right] \left[ 1 + \frac{3}{2} \left( 1 - \alpha \right) x \right] \left[ 1 + \frac{3}{2} x \right] - \frac{3}{2}.
\]

The determination (42) depends on \( \lambda_n \), which is involved in \( C \). The determination (43) is independent of \( \lambda_n \).

Now taking first \( x = 0 \) and consequently \( M = N = 0 \). I find

\[
\theta = 0, \quad \theta = 0 + 1
\]

from (42)

\[
\begin{align*}
\xi &= \frac{0.5346}{0.07641} \\
\text{from (43)} &= \frac{0.5313}{0.07618}
\end{align*}
\]

Difference

\[
\begin{align*}
+0.0033 & \quad +0.0023
\end{align*}
\]

It is evident that in order to get exact agreement of the two determinations of \( \xi \), we must take a large value of \( \theta \), which is entirely outside the limits of possibility. The differences \( \xi - \xi^* \) are however very small, and, as especially the determination by (42) is very sensitive to a small change in \( \xi \), it would be possible to bring them down to zero by slightly altering the adopted value of \( \approx 1 \).

We can present this result in a different way as follows. With the value of \( \xi \) from (43) we compute \( \Delta \) and \( \rho \) from (38) and (39), and then \( \lambda_n \) from (40), or (37). This gives
for \( \theta = 0 \): \( \lambda_{\theta} = +0.0031 \pm 0.0030 \)

\[
\begin{align*}
\theta = +1 & : \lambda_{\theta} = +0.0039 \pm 0.0030 \\
\theta = +2 & : \lambda_{\theta} = +0.0047 \pm 0.0040 \\
\theta = +3 & : \lambda_{\theta} = +0.0057 \pm 0.0050 \\
\theta = +4 & : \lambda_{\theta} = +0.0071 \pm 0.0070 \\
\theta = +5 & : \lambda_{\theta} = +0.0088 \pm 0.0080 \\
\theta = +6 & : \lambda_{\theta} = +0.0109 \pm 0.0100 \\
\theta = +7 & : \lambda_{\theta} = +0.0136 \pm 0.0130 \\
\theta = +8 & : \lambda_{\theta} = +0.0169 \pm 0.0160 \\
\theta = +9 & : \lambda_{\theta} = +0.0211 \pm 0.0210 \\
\theta = +10 & : \lambda_{\theta} = +0.0266 \pm 0.0260 \\
\end{align*}
\]

The probable errors correspond to the adopted probable error of \( \varepsilon^\cdot \). Both values therefore agree with the adopted value (29) within the uncertainty of the data.

It would also be possible to effect exact agreement by a slight change of the adopted depth of the isostatic surface.

We can thus say that the hypothesis of Wiechert with one surface of discontinuity and constant densities below it and between it and the isostatic surface, represents a possible constitution of the earth. The values of \( \rho \) and \( \Delta \) are

<table>
<thead>
<tr>
<th>( \theta = 0 )</th>
<th>( \Delta \delta_{\theta} )</th>
<th>( \Delta \xi )</th>
<th>( \theta = +1 )</th>
<th>( \Delta \delta_{\theta} )</th>
<th>( \Delta \xi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta )</td>
<td>.9160 ± 12 + 5 = 37</td>
<td>.8831 ± 16 + 3 = 60</td>
<td>.7841 ± 5 + 5 = 41</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho )</td>
<td>.8325 ± 4 + 3 = 27</td>
<td>.7841 ± 5 + 5 = 41</td>
<td>.7700</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta_{\theta} )</td>
<td>.8175</td>
<td>.7700</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The probable errors correspond to that of \( \varepsilon^\cdot \). The quantities given under \( \Delta \delta_{\theta} \) and \( \Delta \xi \) are the effects on \( \Delta \rho \) and \( \rho \) of the already mentioned estimated maximum changes in \( \delta_{\theta} \) and \( \xi \) respectively. The effect of \( \delta_{\theta} \) and \( \rho \) is negligible.

