COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

Dynamics of the galactic system in the vicinity of the sun, by J. H. Oort.

Summary.
Various consequences are discussed of the theory of a rotating stellar system, the stars in our neighbourhood being considered as part of just an arbitrary section from the great system. In broad lines the discussion leads to the same conclusions as were previously reached by Lindblad, viz. that the star streaming as well as the systematic motions of the stars of high velocity can be considered as steady phenomena; both are to be expected a priori in such a system.

The mathematical treatment is different from Lindblad's, mainly because it takes account of the observed law of distribution of the peculiar velocities. One of the consequences of this is a different conception of the factors determining the systematic motions of the stars of high velocity (section 8). Thirteen years ago almost the same mathematical problem has been extensively treated by Eddington and Jeans.

The stars are all considered to form part of one rotating system, but with varying peculiar velocities. Different types of stars are treated separately and are sometimes artificially split up into a number of groups with ellipsoidal distributions, but only to make calculations easier. It is not thought that they should be considered as really separate systems.

The view is put forward that the observed absolute limit of the high velocity stars (at about 63 km/sec) may be identical with the difference between the velocity of escape and the circular motion in the galactic system; this is illustrated in figure 2.

The average direction to the centre as derived from the high velocities is toward 325° galactic longitude (section 2), comparing very well with the longitude 324° derived from rotational effects in radial velocities and with the almost identical position determined from the distribution of globular clusters over the sky.

The general equations to be fulfilled in a dynamically steady state are deduced in sections 5 and 6. The ellipsoidal character of the motions parallel to the galactic plane is shown to be directly linked with the differential rotation effects. In this respect a difficulty is presented by the B type stars, the radial velocities of which do not show the ellipsoidal character required. Other difficulties are formed by the inconsistency of the results for the vertex of stream motion and the considerable deviations of this from the presumed direction to the centre. But the most serious difficulty is that, except in systems of a very special form, the average velocity perpendicular to the galactic plane should be equal to the average velocity in the direction of the vertices and not to that in the direction of the smaller axis of the ellipse, as it is approximately observed to be. However, it is possible that the actual conditions approach closely to such a special form, so that the motions perpendicular to the galactic plane might be considered as independent of the motions parallel to that plane.

The observed proportion between the two galactic axes of Schwarzschild's velocity ellipsoid is somewhat smaller than that computed by the present theory, indicating that the absolute value of the constant $B$ which was derived from proper motions, but which remained uncertain, should be decreased. A change as indicated would increase the distance to the centre as computed from the rotation effects $A$ and $B$ in B. A. N. No 132 to a value much closer to Shapley's estimate of the distance of the centre of the cluster system.

In tables 2 and 3 of section 8 the principal data bearing on the systematic motion of objects of high velocity have been collected. In the same tables the density gradients have been computed which would be required to explain this systematic motion as a steady phenomenon.

1. In a number of papers by Lindblad *) and by the author **) it has been shown that the more striking peculiarities in the motions of the stars can be explained by the simple hypothesis that the galactic system is in rapid rotation around an axis perpendicular to the plane of the Milky Way and situated


**) B. A. N. Nos 120, 132 and 133, 1927.
at considerable distance from the sun in the direction of the centre of the system of globular clusters.

Of the various objections against the tentative model of the galactic system considered the most serious one is perhaps that a very excentric position of the sun, as implied by the theory, is contradictory to observations of the density distribution of the stars. It certainly would be so if we assumed that the light of all stars reaches us without obstruction. We know, however, that even in our immediate neighbourhood considerable obstruction does take place in extended regions covered by dark nebulae. The above difficulty may possibly be only apparent and caused by the observed density distribution in the direction of the galaxy being as much determined by the distribution of dark matter as by that of the stars themselves. Some evidence on this point may be found in the peculiar avoidance of galactic regions by globular clusters, a phenomenon for which the hypothesis of light obstruction seems to give the most plausible explanation.

It is my intention to reconsider in the present article some of the consequences of the rotation hypothesis, in particular those with respect to the distribution of the peculiar velocities of the stars in general and to the systematic motions of the high velocity stars.

The main conclusions, namely that the ellipsoidal distribution of velocities and the general systematic trend of the higher velocities follow at least qualitatively, without special assumptions, from the rotation of the galaxy are originally due to LINDBLAD. However the method used to arrive at these conclusions is rather different, and so is to some extent their interpretation.

