1. Consider the functional differential equation (FDE)

\[ \frac{d}{dt} Mx_t = Lx_t, \quad t \geq 0, \]

where \( M \) and \( L \) are continuous linear operators from \( C \) \( \overset{\text{def}}{=} \mathcal{C}([-r, 0], \mathbb{C}^n) \) to \( \mathbb{C}^n \), and \( M \) is “atomic at zero”, i.e., \( M \varphi = \varphi(0) + M_0 \varphi \) where \( M_0 \varphi \) is “essentially” independent of the value of \( \varphi \) at \( \theta = 0 \). Let \( x_t \in C \) be defined by \( x_t(\theta) = x(t + \theta) \).

Define the solution semigroup \( T(t) : C \to C \) by \( T(t) \varphi = x_t, \quad t \geq 0 \), where \( x \) is the (unique) solution of (1) with initial condition \( x_0 = \varphi \). Let \( \Lambda \) be the (possibly infinite) set of those \( \lambda \in \mathbb{C} \) such that \( x(t) = ce^{\lambda t} \) is a solution of (1) for some \( 0 \neq c \in \mathbb{C}^n \). If \( \lambda_0 \in \Lambda \) is a dominant eigenvalue, that is, if there exists \( \epsilon > 0 \) such that \( \lambda_0 \neq z \in \Lambda \) implies \( \text{Re} \ z < \text{Re} \lambda_0 - \epsilon \), then there exists a (unique) non-empty \( T(t) \)-invariant manifold \( \mathcal{M}_{\lambda_0} \) and \( K > 0 \) such that

\[ e^{-\text{Re} \lambda_0 t} \|T(t)(I - P_{\lambda_0}) \varphi\| \leq K e^{-\epsilon t} \| \varphi \|, \quad t \geq 0, \]

where \( P_{\lambda_0} \) is the spectral projection from \( C \) onto the finite dimensional space \( \mathcal{M}_{\lambda_0} \). Formula (2) yields the large time behaviour of solutions of (1). (See Lemma 2.1 and the figure on the cover of this thesis. See also [2].)

2. Explicit formulas for the spectral projections onto the eigenspaces of functional differential equations can easily be obtained using the associated infinitesimal generator and Dunford calculus. This is an algorithmic approach that can be implemented using a computer algebra system such as Maple. (See Chapter 3 and Chapter 4 of this thesis.)

3. Sufficient conditions for the existence of a dominant root of the (characteristic) equation

\[ \Delta(z) = z \left( 1 + \sum_{i=1}^{m} c_i e^{-z \sigma_i} \right) - a - \sum_{j=1}^{k} b_j e^{-z \rho_j}, \]

can be obtained using the scalar-valued positive nondecreasing function \( V(\lambda) \) defined by

\[ V(\lambda) \overset{\text{def}}{=} \sum_{i=1}^{m} |c_i| (1 + |\lambda| \sigma_i e^{-\lambda \sigma_i}) + \sum_{j=1}^{k} |b_j| \rho_j e^{-\lambda \rho_j}, \quad \lambda \in \mathbb{R}, \]

which does not depend on the coefficient \( a \) of \( \Delta(z) \). (See Chapter 5 of this thesis.)

4. Fix \( n, m \in \mathbb{N}, b_i, c_j \in \mathbb{R} \) and \( h_i, \sigma_j \in (0, r] \) with \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \). If \( \lambda_0 = V^{-1}(1) \), then for any \( \lambda > \lambda_0 \), there exists \( a \in \mathbb{R} \) such that \( \lambda \) is a real simple dominant root of the characteristic equation (3). (See the comments on Chapter 5 of this thesis.)

5. Consider a non-linear FDE with only discrete time delays such that \( 0 \in \mathcal{C} \) is an equilibrium and suppose that \( \lambda = 0 \) is the dominant root of the characteristic equation of the linearization around \( 0 \in \mathcal{C} \). One can explicitly compute the ODE that describes the flow on the center manifold, restricted to a neighbourhood of \( 0 \in \mathcal{C} \), using as elements the space \( \mathcal{M}_0 \) and the spectral projection \( P_0 \). (Typically \( P_\lambda \varphi \) depends on \( \varphi(0) \) and integrals involving \( \varphi \).) Actually, the ODE only depends on the terms of \( P_\lambda \varphi \) that involve value of \( \varphi \) at \( \theta = 0 \). (See Chapter 7 of this thesis.)
6. Let $\eta \in \text{NBV}([0, r], C^{n\times n})$ and define the convolution product $d\eta * f$ to be

$$d\eta * f(t) \overset{\text{def}}{=} \int_0^\infty d\eta(\theta)f(t - \theta).$$

The map $f \mapsto d\eta * f$ maps $C_R \overset{\text{def}}{=} \{ f \in C([0, r], C^n) : f(0) = 0 \}$ into itself. However, $f \mapsto d\eta * f$ does not map $C([0, r], C^n)$ into itself. (See item 4 of Remark 1.1 of this thesis.)

7. Let $F$ and $G$ be continuous functions from $[t_0, \infty) \times \mathbb{R}^n$ to $\mathbb{R}^n$, and let $D$ denote the differentiation operator. Equations of the type

$$Dx = F(t, x) + G(t, x)Du, \quad x(t_0) = x_0,$$

where $u$ and the solution $x$ belong to $\text{NBV}([t_0, \infty), \mathbb{R}^n)$ and equality in (4) is considered in the space of distributions $\mathcal{D}(\mathbb{R}^n)$, are called “measure differential equations”. The solutions of (4) are piecewise differentiable with possible “jumps” in the discontinuities of $u$. (See [1, 5].)

8. In a topological space $T$, consider the set functions $A \mapsto \overline{A}$ (closure of $A$) and $A \mapsto A^c$ (complement of $A$). For any set $A \in T$, at most 14 different sets are obtained by repeated applications of closure and complementation to $A$. The subset $A = (0, 1) \cup (1, 2) \cup [2, 3) \cap \mathbb{Q} \cup \{4\}$ of the real line is an example that the 14 sets can be obtained. This result is due to Kuratowski [3]. See also [4].

9. The Netherlands is a great country but not in size. The province of São Paulo in Brazil covers less than 3% of the area of Brazil, but is 6 times the size of The Netherlands.

10. One could conclude the existence of God from the testimony of the children of Israel who were freed from slavery in Egypt in a sequence of amazing events, like the crossing of the Red Sea on foot, among others. Confer the web page of the Israeli government

http://www.mfa.gov.il/MFA/History/History of Israel/

and the book Exodus of the Bible.

References


