STRUCTURE AND DYNAMICS OF MESSIER 3

by J. H. Oort and G. van Herk

An attempt has been made to understand the observed structure of Messier 3. In the region up to about 8 pc from the centre encounters of stars must have set up an approximately Maxwellian velocity distribution as well as an approach to equipartition of energy. The cluster extends to at least 50 and perhaps to 100 parsec. Between 10 and 100 pc the effects of encounters must have been almost negligible. But the fact that the density distribution in this outer region fits so smoothly to that in the nucleus suggests that there has been an important interchange of stars between inner and outer regions. It is shown that a satisfactory dynamical representation of the entire cluster can be obtained by assuming a velocity distribution of the type shown in formulae (9) and (11). At small distances from the centre this approaches the Maxwellian velocity distribution, for larger distances it becomes elongated in the direction of the centre. A suitable choice of the parameter $K'(15)/h^2$ ensures that nearly all stars far from the centre have orbits passing through the central region where the velocity distribution is regulated by encounters. Most of the calculations in this article were made with a value of 7 for the above parameter for stars above the main sequence.

The evolution of the cluster might tentatively be pictured as follows. It may initially have been a compact group of considerably smaller dimension than at present. The heavier stars would have been most concentrated towards the centre. In their transition to the white-dwarf stage these stars must have expelled mass. It is estimated that by this process the total mass of the cluster will have been reduced to something like 40% of its initial value. The expulsion of mass in this way should by itself have caused roughly a doubling of the radius of the cluster. Stellar encounters in the nuclear region will have brought part of the stars into strongly elongated orbits, thus populating the outer regions. This caused a considerable shrinking of the nucleus, which may approximately have counterbalanced the expansion due to gas ejection. The encounters may have been sufficient to approach a semi-equilibrium between the motions of the outer stars and those in the nuclear part. This semi-equilibrium is likely to be of the nature indicated in formulae (9) and (11).

The investigation was based on star counts down to the 22nd photometric magnitude, made by Sandage. The densities were extended to larger distances with the aid of earlier counts by von Zeipel. In this way direct data for the density distribution of about half the total mass in the cluster were obtained. The space densities are given in Table 5. In order to obtain estimates for the distribution of the remaining mass various assumptions were made concerning the luminosity curve and the number of white dwarfs (cf. Table 9). To some extent these may be tested by comparing the observed distribution of stars down to 19th. With that computed from the mass distribution in these various models and assumed velocity distribution (cf. Table 11). The calculated average random velocity in radial direction ranges from 2.6 to 4.9 km/sec for the different models. The ratio of total mass to total light (expressed in the sun as unit) varies from 0.10 to 0.62. The most probable model of the cluster is somewhere between (d) and (e) of Table 9. The corresponding total mass would be about 150 000 solar masses, the mass-to-light ratio about 0.25 and the mean internal radial velocity 2.9 km/sec. The velocity distribution has always been cut off at the velocity of escape.

Detailed data for one of the models are in Tables 5 and 12.

In section 8 a discussion is given of possible differences between the distribution of RR Lyrae variables and general stars in four clusters that are rich in variable stars. It is found that such differences might be explained as a result of loss of mass prior to the RR Lyrae stage and subsequent re-adjustment of equipartition. The mass loss would have to be of the order of 20%. The last section contains a rudimentary discussion of structural differences between clusters depending on the fraction of the cluster in which a Maxwellian velocity distribution has been established. It is shown that on this basis the observed differences between clusters can be explained, at least in a qualitative manner.

1. Introduction

The purpose of the investigation described in this article is to see in how far the observed distribution of the stars corresponds to what we should expect theoretically, and to investigate what the comparison between observation and theory can teach us concerning the number of unobserved faint stars.

When the investigation was half-way completed, an article by von Hoerner (1957) appeared in which much the same aim was pursued. We have nevertheless decided to publish an account of our calculations because of the different way of approach used. In particular, we have made more extensive use of the observational material on star counts throughout the cluster. As regards the mass distribution for a given distance from the centre, this was taken from the direct observations down to $M_V = +6.3$, and only the part referring to still fainter stars was inferred indirectly from the space distribution of bright stars. Furthermore, we have attempted — albeit in a primitive way — to find a theoretical model which is not restricted to the inner, equipartition region, but gives a coherent representation of the entire cluster.
The subject of the dynamics of globular clusters has been previously discussed by many authors. A calculation introducing preferential radial motions of stars in the outer parts of a cluster has been recently published by Woolley and Denise Robinson (1956). In the present investigation this problem is considered in somewhat more detail.

2. Star counts and space densities

The available observational material consists of star counts by von Zeipel (1913), measures of surface brightness by Hertzsprung (1918), and recent star counts by Sandage (1954). Up to a radius \( r = 5'3 \) von Zeipel’s counts were made on positive copies of plates taken with the 40-inch Yerkes refractor. For larger distances, to about 40', they were made on plates taken with the Carte-du-Ciel refractor of the Paris Observatory (von Zeipel 1908). He has taken account of the images which disappeared through overlapping. Comparison with Sandage’s numbers indicates that von Zeipel’s limit must have been roughly between 15m.5 and 16m.0 pv. Hertzsprung made his photographic measures of the brightness distribution on interfocal plates taken with the 80-cm Potsdam reflector. In these measurements the difficulties of counting stars near the crowded centre disappear. The star counts by Sandage were made on 10 plates taken with the 200-inch reflector, with exposure times varying from 5' to 50'; they extend to \( r = 8' \). The actual counts have not been published, but they were kindly put at our disposal by Dr Sandage. The magnitudes in Sandage’s 1954 article require corrections. These were determined in a three-colour photometry of M3 made by H. L. Johnson and Sandage (1956). The magnitudes used in the following will always be the magnitudes corrected to the \( V \) scale with the aid of the data just mentioned. Dr Sandage has informed us that the faintest magnitude, at \( V = 22.0 \), is uncertain by perhaps \( \pm 0.3 \). Table 1 shows the logarithms of the numbers of stars per square minute, \( \sigma \), as found by Sandage. The numbers given refer to counts down to the magnitude indicated at the top of each column, made in rings of 0.5 width around the distance shown in the first column.

| \( r(') \) | \( r(pc) \) | 14.1 | 15.0 | 16.1 | 16.9 | 17.6 | 19.3 | 19.8 | 20.3 | 21.1 | 22.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.125 | 0.44 | 1.952 | 2.267 | 2.608 | \| | | | | | |
| 0.375 | 1.32 | 1.643 | 2.083 | 2.409 | 2.615 | 2.383 | \| | | | |
| 0.75 | 3.05 | 1.322 | 1.667 | 2.135 | 2.361 | 2.048 | \| | | | |
| 1.25 | 4.41 | 0.81 | 1.242 | 1.723 | 1.872 | 2.611 | 2.643 | \| | | |
| 1.75 | 6.18 | 0.37 | 0.79 | 1.428 | 1.589 | 1.732 | \| | | | |
| 2.25 | 7.94 | 0.26 | 0.58 | 1.122 | 1.324 | 1.477 | 2.192 | 2.415 | 2.514 | 2.722 | 2.786 |
| 2.75 | 9.71 | 0.08 | 0.47 | 1.000 | 1.130 | 1.226 | 1.988 | 2.241 | 2.375 | 2.604 | 2.673 |
| 3.25 | 11.5 | \| | | | | | | | | |
| 3.75 | 13.2 | 0.18 | 0.71 | 0.90 | 1.085 | 1.871 | 2.074 | 2.234 | 2.443 | 2.551 | \| |
| 4.25 | 15.0 | \| | | | | | | | | |
| 4.75 | 16.8 | \| | | | | | | | | |
| 5.25 | 18.5 | \| | | | | | | | | |
| 5.75 | 20.3 | \| | | | | | | | | |
| 6.25 | 22.1 | \| | | | | | | | | |
| 6.75 | 23.8 | \| | | | | | | | | |
| 7.25 | 25.6 | \| | | | | | | | | |
| 7.75 | 27.4 | \| | | | | | | | | |
| 8.25 | 29.1 | \| | | | | | | | | |

The main sequence in M3 breaks off around \( M_V = +3.4 \). According to the theory of stellar evolution developed by Schwarzschild, Sandage and others, all stars brighter than this limit will have had practically identical masses when they were on the main sequence. Though presumably they lose mass before they become white dwarfs, this mass loss is likely to occur mainly in a late period of their evolution, which is short compared with their total age. We may reasonably expect that, in general, the mass loss has not much affected their dynamical properties and that, consequently, the space distribution of all stars brighter than \( +3.4 \) absolute visual magnitude will be practically the same. This expectation is fully confirmed by an intercomparison of Sandage’s counts to successive magnitude limits (cf. Figure 1). There is a slight indication that the stars brighter than 14m.1 are more concentrated in the nucleus. The possible existence of a small difference in distribution for the RR Lyrae stars will be discussed in section 8.

In the following we have assumed that all stars
bigger than $M_V + 3.4$ have the same distribution. These stars will be designated as group $A$. Near the centre, only the brighter stars of this group can be counted, while in the outer parts we have to rely on the more numerous fainter stars. With the above assumption the counts to various magnitudes can be fitted together to give a reliable distribution curve over the entire range of $r$ covered by the counts.

Following Sandage (1957a), the apparent modulus $m - M$ was taken to be 15.68. If we assume an interstellar absorption of $a = 0.25$, the corresponding distance is 12.1 kpc. At this distance $1' = 3.53$ pc. The apparent magnitude corresponding to $M = + 3.4$ is 19.1. This being close to the value 19.2 used by Sandage as one of the limiting magnitudes in his counts, we have included in our group $A$ all stars brighter than 19.2.

The star counts to limits brighter than 19.2 were reduced to those in the column 19.2 by appropriate shifts in the log $\sigma$ co-ordinate. The resulting values are shown in Figure 1; the counts to 15.0 and 16.9 were omitted, in order to avoid excessive crowding of points. It may be noted that the agreement in the run of log $\sigma(r)$ with $r$ for the different magnitude limits is quite satisfactory, and confirms the theoretical expectation that there should be no systematic differences.

Very close to the centre the counts are uncertain, both on account of small numbers and of the intense background and crowding. In this part the surface-
brightness measures by Hertzprung (1918) may be more reliable. We have therefore also plotted these measures, the zero point having been adjusted so as to agree with the counts between \( r = 1' \) and \( r = 2' \). The final curve used in the present article, and indicated in Figure 1, has been made to fit Hertzprung's brightnesses for the part within \( r = 1' \). On the whole the agreement between the two kinds of data is quite good.

In these counts there appear to be no reliable indications of steps or waves in the run of \( \sigma \) or \( \log \sigma \) with \( r \), such as Kholopov (1953) believes exist for bright stars.