If instead of assuming the density to be constant below the surface of discontinuity, we allow it to vary according to Roché’s law, we introduce an additional parameter \( \Delta' \) and we have thus one more unknown than there are conditions to be satisfied. The equation (42) gives for the correction to the value of \( \xi \) for \( x = 0 \):

\[
\delta_{\theta}, \xi = -\frac{B}{C} M
\]

and the equation (43) gives for the correction to the other determination of \( \xi \) with sufficient approximation:

\[
(\xi G + E) \delta_{\theta}, \xi = G. N (1 - \xi) \frac{1}{\xi}
\]

The small quantities \( M \) and \( N \) are easily tabulated. We have for small values of \( x \) and \( \alpha \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( M )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.8355</td>
<td>.8205</td>
</tr>
<tr>
<td>+1</td>
<td>.7859</td>
<td>.7718</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x = \cdot025 )</th>
<th>( x = \cdot05 )</th>
<th>( x = \cdot025 )</th>
<th>( x = \cdot05 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.0000 + 00000</td>
<td>00000</td>
<td>00000</td>
<td>00000</td>
</tr>
<tr>
<td>.0042</td>
<td>.042</td>
<td>.010</td>
<td>.010</td>
</tr>
<tr>
<td>.0048</td>
<td>.048</td>
<td>.018</td>
<td>.018</td>
</tr>
<tr>
<td>.0064</td>
<td>.064</td>
<td>.026</td>
<td>.026</td>
</tr>
<tr>
<td>.0080</td>
<td>.080</td>
<td>.034</td>
<td>.034</td>
</tr>
<tr>
<td>.0100</td>
<td>.100</td>
<td>.042</td>
<td>.042</td>
</tr>
</tbody>
</table>

With the aid of this table it is easy to find values of \( x \) and \( \alpha \) which will make \( \xi_{\varepsilon} + \delta_{\theta}, \xi = \xi_{\varepsilon} + \delta_{\theta}, \xi \).

Thus e.g. taking \( x = \cdot025 \) I find:

\[
\begin{align*}
\theta = 0 & : x = \cdot0532, \xi = \cdot05310 \\
\theta = +1 & : x = \cdot0367, \xi = \cdot07618
\end{align*}
\]

It would of course be a very improbable accidental coincidence, if it so happened that these values of \( \xi, x \) and \( \alpha \) were compatible with each other. To decide this we must derive \( \alpha \) by integration:

\[
\alpha = \frac{1}{\beta^3 \varepsilon^\cdot} \int_0^\varepsilon^\cdot \varepsilon' (1 + \beta \varepsilon') d\varepsilon.
\]

I have not thought it worth while to execute this integration, which would involve very long and laborious calculations. It is evident that for every value of \( x \) above a certain, very low, limit *, there is a value of \( x \) bringing the two determinations of \( \xi \) into agreement, and there can thus be no reasonable doubt that the problem to find compatible values of \( x, \xi \) and \( \alpha \) is soluble, even for an arbitrarily assumed value of \( \theta \).

The values of \( \rho, \Delta, \Delta' \) and \( \delta_{\theta} \) corresponding to the values (45) of \( \xi \), are:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \rho )</th>
<th>( \beta_{\theta} )</th>
<th>( \Delta )</th>
<th>( \Delta' )</th>
<th>( \delta_{\theta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.8355</td>
<td>.8205</td>
<td>.8866</td>
<td>.0464</td>
<td>1.4330</td>
</tr>
<tr>
<td>+1</td>
<td>.7859</td>
<td>.7718</td>
<td>.8644</td>
<td>.0318</td>
<td>1.4962</td>
</tr>
</tbody>
</table>

These, of course do not differ very much from (44), and, as the real constitution of the earth can not be assumed to follow exactly either the simple or the amended theory of Wiechert, the exact values of these quantities are not of much interest. The principal point is that we have shown that the hypothesis of Wiechert is not in contradiction with observed facts, and can be varied in several ways without coming into such contradiction.

It may have some interest to compute on the basis of the theory of Wiechert the depression \( \varepsilon \) of the ideal surface below the ellipsoid. We have

\[
(46) \quad x = \delta_{\theta} K - \varepsilon^\cdot \varepsilon' + \delta_{\theta} \varepsilon + \delta_{\theta} \xi + \delta_{\theta} \xi = \delta_{\theta} K - \cdot00000 0264,
\]

and

\[
(47) \quad K = \frac{1}{\beta^3 \varepsilon^\cdot} \int_0^\varepsilon^\cdot \left[ \beta' \varepsilon^\cdot \right] d\beta + \int_0^\varepsilon^\cdot \frac{d}{d\beta} \left[ \beta \varepsilon^\cdot \right] d\beta.
\]

The first integral can be completely solved by means of the values of \( \Delta, \rho, \) and \( \xi \), which have been found above. We have