2. It is very unlikely that the galactic system as a whole should contain a considerable number of stars whose velocities exceed the velocity of escape. Let us call the velocity of escape from a point near the sun $V_e$ and let $\Theta_o$ represent the linear velocity of the rotation, say for the apparently bright stars. Let, further, the velocities of the stars be counted from this rotational velocity as origin (the origin thus co-incides with the centre of gravity of the bright stars around the sun). It is then clear that, if there are any velocities larger than $V_e - \Theta_o$, these velocities must not be directed to a region surrounding the direction of the rotational velocity: they should avoid the area around $54^o$ galactic longitude.)* **

Now this is exactly a state of things as has been shown to exist for the so-called high velocity stars. A large area is avoided completely by the stars with velocities higher than 65 km/sec and it is seen at a glance that the centre of this area lies on the galactic circle in the neighbourhood of the predicted point.***

![Figure 1](image)

**Figure 1.**

Distribution of apices of high velocity stars in galactic longitude.

In order to derive the precise position of the centre of this avoided region I have plotted the numbers of space velocities higher than 80 km/sec against the galactic longitudes towards which they are directed. The stars were taken from table 6 of Groningen Publications No. 40. The drop in the number of stars near the edges of the avoided galactic sector is very sudden and this makes it possible to estimate rather accurately the exact position of this sector. The dots in figure 1 show the numbers for intervals of 5° in longitude; the worst irregularities have been abolished by taking "means of three". A smooth curve has been drawn through these points. If we define the limits of the avoided region as the longitudes where the curve rises to an ordinate 1, we find 342° and 132° for these limits. The mean errors of these numbers are probably less than ±5°. The centre of the avoided hemisphere is thus found at 57° galactic longitude, with a mean error of not more than ±4°.

* Assuming that the centre lies at 324° galactic longitude; compare OORT, B. A. N. No. 132, p. 80, and also PLASKETT, M. N. 88, 400, 1928.

** The galactic co-ordinates used in the present article are computed with NEWCOMB's pole; the longitudes are counted from the ascending node of the galaxy on the equator.

The curve in figure 1 shows two maxima, at 162° and 310° galactic longitude. Later on in this present article we shall refer to the possible significance of these maxima, but it may be remarked already here that they are situated exactly symmetrically with respect to the direction of 57° found above. The fact that the first maximum is more extended than the second one may be due to the selection of stars with large proper motions.

An independent, but somewhat less precise, determination of the position of the avoided region can be obtained from the radial velocity vectors of stars with high radial velocities for which no reliable space motions can be computed. After excluding the planetary nebulae and globular clusters on account of their very uneven distribution over the sky, a plot of the radial velocity vectors of the 82 remaining stars (the catalogue of high velocities *) being supplemented by some recent Mt Wilson and Lick velocities) shows that the galactic longitudes are distributed symmetrically around a mean position of 230° ± 7° m.e. Taking the opposite point as the centre of the avoided region we thus find 50° ± 7° galactic longitude.

Both values derived agree within their mean errors with the position predicted from the rotational hypothesis.

Further corroboration, though not entirely independent from the foregoing, may be obtained from the results found by STRÖMBERG for the direction of the general asymmetry, his latest results **) giving 61.5° galactic longitude and + 9.0° galactic latitude, with an estimated uncertainty of ± 5°. (This result includes the systematic velocities derived from globular clusters and spiral nebulae). Also, the present author has determined a position at 54° longitude and − 1° latitude from the space motions of the nearest stars. ***)

The agreement with the very highest velocities considered separately is not so good. The velocities above 300 km/sec are directed towards one quadrant of the galaxy, centered around 257° longitude. This may be due to an accidental arrangement of the small number of velocities concerned, but it is well to remember this possible difficulty.

On the whole the evidence of a connection between the motions of the high velocity stars and the supposed rotation of the galaxy is rather striking. In fact it appears tempting to suppose that the limit of about 65 km/sec, above which the velocities show this absolute avoidance of the region considered, is just equal to the difference between \( V_r \) and \( \Theta_c \). This hypothesis is illustrated in figure 2 where I have plotted as dots and open circles the projections on the galactic plane of all the space velocities contained in table 6 of Groningen Publications No. 40. In this diagram abscissae are velocity components parallel to the velocity of rotation, ordinates are components parallel to the direction of the galactic centre. The black square represents the velocity of the centre of gravity of the so-called slow moving stars, whereas the motion of the sun is indicated by a dot enclosed in a small circle. The full-drawn circle has a radius of 65 km/sec and contains all the "low" velocities, the large, broken, circle has a radius of 365 km/sec and would, on the above theory, indicate the limit which velocities in the galactic system may not surpass. No well determined velocities are known which surpass this limit; *) the only exceptional velocity in the catalogue, that of Cincinnati 1918—19, has a large uncertainty and may well in reality be contained within the large circle. **)