For comparison we have plotted in the same figure also logarithms of the numbers of stars between \( V = 19.2 \) and 20.3, and between 20.3 and 22.0. In the case of "isothermal" equilibrium the values of \( \log \nu (r) - \log \nu (0) \), where \( \nu \) denotes the space density, are proportional to the mass of the stars [cf. formula (4)]. If over the interval of \( r \) considered \( \log \nu \) can be represented by a linear function of \( r \), the same proportionality must hold for the logarithms of the surface densities. For the range between \( r = 3' \) and \( r = 8' \), over which the faintest stars have been counted, the relation is sufficiently linear, so that the derivatives of the \( \log \sigma \) curves in Figure 1 should be proportional to the masses of the stars considered. We see from Figure 1 that the slopes are indeed smaller for the faint stars. Table 2 gives the absolute

<table>
<thead>
<tr>
<th>( V )</th>
<th>mass</th>
<th>(-d \log \sigma (r)/dr) obs.</th>
<th>comp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;19.2</td>
<td>1.25</td>
<td>.235 (.235)</td>
<td></td>
</tr>
<tr>
<td>19.2-20.3</td>
<td>1.11</td>
<td>.186 .209</td>
<td></td>
</tr>
<tr>
<td>20.3-22.0</td>
<td>0.90</td>
<td>.173 .169</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Comparison of observed values of \( d \log \sigma (r)/dr \) with those expected for isothermal equilibrium. (unit of \( r \) 1 minute of arc)

values of the derivatives in the region between 4' and 8' from the centre. The second column contains the average mass in solar units; for the fainter groups it was taken from Allen, "Astrophysical Quantities", p. 186 (1955). It was used to obtain the computed values of \(-d \log \sigma (r)/dr\) in the last column, the first number being made equal to the observed value. The observed and computed slopes for the fainter stars will be seen to agree satisfactorily. The same conclusion follows if we use directly the space densities given in Table 5 for the points at 8, 15 and 30 pc, where the space densities were derived from the star counts. Computing the mass by means of formula (4) we find that the stars in group BC have a mass 0.89 times that of the stars of group A, or 1.11 solar masses, while the mean mass for group DE is found to be 0.91 solar masses. The two masses are in precise agreement with the values inferred from the absolute magnitudes, as given in Table 2.

Space densities have been derived, using a procedure suggested by A. J. Wesselink. Let the total numbers of stars of a given magnitude interval in rings of unit width be represented by a formula of the following form:

\[
2 \pi r \sigma (r) = N \sum_{i} \theta_i \cdot 2 \pi r^2 e^{-l_i r^2},
\]

where \( N \) is the total number of cluster stars in the magnitude interval, and \( \sum_{i} \theta_i = 1 \). It is easily seen that in the case of spherical symmetry (which we shall assume throughout this article) the numbers of stars in shells of unit thickness are then given by

\[
4 \pi r^2 \nu (r) = N \sum_{i} \theta_i \frac{4 \pi}{\pi} r^2 e^{-l_i r^2}.
\]

It follows that

\[
\nu (r) = N \pi^{-1/2} \sum_{i} \theta_i l_i^2 r^2 e^{-l_i r^2}.
\]

According to (1) the surface density is

\[
\sigma (r) = N \pi^{-1} \sum_{i} \theta_i l_i^2 r^2 e^{-l_i r^2}.
\]

This procedure for obtaining the space densities is more satisfactory than the usual procedure with the aid of counts in strips. With the latter method the density in the nuclear part is always diluted with large numbers of stars in the outer parts, so that the accuracy of the determination of the nuclear density is diminished.

The smooth curve representing the star counts down to 19.2 in Figure 1 can be satisfactorily represented by the set of parameters indicated in Table 3.

<table>
<thead>
<tr>
<th>( \theta_i )</th>
<th>( l_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.207</td>
<td>2.17</td>
</tr>
<tr>
<td>.353</td>
<td>0.93</td>
</tr>
<tr>
<td>.216</td>
<td>.45</td>
</tr>
<tr>
<td>.284</td>
<td>0.20</td>
</tr>
</tbody>
</table>

\( N = 20000 \)

Table 4 shows the comparison between the observed and computed surface densities. The last column gives the space densities computed from (2).

We shall now compare the space densities derived from Sandage's counts down to 19.2 with those
Table 4

Surface and space densities of stars brighter than $19^{m}.2$
(unit of length everywhere $r$)

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\sigma(r)$ obs.</th>
<th>$\sigma(r)$ comp.</th>
<th>$\nu(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0'$.</td>
<td>8510</td>
<td>8510</td>
<td>8720</td>
</tr>
<tr>
<td>0.5</td>
<td>3800</td>
<td>3800</td>
<td>3230</td>
</tr>
<tr>
<td>1</td>
<td>1150</td>
<td>1160</td>
<td>561</td>
</tr>
<tr>
<td>2</td>
<td>234</td>
<td>233</td>
<td>68.4</td>
</tr>
<tr>
<td>3</td>
<td>85</td>
<td>85</td>
<td>16.3</td>
</tr>
<tr>
<td>4</td>
<td>41</td>
<td>41</td>
<td>6.16</td>
</tr>
<tr>
<td>6</td>
<td>13.6</td>
<td>13.9</td>
<td>1.59</td>
</tr>
<tr>
<td>8</td>
<td>4.6</td>
<td>4.4</td>
<td>-0.0</td>
</tr>
</tbody>
</table>

found by von Zeipel (1908 and 1913), and extend them to larger distances with the aid of these latter data. The comparison is shown in Figure 2 for the part within 10 pc from the centre, and in Figure 3 for distances up to 100 pc. In each figure the crosses and the curve give the distribution obtained from Sandage’s data, except for the part beyond $\log r = 1.5$, which is based on von Zeipel’s counts. The dots show von Zeipel’s results. They are the logarithms of the values of $f(r)$ from his Table VIII in the 1913 publication, increased by 1.176 to reduce them to the zero point of $\log \nu$ from our Table 4. It will be seen that the agreement between the two determinations of the space distribution in the cluster is excellent.

von Zeipel did not publish any data on $\nu$ for distances greater than 31.5 pc. His counts, however, give fairly clear evidence of a farther extension of the cluster, to at least 13', or 46 pc (cf. his 1908 article, Table XIV, and the 1913 treatise, Table V). His surface densities, which had been corrected for foreground stars, were plotted against $\log r$. Between $r = 7$ and $r = 50$ pc they could be well represented by

$$\sigma(r) \propto r^{-2.6},$$

while beyond 30 pc the exponent may approach 3. The counts can be extrapolated to $r = 100$; the uncertainty at this point will probably not exceed 25%. We expect to extend faint-stars counts to these large distances, using plates taken with the 48" Palomar Schmidt telescope.

It can easily be shown that if the surface density varies as $r^a$ the space density must vary as $r^{a-1}$. With the aid of this expression and the above values for the exponents we have extrapolated the log $\nu$ curve in Figure 3 to $r = 100$ pc.

It may be noted that the exponent $-3$ for the surface density, corresponding to $-4$ for the space densities, agrees with the limiting value the exponent must assume at large distances in a cluster of finite mass. The theorem that the space density in the
outermost parts of a spherical cluster in dynamical equilibrium must always vary as \( r^4 \), irrespective of the velocity distribution, was first enunciated by Jeans (1915).

In accordance with the calculation of galactic tidal effects as made by von Hoerner, we assume that the cluster does not extend beyond a radius of 100 pc.

The counts made by Sandage for stars fainter than \( 19^m.2 \) were reduced to space densities by the same procedure as used for the stars down to \( 19^m.2 \). For this purpose the numbers of stars within each of these fainter-magnitude intervals were represented by a sum of a few Gaussian functions. However, because the counts did not extend to the part within \( r = 2' \), we could in this way obtain only space densities for the region outside \( 2' \). Extensions to the inner part were computed with the aid of the known density distribution of the brighter stars, assuming that

\[
\log v(r) - \log v(0) \text{ is proportional to the mass of the stars considered. This relation holds in the case of equipartition. We shall see in section 5 that we must expect at least a fair approach to equipartition in the region within } r = 8 \text{ pc. Moreover, we have seen in Figure 1 and Table 2 that in the range where the observed densities can be compared, they are in accord with this assumption.}
\]

The resulting densities, now expressed in solar masses per cubic parsec, are shown in the columns \( \nu \) of Table 5. The letter \( A \) refers to the group of stars brighter than \( 19.2 \), while \( B, C, D \) and \( E \) refer to the stars in the intervals \( 19.2 - 19.8, 19.8 - 20.3, 20.3 - 21.1 \) and \( 21.1 - 22.0 \), respectively. In the table, \( B \) and \( C \) as well as \( D \) and \( E \) have been combined, though in the computations all groups were treated separately.

The densities for the fainter groups at 50 and 100 pc are extrapolations based on the somewhat uncertain hypothesis that in the outer part, where the density of \( A \) approaches the \( r^4 \) law, the same will be true for all other masses. The run of \( \log v \) between 30 and 100 pc was, therefore, taken to be the same for all masses.

In Figure 4 the central densities are plotted against magnitude. Ordinates are logarithms of numbers of stars per pc\(^3\) per magnitude. For \( m < 16.1 \) they were taken from Sandage (1957a, Table 9) after multiplication by the proper constant. For the magnitude intervals from 16.1 to 19.2 they were obtained from the shifts in the log \( \sigma \) curves used in constructing

**Figure 4**

![Graph](image)

**Luminosity distribution at the centre of Messier 3.**

Abscissae: visual absolute magnitudes; ordinates: logarithms of numbers of stars per cubic parsec per interval of one magnitude. The extrapolated parts of the luminosity distribution, marked I and II, correspond to the curves I and II in Figure 5, and are explained in the subscript to that figure.
The faint stars for which we have to extrapolate consist of two kinds: main-sequence stars and white dwarfs. We have tried various assumptions for both. For the main-sequence stars we have first assumed that, integrated over the whole cluster, the frequency of stars below the break-off point at \( M = +3.4 \) would follow van Rhijn's general luminosity function for main-sequence stars. The suggestion that the luminosity curve in globular clusters might be the same as for stars in our surroundings was first made by Sandage (1957). The corrections required to reduce van Rhijn's (1956) luminosity function to main-sequence stars, as well as the smoothed values for the resulting distribution were taken from Sandage's Table 2 and Figure 2. In the following this will be referred to as \( \Phi_{vR} \).

For our calculations it is inconvenient to start from data referring to the cluster as a whole. As preliminary results had shown that the luminosity function at a distance of 15 pc from the centre would be roughly representative of that for the entire cluster, our various hypotheses regarding the luminosity function were actually applied to a point at \( r = 15 \) pc. The different assumptions for the luminosity distribution at this distance are shown in Figure 5. For \( M < +3.4 \) they are the same as the curve in Figure 4, except that the total numbers are lower by the factor 0.149/248, taken from Table 5. For \( M > +3.4 \) curve I approximately follows \( \Phi_{vR} \) with an appropriate shift in ordinates so as to make it fit approximately to the observed densities between +3.4 and +6.3 absolute magnitude. In the second place we have made calculations with a luminosity distribution that is two times lower than \( \Phi_{vR} \) between \( M_v = +6.3 \) and +7.9, and three times lower beyond \( M_v = +7.9 \) (curve II in Figure 5). This case will be referred to as \( \Phi_{vR'} \). The corresponding luminosity curves at the centre of the cluster have been indicated as curves I and II in Figure 4. These were computed from the distributions in Figure 5 assuming that \( \log \left( \frac{v(15)}{v(0)} \right) \) was proportional to the mass of the stars.

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1) The curve actually used in the calculations of this article lies 0.08 below \( \Phi_{vR} \). Moreover, in fitting \( \Phi_{vR} \) to the observed numbers between +3.4 and +6.3, no account was taken of the brightening that the stars now observed at \( M = +3.4 \) had undergone since their origin on the initial main sequence (see below). As a consequence, the stars which are now between +3.5 and +6.3 absolute magnitude were between about +4.4 and +6.3 on the initial main sequence. Per interval of one magnitude they must therefore initially have been \( 2.8 \) times more numerous than they are now. Strictly, the assumption that the original luminosity curve in M3 was the same as Salpeter's initial luminosity curve for the vicinity of the sun, would have required the extrapolated part to be \( 1.5 \) times higher for the filled circles in Figure 5, and \( 1.8 \) times higher than curve I.
As regards the white dwarfs we have again followed a hypothesis made by Sandoage. He suggested that the initial distribution of luminosities for the whole cluster was the same as the initial luminosity distribution in our vicinity as derived by Salpeter (1955). For stars that are so faint that evolution cannot have raised them appreciably above the original main sequence, the observed distribution must evidently be identical with this initial luminosity distribution. All stars brighter than +3.4 have definitely moved off the main sequence. Part of the latter have become white dwarfs, the rest are in the transition stage represented by the upper part of the Hertzsprung-Russell diagram (our group A).