\(* \) We cannot take \( x = 0 \) since, \( \rho \) being negative, \( \varepsilon, \xi \) must be therefore also negative, and numerically larger than \( \delta_{\theta}, \xi \), whereas with \( x = 0 \), we have \( M = 0 \) and \( \delta_{\theta}, \xi = \cdot0 \).
(48) \[ \int_{\theta=0}^{1} \frac{d}{d\beta} \left[ \beta^2 \varphi \right] d\beta = \beta^2 \varphi \left[ \beta + \Delta \rho \left( 1 - \frac{1}{2} \beta^2 + \frac{1}{4} \beta^4 \right) \right] + \int_{0}^{1} \frac{d}{d\beta} \left[ \beta^2 \right] (\beta + 2\beta^3) d\beta, \]

where the second integral can be easily approximated in the same way as the various integrals from \( s \) to \( r \) treated above. We find thus for the integral (48)

\[ \beta = 0 : 000000772 + 0077 = 000000849 \]
\[ + \beta = 1 : 00724 + 0081 = 00805. \]

As to the second integral in (47) we have

(49) \[ \frac{d}{d\beta} \left[ \beta^2 \right] d\beta = \beta \varphi \left[ \beta + \Delta \rho \left( 1 - \frac{1}{2} \beta^2 + \frac{1}{4} \beta^4 \right) \right] + \int_{0}^{1} \frac{d}{d\beta} \left[ \beta^2 \right] d\beta. \]

Now at the surface we have

\[ \beta \frac{d\varphi}{d\beta} = - 4 \xi - \frac{1}{2} \xi \rho + \frac{5}{2} \rho^2 = - 4 \xi + 00000040, \]

by means of which we can approximate the last integral in (49), and we also find

\[ \xi = 0.28 \varphi - 000000008. \]

We thus find for the integral (49)

\[ \beta = 0 : 0456 \varphi + 0321 \varphi + 000000003 \]
\[ + \beta = 1 : 0165 + 00724 + 000000003. \]

Combining the two parts of (47) we have finally

(50) \[ \beta = 0 : x = 0000000053 + 0152 \varphi + 0107 \varphi \]
\[ + \beta = 1 : 0039 + 0165 + 0078. \]

Now we have \( 0 < x < \beta \), and consequently

(51) \[ \beta = 0 : 0000000063 < x < 0000000072 \]
\[ + \beta = 1 : 0047 < x < 0052. \]

It would of course be possible to compute the exact value of \( x \) from the adopted data, but this appears hardly worth doing.

**Summary.**

1. 2. 3. The development to the second order in the compression of the different equations of the theory of Clairaut is found to give very narrow limits for the ratio \( q = J/H = \frac{1}{2} \left| C/M \beta^2 \right| \), if instead of the equatorial radius we use as an independent variable the mean radius of the spheroids. These limits are discussed, and the resulting value of \( q \) is given by formula (30).

4. The deviation of the actual moments of inertia from the values they would have if the earth were up to the surface in hydrostatic equilibrium, are computed on the basis of the theory of isostasy and are found to be negligible. The effect on \( q \) would be negligible even if there were no isostatic compensation.

5. This justifies the use of the value (30) of \( q \) for the determination of the compression \( \varepsilon \) from the precessional constant. The result is

(54) \[ \varepsilon = 29692 \pm 136. \]

The probable error is due almost entirely to the uncertainty of the mass of the moon.

6. 7. Conditions are formed which must be satisfied by any hypothesis regarding the distribution of mass inside the earth. It is shown that Roche's hypothesis does not satisfy these conditions. Wieschert's hypothesis on the other hand can be made to satisfy them within the uncertainties of the data with only one surface of discontinuity and constant densities below it and between it and the isostatic surface, and the agreement can be made exact if the density below the surface of discontinuity is supposed to follow Roche's law. The discussion is carried out for two values of the density between the surface of discontinuity and the isostatic surface. The results are given in the table (44), and are here for convenience repeated, expressing the radii in kilometers and the densities in that of water as a unit. The depth of the isostatic surface at the latitude 35°265 is assumed to be 114.7 km. Taking for the mean density of the earth 5.52 the adopted surface density becomes 2.7324 and the density is assumed to increase regularly from this value to \( \beta \) at the isostatic surface. We then find:

- density above surface of discontinuity \( \beta_s : 2.760 \), 3.312
- density below surface of discontinuity \( \beta_s + \Delta : 7.816 \), 8.187
- mean radius of surface of discontinuity \( \beta_s : 5208 \text{ km}, 4906 \text{ km} \).

It must be left to geologists to decide whether these values are acceptable.