From Table 5 we computed that the total number of stars between $M_V = +3.5$ and +4.6 is 25800, while between +4.6 and +6.3 it is 40500; the total between +3.5 and +6.3 is 66300. The stars that are at present near the break-off point at +3.5 have already brightened considerably compared to what they were when on the original main sequence. Following Sandoage (1957a) we have assumed this brightening to be $\phi^m.94$. At the lower limit of the interval the brightening must have been unimportant. The stars that are now observed in the interval from +3.5 to +6.3 must therefore have occupied the interval from +4.44 to +6.30 on the original main sequence. The initial luminosity function $\Psi(M)$, given in Sandoage's Table 2 (1957a), yields $6.1 \times 10^{-3}$ per pc$^3$ between +4.44 and +6.30. The total number brighter than +4.44 is $9.5 \times 10^{-3}$. If the same initial luminosity curve has existed in the cluster, and if there has been only one generation of stars, the total number of stars which has left the main sequence should be $\frac{9.5}{6.1} \times 66300 = 103400$. The total number in group A, as found from Table 4 or from Table 5, is 22800$^{1}$. The difference, 80600, gives

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1) Sandoage (1957b) gives only 9300 stars in this group. The reason for this being so much lower than the number given above is that he extrapolated his counts of fainter stars to the central part of the cluster by extending smooth curves drawn through the counts at larger distances, while we based our extrapolations on the assumption that all stars to $10^m.2$ have the same distribution. This leads to much higher central densities.
the number of white dwarfs formed in the cluster. This would thus be 3.5 times the number in group A.

By a similar calculation we find that the stars that are now in our group A must have had absolute magnitudes in the interval $+3^{M.52}$ to $+4^{M.44}$ when on the original main sequence. They do not, therefore, form an entirely homogeneous group. The ones furthest on in the evolution will have had larger initial masses than the stars that are still closer to the main sequence; they may thus be slightly more concentrated toward the centre. However, these differences should be small, and we have thought that they could be neglected in the present analysis.

For the mass of the white dwarfs in M3 we have made two alternative assumptions, viz. 1.25 and 0.5 in units of the sun’s mass. The latter is close to the value of 0.56 derived by Greenstein (1958) for the white dwarfs in our surroundings. These stars must have lost an appreciable part of their mass during the transition from the main sequence to the white-dwarf stage. It is not entirely certain that a similar mass loss has occurred in globular clusters. We have, therefore, also made computations with a mass of 1.25 times the sun’s mass, the same as that assumed for group A.

From a dynamical point of view the white dwarfs are likely to form an inhomogeneous group, because part of them have reached the small-mass stage at a fairly late epoch, so that they would not have had enough encounters with other stars to adjust their motions to these new masses. In view of the uncertainty regarding their masses it did not seem efficient to attempt computation of the resulting velocity distributions. We have therefore limited ourselves to computing models with different proportions and different masses for the white dwarfs, assuming that they have adjusted themselves to the “equilibrium” distribution corresponding to their present masses.

The data used in the various solutions are indicated in Table 9. In (a) the frequency of white dwarfs was taken to be 4.8 times that of group A, while in (b), (c) and (d) it was equal to that of A. In these four solutions the white dwarfs were assumed to have a mass of 1.25 $\odot$. In solution (e) the number of white dwarfs was 2.5 times the number in A, while their mass was taken to be 0.5 $\odot$.

4. Theory

For an average star in a globular cluster like M3 the time of revolution is of the order of ten million years. This is so short compared to the ages of the globular clusters that a thorough mixing of the stars must have taken place. The clusters will long ago have reached a state of “dynamical”, equilibrium. This well-mixed state will have been attained before there had been any important exchange of energy between individual stars. However, such exchange of energy must have played an essential role in the further evolution of the cluster. Various authors have estimated that the time of relaxation in globular clusters is of the same order as their age. The encounters will tend to set up a Maxwellian velocity distribution. According to von Hoerner’s calculations “relaxation” should be nearly complete up to 10 pc from the centre. This is confirmed by the calculations described below.

It must be expected that this inner region will be approximately in isothermal equilibrium. The density distribution for such an equilibrium state can be readily computed as soon as we know the distribution function of the masses at one point in the cluster. The computation may be made as follows. We start from the density distribution $\rho_A(r)$ of the stars in group A as given in Table 5. Let the mass of these stars be $M_A$. For stars of another mass $M_X$ the density is then given by

$$\log \rho_X(r) = \frac{M_X}{M_A} \log \rho_A(r) + f(M_X).$$

The term $f(M_X)$ is to be adjusted so as to reproduce the assumed mass distribution at $r = 15$ pc. By means of (4) we can derive the total mass density $\rho$ as a function of $r$. The force $K(r)$ may then be computed from

$$K(r) = - \frac{4\pi G}{r^2} \int_0^r r^2 \rho(r) \, dr,$$

where $G$ is the constant of gravitation. The potential $\Phi(r)$ is

$$\Phi(r) = \Phi(\rho) - \int_0^r K(r) \, dr.$$

If the velocity distribution follows Maxwell’s law, we have the following relation between $\nu$ and $\Phi$

$$\frac{\nu(r)}{\nu(\rho)} = e^{-\frac{1}{2} \frac{kT}{\mu(\rho)}} \Phi(\rho) - \Phi(\rho_0),$$

where $2kT$ is the reciprocal of the mean square velocity in one co-ordinate.

If our assumptions were correct the density computed from (7) should agree with the density from which we started. This test can best be applied to group A, for which the density is relatively well known over a large range of $r$.

In reality the cluster will deviate in several respects from strict isothermal equilibrium. In the first place the cluster cannot for an appreciable length of time contain stars with velocities above the velocity of escape, $V_e$. We must therefore cut off the velocity distribution at or below $V_e$. In the following we have
assumed this cut-off to be exactly at the velocity of escape and to be gradual. Actually, it will be gradual. Already somewhat below $V_e$ the velocity distribution will fall below that corresponding to thermodinamical equilibrium. This effect has recently been discussed by Spitzer and Harm (1958). It is of importance in connection with the rate of evaporation of clusters. We have not tried to take it into account, as it would have still further complicated the calculations, and because it turned out that within one mass group the cut-off factors varied relatively little with the distance from the centre.

With a sharp cut-off at the velocity of escape the cut-off factor $c(r)$ by which the density will be reduced as compared to the strictly Maxwellian case is given by

$$c(r) = \frac{2}{\sqrt{\pi}} \left( \int_0^{BV_e} e^{-x^2} dx - h V_e e^{h^2V_e^2} \right).$$ (8)

In the second place there is no reason to expect that in the outer parts, where the encounters are ineffective, the velocity distribution has become Maxwellian. Most of the following discussions will be based on the working hypothesis that these outer regions are populated by stars whose orbits pass through the more central part where the encounters are effective. It is possible, of course, that there are also stars which were born in these outer parts and endowed with sufficient transverse motion to prevent them from falling in towards the nuclear region. But it seemed of interest to investigate how far we can get in explaining the observed structure of globular clusters without introducing that additional complication.

If the stars at large $r$ are to move through the inner region of the cluster, their transverse velocities must be small compared with the radial motion. The following procedure may be used to give a simple mathematical representation of conditions such as proposed above. We represent the velocity distribution at a distance $r$ from the centre by

$$\varphi(u,v) = L e^{-h^2u^2 - k^2v^2},$$ (9)

where $u$ is the radial and $v$ the transverse component of the velocity, and $k \neq h$. $L$ is a normalizing factor; without cut-off this is

$$L = \frac{2h}{\sqrt{\pi}} \frac{kw^2}{\sqrt{\pi}}.$$ (10)

Now it can easily be shown that if (9) holds at a certain distance $r$ and the cluster is in dynamical equilibrium, the same type of velocity distribution will exist at any other distance from the centre. The parameter $h$ is a constant throughout the cluster, while

$$k^2 = h^2 (1 + \rho^2 r^2),$$

$\rho$ being an arbitrary parameter. In our calculations this parameter has been given by fixing the ratio $k^2/h^2$ for $r = 15$ pc. Writing this ratio as $k^2(15)/h^2$ we have, then,

$$c(r) = \frac{k^2(15)}{h^2} - 1 = \left( \frac{r}{15} \right)^2.$$ (11)

The density distribution is

$$\nu(r) = \frac{k^2(15)}{k^2(15)} e^{-2h^2(15) \Phi(r) - \Phi(15).}$$ (12)

The above generalization of the Maxwellian velocity distribution is not new. It was already used by Eddington (1915) in his pioneer discussion of the dynamics of globular systems, though he did not apply it to actual globular clusters. A somewhat different method to ensure that all “outer” stars move through the nuclear region has recently been used by Woolley and Denise Robertson (1956). They assumed inside a certain radius $R$ a Maxwellian velocity distribution cut-off at the velocity of escape. They further postulated that the region outside the sphere with radius $R$ would be populated solely by stars having orbits penetrating through the inner region.

When using velocity distributions of the type given in formula (9) we shall again have to take account of the cut-off near the velocity of escape. The factor by which the density will be reduced as compared to the case when $V_e = \infty$, is given by

$$c(r) = \frac{2}{\sqrt{\pi}} \left( \int_0^{BV_e} e^{-x^2} dx - \frac{h}{\sqrt{k^2 - h^2}} e^{-k^2V_e^2} \int_0^{V_e} e^{-x^2} dx \right).$$ (13)

As can easily be verified, $c(r)$ becomes 1 for $V_e = \infty$.

5. Effects of stellar encounters

In order to decide as to the most appropriate value for the parameter $k^2(15)/h^2$, we should first have some knowledge about the relaxation times at different distances from the centre.

We begin by considering the simplest case, when $k/h = 1$ throughout the cluster. As an example we take model (c) of Table 9. We define the time of relaxation $t_e$ as the time needed for a transfer of kinetic energy equal to the average kinetic energy of a star. Let us denote by $N_1, N_2, \ldots$ the numbers of stars of masses $m_1, m_2, \ldots$ per cubic parsec, by $E_1, E_2, \ldots$ their kinetic energies, and by $v_1, v_2, \ldots$ their space velocities. Let, further, $v_{ij}$ represent the relative velocity of a star of mass $m_i$ with respect to a star of mass $m_j$. If $N$ is measured in (pc)$^{-3}$, $v$ in km/sec and $m$ in solar masses, we have then, per 10$^9$ years,

$$\frac{\Delta E_i^2}{E_i^2} = 0.221 (v_{11}^2)^2 (\ln Y - 0.6) \sum_i \{N_i m_i^2 (v_{11}^2)^{1/2}\},$$ (14)
the sum to be extended over the various mass groups used (including \( m_1 \)); the bars indicate averages. In this expression \( \Delta E_i \) is the change in kinetic energy caused by the encounters, and

\[
Y = 77 N^{-3/2} m^{-1} v^2.
\]

As \( Y \) is a large number, (14) is relatively insensitive to the exact value of \( Y \), so that a rough estimate suffices. We obtained this by inserting for \( N \) the total density in solar masses per pc\(^2\), for \( m \) one solar mass and for \( v \) the average relative space velocity of stars of one solar mass. The time of relaxation, \( t_r \), will, as usual, be defined as the time in which \( \Delta E_i^2 \) becomes equal to \( E_i^2 \), \( E_i^2 \) being the average square kinetic energy which the stars of this mass would have in the case of a Maxwellian velocity distribution with complete equipartition. In units of 10\(^9\) years we have, therefore,

\[
t_r = E_i^2 / \Delta E_i^2.
\]

The above expressions are practically identical with those given by Chandrasekhar (1942, p. 67), if due account is taken of differences in notation.

We combined the stars into the following groups; the mean masses in the last column are in units of the sun’s mass.

<table>
<thead>
<tr>
<th>( i )</th>
<th>group</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>1.25</td>
</tr>
<tr>
<td>2</td>
<td>B-E</td>
<td>1.00</td>
</tr>
<tr>
<td>3</td>
<td>F-G</td>
<td>0.60</td>
</tr>
<tr>
<td>4</td>
<td>H</td>
<td>0.25</td>
</tr>
<tr>
<td>5</td>
<td>I</td>
<td>0.125</td>
</tr>
</tbody>
</table>

Formula (14) refers to mass group \( m_1 \); in order to apply it to another mass group the index \( i \) should be replaced by the index corresponding to that group. The densities for groups 1 and 2 were taken from Table 5, and are the observed values, except that for group 1 they were doubled in order to take account of the white dwarfs; for groups 3, 4, and 5 they were inferred from solution (c) (Table 9). The relevant values of \( N_i m_i \) are summarized in Table 6.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( N_1 m_1 )</th>
<th>( N_2 m_2 )</th>
<th>( N_3 m_3 )</th>
<th>( N_4 m_4 )</th>
<th>( N_5 m_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>496</td>
<td>199</td>
<td>22.6</td>
<td>2.3</td>
<td>1.0</td>
</tr>
<tr>
<td>4</td>
<td>24.2</td>
<td>15.9</td>
<td>4.84</td>
<td>1.12</td>
<td>0.63</td>
</tr>
<tr>
<td>8</td>
<td>2.48</td>
<td>2.45</td>
<td>1.55</td>
<td>0.64</td>
<td>0.41</td>
</tr>
<tr>
<td>15</td>
<td>0.298</td>
<td>0.481</td>
<td>0.546</td>
<td>0.359</td>
<td>0.271</td>
</tr>
</tbody>
</table>

The root-mean-square space velocity for group 1 was taken 5.99 km/sec, in accordance with solution (c); the mean velocities assumed for the other groups were the equipartition values corresponding to this velocity. Table 7 shows the resulting values of \( \ln Y \) and the relaxation times; the latter are expressed in units of 10\(^9\) years. The group with mass 0.125 \( \odot \) has been omitted as being of little importance for the actual cluster.

The relaxation time for a given distance \( r \) has been computed on the simplified assumption that the star density through which the star moved was always equal to the density at \( r \). In reality the star’s orbit will not be circular. For example, a star which at \( r = 4 \) has a radial velocity equal to the average radial velocity and whose transverse velocity is likewise equal to the average will move in an orbit of which the pericentre lies at 2.2 and the apocentre at 4.7 pc from the centre. A rough computation shows that the average transfer of energy to a star moving in such an orbit does not differ appreciably from that to a star describing a circular orbit at \( r = 4 \). At larger distances from the centre, where, in the model that we shall finally adopt, the orbits are all very much elongated, conditions are different. But in these outer parts, the entire exchange of energy is negligible, so that this complication is irrelevant for our present discussion.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \ln Y )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5.9</td>
<td>0.25G</td>
</tr>
<tr>
<td>4</td>
<td>6.8</td>
<td>1.00G</td>
</tr>
<tr>
<td>8</td>
<td>7.5</td>
<td>0.60G</td>
</tr>
<tr>
<td>15</td>
<td>7.9</td>
<td>0.25G</td>
</tr>
</tbody>
</table>

We see from the table that at the centre of the cluster all relaxation times are less than the age of 6 to 10 \( \times 10^9 \) years that may be attributed to the cluster. Accordingly, complete equipartition of energy should have been established. At \( r = 4 \) pc this conclusion will be valid for all groups except 4 and 5. In the solution (d) groups 4 and 5 contribute less than 10\% to the total mass within \( r = 15 \), so that their exact distribution is rather unimportant for the model of the cluster. To a fair approximation we may therefore assume Maxwellian velocity distributions and equipartition to be established up to at least \( r = 4 \) pc. For group A the relaxation time becomes 7 \( \times 10^8 \) years at \( r = 6.4 \); up to that distance the Maxwellian law should be a good approximation for this group.

There is some reason to believe that stars expel an
important fraction of their mass before becoming white dwarfs. If this happens during the time spent above the main sequence the masses of stars in the horizontal branch might be less than those in the other parts above the main sequence. If the mass were expelled in the bright red portion of the evolutionary track the time elapsed since this expulsion would be of the order of 0.2 to 0.4 × 10⁷ years. It is just possible according to Table 7 that in this time the velocity distribution near the centre of the cluster would have partly adjusted itself to the decreased mass and that the concentration towards the centre would have become less than that for the other bright stars. The effect will be discussed in section 8.

At distances as far out as 15 pc the transfer of energy will evidently have been negligible. On the hypothesis that the stars in these regions have come from the parts where the encounters are effective, the stars observed in the outer regions must have orbits passing through the parts where the relaxation time is shorter than the age of the cluster, say within r = 5 pc. As an example let us consider the stars in an element of volume at r = 15. The first requirement is that the parameter k²(15)/h² be chosen in such a way that the above condition is fulfilled for the majority of the stars in this element of volume.

For the following orbit calculations we use the model corresponding to solution (d) of Table 9. Had we used model (e) the results would have been practically identical. The gravitational potential Φ(r) for solution (d) is given in Table 12 and Figure 6. If Φ(r) is in (km/sec)² and the velocities are in km/sec, we have the following expression for the radial velocity u as a function of r:

\[ u^2(r) = -2\Phi(r) + 2\Phi(15) + u^2(15) + \left(1 - 225/r^2\right)v^2(15). \]  

The peri- and apocentres r_p and r_a are obtained by putting the left-hand member equal to zero. If we now insert \( u(15) = |u| = 3.23 \) km/sec (corresponding to group A in solution (d)), we find that in order to get r_p = 5.0 we must take v(15) = 3.44 km/sec. If we make the average transverse velocity equal to half this amount, and take a velocity distribution of the type given in formula (9), 92% of the stars at r = 15 will have transverse velocities less than 3.44 km/sec and pericentre distances smaller than 5 pc. We consider that in this manner our requirement is sufficiently fulfilled. If \( v(15) = 1.72 \) km/sec one component of the transverse velocity averages 1.22 km/sec. Accordingly,

\[ \frac{k(15)}{1.22} = 2.65, \text{ and } \frac{k^2(15)}{h^2} = 7.0. \]  

We shall provisionally adopt this value for the parameter of the velocity distribution. It ensures that the bulk of the stars at r = 15 move through the equipartition region. We must still investigate whether it ensures the same for stars at still larger distances from the centre. Let us consider a volume at r = 30. From (18) and (11) we find \( k(30)/h = 5.0. \) If we consider again a star with \( u(30) = |u| = 3.23 \) km/sec and \( v(30) = 2v(30) = 1.83 \) km/sec, we find r_p = 4.9 pc. This shows that the great majority of the stars at r = 30 move likewise through the central region.

If we want the less massive stars to have orbits passing through the same nuclear region, their average transverse velocity must be approximately the same as that for stars of group A. In as much as their radial velocities are higher this means that the parameter \( k^2(15)/h^2 \) must be larger in inverse proportion to their mass. In the following we have assumed this to be the case.

It is evident that the above considerations are insufficient to fix the parameter \( k^2(15)/h^2 \) with any precision. It seems unlikely that it could be much smaller than the value proposed, but the actual value may well be larger. However, even a doubling of the parameter would have made little difference for the considerations given in the following, nor would it have greatly affected the representation of the observed density distribution.

We shall now consider an average star at r = 15 pc. For this star we assume \( u(15) = |u| = 3.23 \) km/sec, \( v(15) = v(15) = 1.72 \) km/sec. The pericentre of the orbit is then at \( r_p = 2.04 \) pc, the apocentre at
\( r_a = 18.3 \text{ pc} \). If \( r_1 \) and \( r_2 \) are two distances between \( r_a \) and \( r_p \), the fraction of time which a star passes between these distances is given by:

\[
f(r_1, r_2) = \frac{2}{T} \int_{r_1}^{r_2} \frac{dr}{u(r)}.
\]

(19)

In this expression \( T \) is the “time of revolution”, defined as the time between two successive passages through the pericentre; \( u(r) \) is given by (17). If \( r \) is in pc and \( u \) in km/sec, the corresponding unit of \( T \) is \( 0.978 \times 10^6 \text{ years} \). For the “average” star at 15 pc we find \( T \) to be \( 8.7 \times 10^6 \text{ years} \). The only periods of interest for our present purpose are those which the star has spent in the region where it experienced appreciable exchange of kinetic energy with other stars. These periods may be dissected as indicated in Table 8.

<table>
<thead>
<tr>
<th>( r ) (pc)</th>
<th>( f(r_1, r_2) )</th>
<th>( \bar{r} )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-04 - 3</td>
<td>0.044</td>
<td>2.3</td>
<td>0.7 \times 10^6</td>
</tr>
<tr>
<td>3 - 6</td>
<td>0.083</td>
<td>4.5</td>
<td>2.5 \times 10^6</td>
</tr>
<tr>
<td>6 - 10</td>
<td>0.135</td>
<td>8</td>
<td>14 \times 10^6</td>
</tr>
</tbody>
</table>

The third and fourth columns give the average distance from the centre in the corresponding part of the orbit and the time of relaxation at that distance. The evolution of the elongated orbits might roughly be pictured as follows. If we consider, first, the shell between 2.04 and 3 pc, we see that an approximately Maxwellian distribution must have developed in about \( 0.7 \times 10^9 \text{ years} \). However, there will be deviations from the exponential law at the higher velocities. Velocities higher than the velocity of escape will be virtually absent; in addition, there will be a depletion over a considerable range below \( V_c \). For, on one hand, stars which have attained sufficiently high velocities will move away in orbits where they will spend most of their time outside the central part. This will cause the central region to be depleted of these high velocities. It is true that after another relaxation time they will be replenished, but the newly created high-velocity stars will again move away. This process will continue until the orbits are completely “filled”, i.e., until the number of stars returning per unit of time to the central region will balance that moving out of it. This, in fact, is the steady state corresponding to the velocity law (g). We shall investigate below in how far there is theoretical justification for the supposition that this state has been approached.

It is clear that there can be no strictly steady state of this type. In the first place, stars will be continually escaping from the cluster; however, the rate of escape will be slow and will hardly affect our model. In the second place, there will be some effects of encounters in regions further away from the centre, causing a few stars to remain in these regions. The influence of this will probably be relatively small. At \( r = 15 \), for instance, Table 7 gives a relaxation time of \( 61 \times 10^9 \text{ years} \) for stars of group A. In \( 7 \times 10^9 \text{ years} \) the exchange of energy will therefore be almost negligible. The time quoted refers to the isothermal case. In the model we are considering the mean transverse velocity is lower, so that the time needed for a transfer of energy equal to the average energy is reduced to \( 28 \times 10^9 \text{ years} \). But this is still so long that it seems safe in the first approximation to neglect the effects of encounters at these large distances from the centre.

As indicated in the next section the outer regions have initially probably been populated by another process, which, however, still made the stars move through the central part, and therefore made them enter into an exchange of energy with the nuclear stars. These “outer” stars moving through the nuclear region may originally have been either more abundant or less abundant than what is required by the ellipsoidal velocity distribution. After sufficient time the encounters in the central region would again have set up a semi-equilibrium with the outer stars; this state would correspond approximately to that described by formula (g). The time needed would be roughly the same as that required to attain this state by “evaporation” from the nuclear region.

We shall now estimate how far the semi-equilibrium condition will have been approached for the “average” star at \( r = 15 \) that we have considered above. According to Table 8 the stars in the shell between 2.04 and 3 pc with velocities corresponding to that of the “average” star at \( r = 15 \) will spend only 0.044 of their time in this shell. After one relaxation time the average frequency of these velocities in the shell considered will, therefore, be only 0.044 times that corresponding to a Maxwellian or ellipsoidal law. After a second relaxation time this fraction will have been nearly doubled; after \( 7 \times 10^9 \text{ years} \) the fraction of these long orbits filled through encounters in the shell between 2.04 and 3 pc will be 0.36. The encounters in the shell between 3 and 6 pc will fill another fraction of about 0.24, and those between 6 and 10 pc about 0.07 more. In total, therefore, we may assume that about 65% of the orbits concerned will be “filled”. This means that these orbits must have approached the equilibrium state considered above.

We consider, next, the stars at \( r = 30 \text{ pc} \). We take again an “average star”, with \( u(30) = \bar{u} = 3.23 \text{ km/sec} \) and \( v(30) = \bar{v}(30) = 0.91 \text{ km/sec} \). It has its
pericentre at \( r_a = 1.96 \) pc and its apocentre at \( r_a = 38.5 \) pc; the period of revolution is \( 21 \times 10^6 \) years. In this case we find that the encounters in the inner parts of the cluster could have filled only about 30% of these orbits. We are, therefore, a factor of 3 short of the time required to establish the equilibrium conditions envisaged above. At \( r = 50 \) pc the discrepancy becomes still larger.

The conception that all the “outer” stars have been evaporated from the “nucleus” meets with still another, and more serious difficulty. In solution (d) the total mass outside \( r = 8 \) is about twice that within 8 pc. If all this mass would have come from the nucleus the latter would not only have had to be three times more massive than the present nucleus but it should also have had a larger radius. A very rough estimate of the factor by which the radius would have been larger may be obtained in the following way. Most of the stars evaporating to the outer parts must have had velocities not too far below the velocity of escape. Their energies will not, therefore, have differed much from zero. It follows that the total energy of the reduced nucleus would have remained practically equal to its initial value. If, for our rough estimate, we apply the virial theorem to the remaining nucleus, we see that also the total potential energy remains unaltered. This means, that, as the membership of the nucleus is reduced, it contracts. Let \( \alpha \) be the fraction of the members remaining at a certain time; then, if the radius is supposed to have shrunk to \( x \) times its original value, it can easily be seen that the potential of the cluster has become \( \alpha^2/x \) times its original value. As it must remain unchanged, we see that \( x = \alpha^2 \).

A similar relation has been given by Ivan King (1958). In the case considered above we would have \( \alpha = 1/3 \) and, therefore, \( x = 1/9 \). The radius of the original nucleus would thus have been about 9 times the present radius. As the time of relaxation is proportional to \( n^{1/2} R^{3/2} \) (\( n \) being the number of stars in the nucleus) the relaxation time in the original nucleus would have been larger by a factor of the order of 50. It is clear that in this case there could hardly have been any evaporation.

It might appear from the foregoing that we have to give up the idea that the stars in the regions where encounters have little influence have come from the regions where they are effective, and that we must conclude that the stars now observed in the outer parts have originated there and have had little exchange with stars of the equipartition region. In our opinion this does not yield a satisfactory picture of the cluster. The very smooth density variation from inner to outer parts in particular suggests an evolution that was more coherent for the various parts. This is also suggested by the fact that, as shown in section 7, the observed densities at \( r = 30 \) and 50 are close to the values corresponding to the velocity distribution (g), i.e., to a state in which the long orbits are completely filled.

Upon closer inspection we find that the true ratio between the relaxation times in the past and those computed from the present structure must be rather less than the factor of 50 indicated above. In the first place the hypothesis that all stars outside \( r = 8 \) pc would have come there by “evaporation” from a cluster originally confined to the region inside a radius \( r_c \) corresponding to \( r = 8 \) in the present cluster and having zero density outside \( r_c \), gives undoubtedly an exaggerated picture of the “evaporation” process. If the present cluster was approximately in isothermal equilibrium up to \( r = 8 \), it would have extended considerably beyond this radius without any “evaporation” into strongly elongated orbits. The ratio of the mass escaped from the central region into these long orbits to that originally in the isothermal nucleus is therefore certainly less than the factor 2 estimated in the preceding column. In the second place the assumption that the stars in the exterior regions have zero energy is also certainly too extreme. Their mean negative energy may be of the order of one third of that of the stars in the nucleus. Thirdly, our “evaporated” stars do not disappear permanently from the nuclear region: their returns will contribute to keep the nucleus larger than it would have become in the case of real escapes of stars.

It seems impossible to obtain a reliable estimate of the above effects without making very extensive calculations. As we were not in a position to undertake such computations, we have very tentatively supposed that the actual factor by which the nucleus would be reduced by the expulsion of stars into strongly elongated orbits might be about 4 instead of 9. This factor would apply if there were no other important effects influencing the dynamics of the cluster. There is, however, another effect, which may well have been sufficient to counterbalance the entire shrinking process just discussed, viz. the ejection of gas from stars passing into the white-dwarf stage. This process will be considered in the following section.

6. Effects of mass loss in the transition to the white-dwarf stage

It was estimated on p. 308 that the total number of stars in M3 which have passed into the white-dwarf stage is \( 8000 \). These are the stars whose original absolute magnitudes on the main sequence were brighter than \(+3.52\). Using the data given by Sandage (1957a), Tables 1 and 2, we find that the average original mass of these stars was \( 4.44 \). If, following Greenstein (1958), we assume that their
present mass is 0.56 $\odot$, the mass of gas expelled by the stars during transition into the white-dwarf stage is found to be 313 000 times the mass of the sun. From the model computations given in the next section we estimate that the present mass of the cluster with 80660 white dwarfs of mass 0.56 $\odot$ would be about 210 000 $\odot$. From dynamical considerations as well as from observations it seems quite improbable that there would at the present time be an overabundance of interstellar gas in the cluster. If we suppose that there is not, the gas ejected by the stars that have become white dwarfs must have largely disappeared from the cluster. It may have disappeared in various ways, for instance because the velocities with which it was expelled exceeded the velocity of escape from the cluster, or by collision with interstellar clouds at the times when, about every 100 million years, the cluster moved through the galactic gas layer. In whatever way the gas left the cluster, the effect of this expulsion on the dynamical evolution of the cluster must have been very important. The importance of gas ejection for the evolution of clusters has also been pointed out by Ivan King (private communication to Sidney van den Bergh).

As we shall see below, the effect of gas ejection is opposite to that of the ejection of stars by stellar encounters. The former will increase the radius of the cluster, while the latter will decrease it. We shall first estimate the result of gas ejection for the central part of the cluster, in which the massive stars that lost gas were presumably situated. Let us consider a time interval of the order of a few times the orbital period of an average star, i.e., a time very short compared to the life of the cluster but long enough to establish an approximate dynamical equilibrium. Suppose that in this interval the mass of the cluster is reduced to $\gamma$ times the initial mass. Let $M, R, T$ and $\Omega$ denote mass, radius, kinetic and potential energy of the cluster at the beginning of this interval of time and let the same quantities at the end of the interval be indicated by double primes. We define the radius $R$ by the following expression

$$1/R = (1/r_{II}),$$

where $r_{II}$ is the distance between two stars in the cluster; the average is to be taken over all combinations. In suitable units we have then

$$\Omega = -\frac{GM^2}{R}, \quad \Omega' = -\gamma^2 \frac{GM^2}{R'}.$$  

(20)

According to the virial theorem we have, further,

$$T = -\frac{1}{2} \Omega.$$  

(21)

As a further simplification let us suppose that all the gas is expelled suddenly at the beginning of the interval considered. Denoting the kinetic and potential energy directly after this expulsion by $T'$ and $\Omega'$, we have, evidently,

$$T' = \gamma T, \quad \Omega' = \gamma^2 \Omega.$$  

(22)

As a consequence of this ejection the equilibrium of the cluster will be disturbed. We suppose that during the remainder of the time interval considered, in which it was assumed that there would be no further loss of mass, the equilibrium will be restored, so that

$$T'' = -\frac{1}{2} \Omega''.$$  

(23)

In this interval the total energy of the cluster does not change; hence,

$$T'' + \Omega' = T' + \Omega'.$$  

(24)

If we insert in (24) the expressions (22) for $T'$ and $\Omega'$, and then eliminate $T$ and $T''$ with the aid of formulae (21) and (23), we obtain

$$\frac{1}{2} \Omega' = \frac{1}{2} \Omega (2\gamma^2 - \gamma).$$

Using (20) we then find

$$R''/R = \gamma/(2\gamma - 1).$$  

(25)

If we now imagine the total expulsion of mass to have occurred in $n$ steps, the mass in each step being reduced by a factor $\gamma$ close to unity, the ratio of the final radius, $R_f$, to the initial value will be $(2 - 1/\gamma)^n$. For large values of $n$ this will approach to

$$R_f/R = (1/\gamma)^n = 1/\Gamma,$$  

(26)

if $\Gamma$ is the total reduction in mass. The radius will, therefore, increase in the same proportion as the mass decreases. If, as estimated in the preceding column, the mass of the cluster has decreased to 0.40 times its initial value by the transformation of main-sequence stars to white dwarfs, the radius of the remaining cluster will be 2.50 times the original radius.

The expulsion of gas by the stars thus causes the radius of the whole cluster to increase by approximately the same factor by which the radius of the nucleus is reduced owing to the escape of stars into the non-isothermal outer regions. As a very rough model we may, therefore, picture the nucleus as keeping roughly the same dimensions throughout the evolution of the cluster. The principal obstacle to the theory that the outer regions have mainly been populated from the nuclear part has thereby been removed.

There remains the difficulty that, with the relaxation times given in Table 7, the most strongly elongated orbits could not have been completely filled (cf. p. 312). It may be that the true relaxation times were slightly shorter due to the larger masses of the stars initially populating the cluster. But this effect cannot have been very important, because of the relatively short lives of the massive stars. In fact, most of the loss of mass due to transitions into white dwarfs must have occurred in the early stages of the cluster's evolution. A rough estimate, based on
Salpeter’s initial luminosity distribution, shows that about 80% of the gas should have been expelled in the first tenth of the life of the cluster. In this period nearly 50% of the white dwarfs would have been formed.

Other mechanisms, which we have thus far neglected, may have contributed materially to increase the density in the outer regions. Such mechanisms are the perturbations of the orbits of the stars in the cluster caused by passing complexes of interstellar clouds and, in particular, by the gravitational field of the Galactic System as a whole.

A complete theory of the evolution of a globular cluster should take all these factors into account. In addition, it should deal with the problem of the real escape of stars. Among other things it should consider in how far this will have affected the luminosity curve and the value for the ratio of mass to light. The problem of the filling of the elongated orbits should be studied much more completely than has been done in the preceding section. In particular this should also be investigated for the less massive stars.

The estimates and suggestions given in this and the preceding section should be considered only as a first, and quite uncertain reconnaissance of the subject. They have been indicated mainly to serve as a working hypothesis for the model computations presented in this article. It is quite possible that more adequate computations will show this working hypothesis to be incorrect. A full calculation of the evolution of a cluster under the combined influence of encounters and gas expulsion by evolving stars was, however, far beyond the scope of the present investigation.

7. Comparison of the theoretical models with observations

The data used in computing various models of M3 are summarized in Tables 5, 9 and 10.

Table 9

<table>
<thead>
<tr>
<th>Model</th>
<th>( \Phi(M)_{15} )</th>
<th>proportion of white dw.</th>
<th>mass of wh. dw.</th>
<th>( k^2(15)/h^2 ) for ( A )</th>
<th>( \overrightarrow{v}_A )</th>
<th>( M_r )</th>
<th>( M_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>vRbh</td>
<td>4.8 A</td>
<td>1.25</td>
<td>7</td>
<td>4.93</td>
<td>415 000</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>vRbh</td>
<td>1.0 A</td>
<td>1.25</td>
<td>7</td>
<td>3.57</td>
<td>311 000</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>vRbh</td>
<td>1.0 A</td>
<td>1.25</td>
<td>1</td>
<td>2.76</td>
<td>881 000</td>
<td></td>
</tr>
<tr>
<td>(d)</td>
<td>low</td>
<td>1.0 A</td>
<td>1.25</td>
<td>7</td>
<td>3.23</td>
<td>190 000</td>
<td></td>
</tr>
<tr>
<td>(e)</td>
<td>zero</td>
<td>2.5 A</td>
<td>0.50</td>
<td>7</td>
<td>2.57</td>
<td>120 000</td>
<td></td>
</tr>
</tbody>
</table>

The two extrapolations for \( \Phi(M) \) at \( t = 15 \) have been explained on page 305 and are shown in Figure 5. In the calculations the extrapolated stars were divided into four groups, as indicated in Table 10. This table also gives the mean mass attributed to the stars in each group and the mass densities assumed for \( r = 15 \) (in solar masses per \( pc^2 \)). The proportion of white dwarfs, in Table 9, is given in terms of the total number of stars of group \( A \). In the first four solutions the space distribution of the white dwarfs was assumed to be the same as that of group \( A \). In the fifth solution it was supposed that white dwarfs with masses equal to half the sun's mass had a distribution corresponding to equipartition of energy. The proportion of white dwarfs in (e) refers to the region at \( r = 15 \). It was taken slightly less than that estimated on p. 306, because part of these stars will have escaped from the cluster. The complete omission in solution (e) of all main-sequence stars fainter than \( +6.3 \) absolute magnitude is certainly too extreme. The true model is more likely to be somewhere between (d) and (e).

Table 10

<table>
<thead>
<tr>
<th>Group</th>
<th>( M_r )</th>
<th>Mass ( \Phi_{VRbh} )</th>
<th>Mass dens. at ( r = 15 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>6.4</td>
<td>7.9</td>
<td>.707</td>
</tr>
<tr>
<td>G</td>
<td>7.9-10.5</td>
<td>.50</td>
<td>.320</td>
</tr>
<tr>
<td>H</td>
<td>10.5-13.0</td>
<td>.25</td>
<td>.359</td>
</tr>
<tr>
<td>J</td>
<td>13.0-15</td>
<td>.125</td>
<td>.271</td>
</tr>
</tbody>
</table>

The space distribution of the extrapolated stars was estimated on the basis of the observed distribution for stars of group \( A \). In a state of isothermal distribution the relation between the density distribution of stars of mass \( M_A \) and that of stars of mass \( M_X \) would be given by formula (4). Actually, we have to take account of the differences in cut-off in the two mass groups, as well as of the differences in \( k^2/h^2 \). Though this complicates the computations, it does not introduce an essential difficulty. For stars of mass \( M_X \), the value of \( k^2(15)/h^2 \) was taken \( \frac{M_A}{M_X} \) times the values for group \( A \) given in Table 9 (cf. p. 310). The density for stars of group \( X \) may be computed by means of (27). This relation is obtained by applying formulae (12) and (13) to groups \( A \) and \( X \) respectively.

\[
\log \frac{v_X(r)}{v_X(15)} - \log \frac{c_X(r)}{c_X(15)} - \log \frac{k^2(15)}{k^2(r)} = \frac{M_A}{M_X} \left( \log \frac{v_A(r)}{v_A(15)} - \log \frac{c_A(r)}{c_A(15)} - \log \frac{k^2_A(15)}{k^2_A(r)} \right),
\]

For the solution with \( k^2/h^2 = 1 \) the third term on both sides drops out. For \( r > 30 \) the run of \( \log v \) with \( r \) was assumed to be identical for all masses (cf. p. 304), except in model (c) where it was computed from (27).

In order to compute the cut-off factors \( c_A \) and \( c_X \) from formulae (8) or (13) we must first know \( V_t \). As this can only be found when the complete potential field is known, it was necessary to use successive approximations. As a rule, two approximations sufficed. It may be well to point out that the uncer-
tainty arising from our lack of knowledge about the faint stars in the cluster is not so large as would appear at first sight. This is because Sandage’s counts give us direct observational information on more than half of the mass density within \( r = 20 \) pc. The uncertainty concerning the distribution of the remaining mass is relatively small.

Once the potentials have been found in this manner, we can investigate in how far the assumed velocity distribution and luminosity law fit the actual cluster, by computing the distribution of stars of group \( A \) from this potential and velocity distribution. If the agreement between this computed distribution and the counts is satisfactory, we may conclude that we have obtained a satisfactory dynamical model of the cluster.

The comparisons are shown in Table 11. In each case the average radial velocity for group \( A \), \( |u_A| \), was determined in such a way that it roughly reproduced the ratio of the observed density at \( r = 1 \) to that at \( r = 8 \) pc. The values of \( |u_A| \) found are given in Table 9.

It will be seen that all solutions, except that with \( k/b = 1 \), give a rather satisfactory representation of the observed counts. This in itself is quite remarkable, in view of the million-fold range in density involved.

---

**Table 11**

Comparison between observed and computed densities for group \( A \)

<table>
<thead>
<tr>
<th>( r ) (pc)</th>
<th>observed</th>
<th>( (a) )</th>
<th>( (b) )</th>
<th>( (c) )</th>
<th>( (d) )</th>
<th>( (e) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10 900</td>
<td>12 200</td>
<td>12 300</td>
<td>11 700</td>
<td>12 600</td>
<td>12 800</td>
</tr>
<tr>
<td>1</td>
<td>7 820</td>
<td>7 820</td>
<td>7 820</td>
<td>7 820</td>
<td>7 820</td>
<td>7 820</td>
</tr>
<tr>
<td>2</td>
<td>3 060</td>
<td>3 110</td>
<td>3 300</td>
<td>2 610</td>
<td>3 150</td>
<td>3 210</td>
</tr>
<tr>
<td>4</td>
<td>534</td>
<td>551</td>
<td>589</td>
<td>427</td>
<td>563</td>
<td>567</td>
</tr>
<tr>
<td>8</td>
<td>54.6</td>
<td>55.3</td>
<td>52.6</td>
<td>54.7</td>
<td>54.7</td>
<td>53.4</td>
</tr>
<tr>
<td>15</td>
<td>6.56</td>
<td>5.76</td>
<td>4.73</td>
<td>12.6</td>
<td>5.59</td>
<td>5.20</td>
</tr>
<tr>
<td>30</td>
<td>0.550</td>
<td>0.512</td>
<td>0.347</td>
<td>1.92</td>
<td>0.459</td>
<td>0.434</td>
</tr>
<tr>
<td>50</td>
<td>0.081</td>
<td>0.075</td>
<td>0.063</td>
<td>0.352</td>
<td>0.090</td>
<td>0.084</td>
</tr>
<tr>
<td>100</td>
<td>0.006</td>
<td>0.007</td>
<td>0.012</td>
<td>0.011</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Model \( (d) \), with \( \Phi(M) \) lower than Van Rhijn’s, gives a slightly better fit between 10 and 40 pc than model \( (b) \) with \( \Phi_{VR} \). A supplementary solution, identical with \( (b) \) except that the white dwarfs were given the more probable mass and proportion assumed in \( (e) \), gave densities between \( r = 15 \) and 100 which were as much as two times too low (this solution is not shown in the table). If we adhere to the elongated velocity distribution this discrepancy can only be eliminated by lowering \( \Phi(M) \) for the faint stars. Solution \( (e) \) shows the extreme case, with \( \Phi(M) = 0 \) for \( M > + 6.3 \); here the discrepancy has greatly diminished, though it has not entirely disappeared.

That the proportion of stars of smaller mass would be less than in the surroundings of the sun might possibly have been caused by the escapes of such stars. The white dwarfs may have been less affected by such escapes than main-sequence stars of comparable mass, because they started with large masses and presumably low velocities.

The model \( (e) \), with spherical velocity distribution, may be seen to give much too high densities in the outer regions.

It would seem from theoretical considerations as well as from the comparisons shown in Table 11 that a solution somewhere between \( (d) \) and \( (e) \) would be the most plausible.

It appears improbable from the computations discussed above that an important fraction of the cluster’s mass could consist of intrinsically faint stars. The estimates given below for the ratio of mass to luminosity are thus not likely to be seriously in error because of the presence of unknown stars of small mass.

---

**Table 12**

Data for model \( (d) \) \( |u_A| = 3.23 \) km/sec

<table>
<thead>
<tr>
<th>( r ) (pc)</th>
<th>( A/k )</th>
<th>( \Phi/\Phi_g )</th>
<th>mass within ( r ) pc</th>
<th>( -\Phi(r) ) (km/sec)(^2)</th>
<th>( V ) (km/sec)</th>
<th>density ( (\odot/pc^3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>296</td>
<td>676</td>
<td>69</td>
<td>45</td>
</tr>
<tr>
<td>1</td>
<td>0.951</td>
<td>0.797</td>
<td>386</td>
<td>3370</td>
<td>640</td>
<td>390</td>
</tr>
<tr>
<td>2</td>
<td>0.877</td>
<td>0.897</td>
<td>10100</td>
<td>9800</td>
<td>2200</td>
<td>1500</td>
</tr>
<tr>
<td>4</td>
<td>0.608</td>
<td>0.818</td>
<td>1600</td>
<td>19500</td>
<td>4900</td>
<td>3500</td>
</tr>
<tr>
<td>8</td>
<td>0.377</td>
<td>0.899</td>
<td>2000</td>
<td>30100</td>
<td>7700</td>
<td>6100</td>
</tr>
<tr>
<td>15</td>
<td>1.000</td>
<td>1.000</td>
<td>24700</td>
<td>44900</td>
<td>11000</td>
<td>9500</td>
</tr>
<tr>
<td>30</td>
<td>0.961</td>
<td>0.937</td>
<td>28500</td>
<td>64100</td>
<td>16400</td>
<td>15800</td>
</tr>
<tr>
<td>100</td>
<td>1.000</td>
<td>1.000</td>
<td>19200</td>
<td>44900</td>
<td>11000</td>
<td>9500</td>
</tr>
</tbody>
</table>

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Table 12 shows the detailed distributions for one model, for which we have selected (d). In the part of Table 12 giving the mass of the various groups up to different values of \(r\), the column "Total" contains the sum of groups \(A\) to \(I\) plus the white dwarfs. The total mass of the latter is the same as for group \(A\). For model (e) the potentials are quite similar, except that they are lower by a small factor.

The approximate cut-off factors for model (d) are given in Table 13. Except for group \(I\) they are moderately close to one, so that the cut-off does not produce too improbable velocity distributions. It is also satisfactory to note that the cut-off remains moderate even at large distances from the centre. This is entirely due to the elongation of the velocity distributions. With a spherical velocity distribution, as in (e), the factors become prohibitive beyond about \(r = 20\) pc.

\[\begin{array}{|c|cccc|}
\hline
r \text{ (pc)} & A & F & G & H & I \\
\hline
1 & .88 & .86 & .82 & .60 & .36 \\
8 & .87 & .85 & .82 & .70 & .55 \\
15 & .86 & .80 & .76 & .60 & .46 \\
50 & .74 & .62 & .55 & .41 & .30 \\
\hline
\end{array}\]

Table 13 indicates how far the total numbers of stars of various absolute-magnitude groups in the cluster conform to the distribution of absolute magnitudes at \(r = 15\) which was used as basis for the model.

Table 14 indicates how far the total numbers of stars of various absolute-magnitude groups in the cluster conform to the distribution of absolute magnitudes at \(r = 15\) which was used as basis for the model.

\[\begin{array}{|c|cc|}
\hline
\text{Group} & \text{rel. mass dens.} \text{ at } r = 15 & \text{rel. total mass} \\
\hline
A & -.310 & .445 \\
B-E & 1 & 1 \\
F & -.235 & -.256 \\
G & .222 & .246 \\
H & -.249 & .314 \\
I & .187 & .254 \\
\hline
\end{array}\]

It will be seen that, relative to the groups \(B-E\), the total numbers of stars of group \(F\) and \(G\) in the entire cluster are practically the same as at \(r = 15\), while the numbers in groups \(H\) and \(I\) are about 25 and 30% higher.

The most important test of the model of \(M_3\) would be a comparison with measured radial velocities. The only measures of internal velocities in a globular cluster made so far are those published by O. C. Wilson and Mary F. Coffeen (1954). They measured 15 red stars in \(M_92\), from which they derived a true average random velocity of 3.5 km/sec. However, this value is extremely uncertain, because the observational errors are of the same order as the velocities. If we try to take account of the slight difference in absolute magnitude and size between \(M_92\) and \(M_3\), we find 2.5 km/sec for the average random velocity in \(M_3\). This agrees within its uncertainty with all the values in Table 9, except perhaps that for model (a).

An accurate determination of internal radial velocities in a globular cluster would evidently be of great importance. In particular, measurements of velocities of stars at various distances from the centre would make it possible to determine in a direct manner the ratio between the average transverse and radial motions in the cluster. But the smallness of the transverse motions at distances of more than a few minutes from the centre of \(M_3\) makes this an extremely difficult task.

It is of interest to consider the ratios of total mass to total luminosity for some of the models. Expressing these in the mass and visual luminosity of the sun as units, we find:

![math equations]

The visual absolute magnitude of the sun was assumed to be +4.84 (Steebplns and Kron 1957). The \(\mathcal{M}/L\) ratio for model (a) has not been computed. A rough estimate makes it 1.3 times higher than for model (b), or 0.62. At the centre of the cluster the \(\mathcal{M}/L\) ratio is considerably lower. In model (b) it is 0.17, in model (d) 0.15.

8. The distribution of RR Lyrae variables

Several authors (W. Chr. Martin 1937; Oosterhoff 1941) have remarked that the RR Lyrae variables in globular clusters are less concentrated towards the centre than the stars in general. Martin made strip counts on plates of \(\omega\) Centauri taken with the Yale telescope in Johannesburg (focal length 10.93 metres). On the short exposures the cluster can be resolved even in the centre (cf. W. Chr. Martin 1938, Plate II). A comparison between his general star counts and the counts of variables (which were investigated on plates taken with the same telescope) is shown in Figure 7. In this graph \(F(r)\) is the number of stars in a strip the centre of which lies at a distance \(r\) from the centre of the cluster. All strips were reduced to the same surface area. The counts are given to two magnitude limits (15.0 and 16.2 photographic). The two sets of counts may be seen to agree quite
well. It is evident that the variable stars are relatively less frequent in the central part. As the blink results indicated that only few variables with amplitudes over $0.4$ had remained undiscovered, Martin concluded that the difference could not be due to incompleteness. He also pointed out that the $c$ variables are less concentrated to the centre than the $a$, $b$ and $d$ Cep variables, but this result cannot yet be considered as definitely established.

Oosterhoff found a similar phenomenon in $M_3$, comparing the RR Lyrae variables with star counts by E. Strömgren and Drachmann. He mentions also $\omega$ Centauri, for which he gives a comparison of the distribution of the short-period variables with star counts by Bailey, and shows that the variables have a much wider distribution. For both clusters the counts were made in rings.

There are two other clusters, $M_3$ and $M_{15}$, in which more than 80 RR Lyrae variables are known, and for which data on star counts, or surface brightness, are available. Both show the same phenomenon.

The distribution of variable stars has been the subject of extensive studies by Kholofov (1953 and 1957). In the first article he compared their distribution in $M_3$ with that of four groups of stars in different parts of the colour-magnitude diagram. He used colours and magnitudes for 848 stars from the 1920 article by Shapley and Miss Davis. The author considered that there is evidence that the various groups have different concentrations towards the centre (cf. Table 6 of the 1953 article). However, the numbers of stars are rather small and the results are based on a range in $r$ of only $1.6 : 1$. In the second article Kholofov extends the discussion of RR Lyrae variables to a larger part of the cluster.

In Table 15 we have collected the data on RR Lyrae variables for the four clusters mentioned above. Under $n_{var}$ the total number of RR Lyrae variables is given for rings, the inner and outer radii of which are given in the first column. The outer limit of the outermost ring differs from cluster to cluster; it is 40 pc for $\omega$ Cen and $M_{15}$, 45 pc for $M_3$, and 50 pc for $M_{15}$. The outer radii of the rings in minutes of arc are shown under $r'$. The distances assumed were: $M_3$ 12.1 kpc, $\omega$ Cen 4.6 kpc, $M_{15}$ 12.0 kpc, and $M_5$ 9.1 kpc. Except for $M_3$ these were taken from Lohmann (1952). The RR Lyrae variables were counted from the catalogue compiled by Helen B. Sawyer (1955). The columns marked $n_*$ show the total numbers of stars in the same rings, multiplied for each cluster by a constant factor chosen such as to equalize the total numbers of stars and variables in those parts where variables could be discovered with some completeness. These totals are given at the bottom; in $M_3$, $M_{15}$ and $M_5$ the parts within 2 pc from the centre were omitted in forming the totals.

For $M_3$ the distribution of stars was taken from Figure 1, the quantity $r\sigma(r)$ being integrated over the interval in $r$ covered by each ring. The same procedure was used in the other clusters. For $\omega$ Cen $\sigma(r)$ was taken from the photo-electric measures of surface brightness made by Gascoigne and Burr (1956). In the case of $M_{15}$ photographic surface brightness measures by Lohmann (1936) were combined with star counts by von Zeipel (1913); the counts and the surface brightness observations agree satisfactorily. For $M_5$ we have again used surface-brightness results by Lohmann (1936), which extend to $r = 3' 7$. Up to this distance the run of $\log \sigma$ with $r$ is exactly the same as that in $M_{15}$ if the distances in minutes for $M_{15}$ are multiplied by 1.68. In order to extend the curve for $M_5$ beyond $r = 3' 7$, we have assumed that the same relation between $M_5$ and $M_{15}$ would hold for larger distances.

Before comparing the numbers in the table we must consider the incompleteness of the variable stars. The case of $\omega$ Centauri has been discussed above; it is unlikely that there is serious incompleteness in this cluster. In Messier 3 the nucleus is too dense to identify individual stars of $15'' 7$, which is the median magnitude of the RR Lyrae variables. But it seems likely that there is no serious incompleteness beyond

<table>
<thead>
<tr>
<th>$r$ (pc)</th>
<th>Messier 3</th>
<th></th>
<th></th>
<th>Messier 15</th>
<th></th>
<th></th>
<th>Messier 5</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>$r'$</td>
<td>$n_{var}$</td>
<td>$n_*$</td>
<td>$r'$</td>
<td>$n_{var}$</td>
<td>$n_*$</td>
<td>$r'$</td>
</tr>
<tr>
<td>0–1</td>
<td>0.28</td>
<td>(2) 25</td>
<td>0.75 2 4</td>
<td>0.29</td>
<td>(1) 9</td>
<td>0.38</td>
<td>(2) 9</td>
</tr>
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<td>1–2</td>
<td>0.57</td>
<td>(13) 37</td>
<td>1.50 5 11</td>
<td>0.57</td>
<td>(15) 16</td>
<td>0.76</td>
<td>(4) 13</td>
</tr>
<tr>
<td>2–4</td>
<td>1.13</td>
<td>26 48</td>
<td>3.0 20 26</td>
<td>1.14</td>
<td>19 23</td>
<td>1.51</td>
<td>20 25</td>
</tr>
<tr>
<td>4–8</td>
<td>2.27</td>
<td>40 40</td>
<td>6.0 35 40</td>
<td>2.29</td>
<td>22 24</td>
<td>3.02</td>
<td>30 27</td>
</tr>
<tr>
<td>8–15</td>
<td>4.25</td>
<td>40 31</td>
<td>11.2 42 31</td>
<td>4.29</td>
<td>21 15</td>
<td>5.66</td>
<td>24 20</td>
</tr>
<tr>
<td>15–30</td>
<td>8.59</td>
<td>40 23</td>
<td>22.4 32 19</td>
<td>8.58</td>
<td>9 8</td>
<td>11.3</td>
<td>7 12</td>
</tr>
<tr>
<td>30–</td>
<td>14.2</td>
<td>8 12</td>
<td>30.0 1 5</td>
<td>11.44</td>
<td>1 1</td>
<td>17.0</td>
<td>6 3</td>
</tr>
<tr>
<td>Total</td>
<td>154</td>
<td>154</td>
<td>137 136</td>
<td>72 71</td>
<td>87 87</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
o'6 from the centre. The cluster has been searched very intensively for variable stars. The most important additions to the search made by Bailey in 1913 were those by Müller in 1933, who found 42 new variables, and by Greenstein (1935), who added another 16. But for 3 variables the latter 16 are all within o'6 of the centre. It would seem that the variables outside this distance had been almost completely discovered by the earlier investigators. Greenstein, however, seems to assume that the apparent shortage of variables within 2' from the centre is due to incompleteness. For Messier 15 Rosino (1950) states that on his plates (taken in the Cassegrain focus of the 82-inch McDonald telescope) stars can be observed without serious blending down to 30" from the centre. In the ring between o'57 and 1'14 the variables will probably be complete. As regards Messier 5, this appears to be well resolved beyond o'6 from the centre (cf. Oosterhoff 1941); it seems reasonable to assume that the variable stars are fairly complete for the rings outside r = o'76.

In Table 15 the numbers referring to rings where the variable stars are expected to be seriously incomplete are in parentheses. Leaving these out of consideration we see that in M3 and ω Cen there remains a difference between the distribution of the variables and the stars in general, in the sense that the variable stars have a greater spread. An exception must be made for the region beyond 30 pc, where the variable stars have about the same relative frequency as the non-variable stars. In M15 and M5 the difference between variables and non-variables is quite uncertain.

A difference in the observed direction could exist if the average mass of the variable stars is smaller than that of the bulk of the stars situated above the main sequence. In the foregoing discussions we have assumed that the masses of all stars above the main sequence were equal. It is conceivable, however, that during the evolution in the upper part of the Hertzsprung-Russell diagram the stars lose mass, so that they arrive on the horizontal branch with less mass than they started with. If the mass loss occurs in the brighter part of their evolution, above $M_r = +1$ for instance, the great majority of the stars would still have their initial mass of 1.25 $σ$, while only the relatively small fraction brighter than this limit would have smaller masses. Evidently, a difference between the distribution of the variables and the other stars can develop only if there has been sufficient time after the loss of mass occurred, to re-establish approximate equipartition of energy. Estimates of times involved in the various parts of the evolutionary path have been given by Sandage (1957b). If we suppose that the loss of mass takes place in the brightest part of the giant branch, his data would indicate that the RR Lyrae stars may have lived about $0.3 \times 10^9$ years with the decreased mass. Sandage's evolution times are based on an age of $5.1 \times 10^9$ years for the cluster. The actual age may be closer to $9 \times 10^9$ years, in which case the low-mass life of the RR Lyrae stars might have lasted $0.5 \times 10^9$ years. It may be that a mass loss takes place already in the vertical branch of the HR diagram, so that the low-mass life of the variables might be increased still further.

The times which the RR Lyrae stars would need for the relatively minor re-adjustment of their kinetic energy would be sensibly less than the full relaxation times. Assuming that they would be 2 or 3 times shorter, we see from Table 7 that in the case of Messier 3 the re-adjustment may have been effectuated up to 3 or 4 pc from the centre. M15 and M5 would be similar. In ω Centauri the relaxation time in the centre is estimated to be about 5 times that in the centre of M3, while at $r = 3$ pc it will be about the same as in M3 (cf. section 9).

We may conclude that it is not entirely impossible that the observed differences, if they are real, would have been caused by mass loss and subsequent redistribution.

The re-adjustment of the velocities in the cores of the clusters would necessarily have caused an increase in the relative numbers of variables in the shells somewhat farther from the centre. However, it is evident that the time available was insufficient to fill more than a small fraction of the elongated orbits discussed in section 5. The rings beyond $r = 15$ pc can hardly have been affected by the re-distribution.

It is of some interest to estimate whether such differences as are indicated by Table 15, would correspond to reasonable values of the mass ratio of RR Lyrae variables to the other stars. For a given total number of variables in a cluster, the density in the inner parts will decrease as a consequence of loss of mass, while in the outer part it will increase. There must be a "half-way" point where the density remains constant, even after complete re-adjustment of the velocity distribution would have occurred. This half-way point is a suitable point of reference. We have assumed that it lies near $r = 15$ pc. After reducing the numbers in Table 15 to surface densities, $σ$, we have now computed the following ratio

$$\frac{\log σ(r) - \log σ(15)}{\log σ(r) - \log σ(15)} \quad \text{for} \quad r = 8 - 15 \text{ and } 15 - 30 \text{ pc},$$

in which $\log σ(15)$ was obtained by taking a straight average of the values of $\log σ$ in the rings $8 - 15$ and $15 - 30$ pc, respectively. The values found for the above ratio are shown in Table 16, the numbers of variables in the corresponding rings being added in parentheses.
Table 16
Values of the ratio \( r \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( M_3 )</th>
<th>( \omega \text{ Cen} )</th>
<th>( M_{15} )</th>
<th>( M_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–1</td>
<td></td>
<td>.72( 2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1–2</td>
<td></td>
<td>.69( 5)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2–4</td>
<td>.74(26)</td>
<td>.80(20)</td>
<td>.71( 9)</td>
<td>1.00(20)</td>
</tr>
<tr>
<td>4–8</td>
<td>.82(40)</td>
<td>.78(35)</td>
<td>.86(20)</td>
<td>1.12(30)</td>
</tr>
</tbody>
</table>

For a range of \( r \) where the relation between \( \log \sigma \) and \( r \) is approximately linear, the ratio \( (28) \) should, in the case of equipartition, be equal to the mass ratio of the two types of stars (cf. page 307). Although the condition of linearity is certainly not fulfilled over the entire range from 0 to 15 pc, we have made our calculation as though it were fulfilled; the error involved is likely to be small. Similarly, an error in the assumed distance of the half-way point is probably unimportant. Assuming that equipartition has approximately been re-established for the variables up to \( r = 4 \) pc, we derived the average mass ratio from the upper three lines of Table 16. Using the numbers of variable stars as weights, we obtained 0.81, with an estimated uncertainty of \( \pm 0.06 \). This corresponds to a mass of 1.016. For the variables, a value which does not conflict with other data on RR Lyrae stars. The numbers in the last line of the table were not included in this average, because there could not have been time to re-establish equipartition in the shell between 4 and 8 pc.

Figure 7 shows a plot of the strip counts made by W. Chr. Martin in \( \omega \) Centauri. The variables have been indicated by crosses, while dots and small circles give general star counts down to \( m_\text{ap} = 15.0 \) and 16.2, respectively. The ordinates have been shifted so as to make the three distributions co-incide approximately for radii larger than 7, or 9 pc. A comparison of the slopes for general stars and variables in the part within 6 pc gives a mass ratio of 0.68. This value is certainly too small, because the equipartition will not extend to 6 pc and the density at 6 pc will therefore be too high. A rough estimate shows, however, that the result from Martin's curves agrees within its uncertainty with that derived from Table 16 for the same range of \( r \).

The above results should be considered with very much reserve. A somewhat contradictory feature, for instance, is the large number of variables which Rosino has discovered in the 1–2 pc ring in \( M_{15} \), a ring which has not been used in the computations because the variables were supposed to be incomplete (cf. Table 15).

We wish to stress that the entire phenomenon of the differences in distribution between variables and general stars rests on a very meagre and uncertain observational basis. It is only because of its evident importance for the theory of stellar evolution that we have discussed it at some length.

The phenomenon appears sufficiently promising to warrant a new search for differences in distribution between stars on the horizontal branch of the Hertzsprung-Russell diagram, those on the vertical branch and those in the region of bright yellow giants. The effects to be expected are small. An attempt to find such differences has been made by Kholopov. However, more numerous photometric data of high systematic accuracy would be required to obtain significant results. The best chance for success may be expected for clusters of low concentration and small linear radius.

9. Structural differences between clusters

For the following, cursory discussion of differences to be expected between clusters we shall assume that they all have the same composition and that rotation is negligible. We also neglect a possible variation with distance from the centre of the cut-off factors in the inner parts.

Under these conditions the parts where encounters have been sufficiently effective to bring about an approximately Maxwellian velocity distribution must be similar in all clusters. Except for the effect of the cut-off factors they will all have the structure of an isothermal sphere of gas. Because different clusters will have started with different masses and different radii, there will be differences in the absolute values of the density and in scale. We shall use the space density at the centre, \( \nu(0) \), and the radius, \( r_{0.1} \), at
which the density in the Maxwellian core has dropped to a tenth of the central density, to specify the two parameters on which our simplified picture the structure of a cluster must depend. If the densities are expressed in the central density, and distances from the centre in \( r_{c1} \) as unit, the distributions in the Maxwellian cores should become identical for all clusters. For the entire clusters, however, structural differences will still occur, as a consequence of the fact that the Maxwellian cores will extend over different fractions of the clusters. In a large and dense cluster the relaxation times will be longer than at "corresponding" radii in a small and poor cluster (corresponding radii being defined as radii expressed in \( r_{c1} \) as unit). In the small and poor cluster the Maxwellian distribution will be realized over a larger part of the cluster.

From formulae (5) and (6) we see that, for corresponding radii, \( \Phi(r) - \Phi(0) \) is proportional to \( p(0)r_{c1}^2 \), or also to \( v(0)r_{c1}^2 \), if the mass functions are supposed to be identical in the various clusters. It follows from (7) that the mean of the squares of the velocities is likewise proportional to \( v(0)^2 r_{c1}^2 \). According to formulae (14) and (16) the time of relaxation is approximately proportional to the three-halves power of the mean square velocity and inversely proportional to \( v(0) \). Inside the Maxwellian core we get, therefore, the following relation for the time of relaxation at corresponding radii

\[
 t_c \propto v(0)^{1/2} r_{c1}^3. \tag{29}
\]

Outside the isothermal, or Maxwellian, core the density will be reduced compared to that which would occur in the case of a Maxwellian velocity distribution. Let us compare two clusters (a) and (b), the first being a rich and large cluster, the second poor and small. In (a) the Maxwellian core will occupy a smaller fraction of the cluster than in (b). If we now adjust the scales and densities so as to make the density distributions identical for those parts in which both clusters have a Maxwellian velocity distribution, the density outside this part will fall off more steeply for (a) than for (b), because in the latter the velocity distribution remains Maxwellian, while in (a) it becomes of the elongated type discussed in section 4. Compared to (a), (b) will have more extended outer parts. It will make the impression of being more concentrated towards the centre than (a). Had we compared two clusters containing the same total number of stars but having different radii, we would have found that the smaller cluster has not only a higher central density, but that also the Maxwellian velocity distribution extends over a larger fraction of the cluster. The core will therefore appear more compressed in comparison with the outer part. It will presumably be classified in a lower "concentration class" in Shapley's terminology. The difference between a cluster where the Maxwellian distribution extends to the outer part and one in which the velocity distribution becomes strongly elongated outside 10 pc may be illustrated by Table 11. In model (c) the distribution is Maxwellian throughout the cluster, while in (\( b \)), which in all other respects is identical with (c), the velocity distribution is of the elongated type used in this article. The difference beyond \( r = 10 \) pc is considerable.

A general comparison of clusters on the basis of these considerations cannot be given because of lack of data. We shall only consider a few examples. We first compare M3 and \( \omega \) Cen. Using for M3 the distribution of surface density found in the present paper, for \( \omega \) Cen that derived by Gascoigne and Burr (1956), and assuming the same distances as in section 8, we find that \( r_{c1} \) for \( \omega \) Cen is 2.1 times that for M3. For M3 we have \( r_{c1} = 4.0 \) pc, so that for \( \omega \) Cen \( r_{c1} \) would be 8.4 pc. In order to obtain a rough idea of the ratio of \( v(0) \), we assume that the same scale difference would hold for the whole clusters. According to Lohmann (1952) the integrated absolute magnitude of \( \omega \) Cen is 1.07 brighter than that of M3. With the above assumption this would lead to a central density in \( \omega \) Cen that is 0.29 times that in M3. According to (29) the relaxation times at corresponding radii would then be 5 times as long as in M3. From Table 7 we find that in M3 relaxation must be practically complete up to \( r = 8 \) pc. Applying the above factors we find that in \( \omega \) Centauri it should be complete up to about 10 pc, corresponding to 4.5 pc in M3. Reducing the \( r \) values in \( \omega \) Cen to the corresponding values in M3, we should expect the two distributions to agree up to about 4.5 pc, while for larger radii the density in \( \omega \) Centauri should fall off more rapidly. In a qualitative way this agrees with observation. Up to about 6.5 pc in M3 the reduced surface-brightness distributions correspond accurately; at larger distances the densities in \( \omega \) Cen become appreciably lower than the corresponding densities in M3.

Gascoigne and Burr made a comparison between \( \omega \) Centauri and 47 Tucanae. Assuming distances of 4.6 and 5.2 pc, respectively, (from Lohmann 1952), we find from their data that \( r_{c1} \) for \( \omega \) Centauri is 4.2 times that for 47 Tucanae. With the aid of the above comparison between \( \omega \) Centauri and M3 this gives \( r_{c1} = 2.0 \) pc for 47 Tuc. This cluster is therefore built on a scale that is about half that of M3. The integrated brightness of 47 Tuc is about the same as that of M3, or 1/3rd that of \( \omega \) Cen. From these data we find that the relaxation times in 47 Tuc are 15 times shorter than at corresponding radii in \( \omega \) Centauri and 2.8 times shorter than in M3. Accordingly, the difference between the density distribution in 47 Tuc and that in \( \omega \) Cen should be still
more pronounced than that between M₃ and ω Cen. Indeed, Gascoigne and Burr find a very striking difference between the two clusters, in the expected direction. We note, further, that Shapley assigns 47 Tuc to concentration class III, ω Cen to class VIII; low numerals indicate high concentration. M₃ is in between, with concentration class VI.

We wish to express our gratitude to Dr. Sandage for giving us the detailed data on his star counts which provided the observational basis for the above article. We are indebted to Professor Spitzer for some constructive comments and to Professor Oosterhoff for a valuable discussion on the distribution of the variable stars.

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