A plot of the velocities contained in the catalogue of high velocities necessarily shows a very inhomogeneous picture; not only are the lower velocities lacking, but there has also been a quite considerable preference for very high velocities so that, for instance, the relative number of velocities between 65 and 100 km/sec is much smaller than it would be in a representative sample. As it is of special interest to see how the transition of the high velocities to the lower ones takes place when there is a considerable proportion of high velocities, and yet no artificial selection, I have singled out a set of stars, indicated by the dots in figure 2, for which a nearly complete picture of the velocity distribution could be obtained. For this purpose it is best to limit oneself to a sample of near-by stars, though this must unavoidably cut down the high values.

*) Professor HERTZSPRUNG draws my attention to the RR Lyrae variable RZ Cephei for which SHAPLEY has found the exceptionally high velocity of 1100 km/sec. If we adopt SHAPLEY's distance of 1170 parsecs (Harvard Bulletin No. 773, 1922) and neglect the unknown radial velocity, the space velocity of this star with respect to the centre of the galactic system is found to be 1020 km/sec, which is about three times the velocity of escape. If the star is to be considered as a member of the great galactic system it would be necessary to reduce the adopted distance by a factor of at least three, corresponding to a change of 2.4 in the magnitude.

**) Though the exact value of the velocity of this wide double star (\( 15^{h}44^{m}7; - 15^{o}54' \) (1900)) is still uncertain it is very probable that it has the highest space velocity known to us at present. The Mt. Wilson spectroscopic parallax is 0.018, the trigonometric parallax, as given in SCHELESINGER's catalogue, is \(+ 0.34 \pm 8\) (p.e.). The radial velocity amounts to +366 km/sec, the proper motion to 3.68 annually, so that the space velocity comes out to be 1014 km/sec with respect to the sun if we use the spectroscopic parallax and 596 km/sec if we use the trigonometric value.

---

***) Groningen Publ. No. 40, p. 65.
velocities to a pretty small number. I don't believe there is much to be gained by considering stars with parallaxes smaller than $0.050$, so the dots represent only stars with parallaxes above this limit. Except for the absence of the velocities within the small broken circle of $20$ km/sec radius around the solar velocity the diagram of the dots may be considered as complete; the selection of large velocities for the parallax programmes has been counterbalanced artificially by somewhat increased radii of the dots near the solar velocity. The density of the velocities in an arbitrary region of the diagram should be judged from the total black area, not from the number of dots. For details concerning the construction of the diagram and the computation of the radii of the dots the reader is referred to an appendix following this article.

Contrary to the ideas developed in the present section I had formerly thought that the limit of the high velocity stars might represent the velocity of escape from a "local" cloud of stars. But according to our present starting point such a cloud, if it exist at all, is not likely to exert any appreciable influence on the motions of the stars: the rotation effects would seem to show that these motions are governed mainly by the larger galactic system. On the whole the suggestion considered above seems to be much more satisfactory; it not only explains why velocities above a certain limit do not occur in certain directions but it affords a perfect explanation of why just the observed directions are avoided.

3. As the circular velocity is probably of the order of $300$ km/sec, and may even be considerably higher, the above hypothesis would make the difference between the velocity of escape and the circular velocity very small in proportion to the velocity of escape itself. We may ask ourselves whether in the actual galactic system such a proportion would be at all likely to occur. We have seen (*) that there is probably a very strong concentration of mass towards the centre of the larger galactic system. Now if the entire mass were concentrated near this centre the ratio between the velocity of escape and the circular velocity would be $\sqrt{2}$ and thus much larger than the ratio $1.22$ resulting from the above hypothesis. In order to obtain so small a ratio there must be a considerable amount of matter outside the central regions and this mass must have a strongly flattened shape. Taking, as an example, the model considered in B. A. N. No. 133 (p. 92) in which the gravitating matter is arranged in ellipsoids whose short axes, perpendicular to the galactic plane, are $\frac{1}{100}$ of the axes in the galactic plane, we compute that the outer ellipsoid may extend $440$ parsecs beyond the sun before the velocity of escape from a point near the sun reaches a value equal to $1.22$ times the circular velocity. We conclude that even with a strong central condensation flattenings of about $\frac{1}{100}$ or $\frac{1}{150}$ would suffice to explain the close approach of the circular velocity to the velocity of escape.

Though flattenings of this order in the case of a fluid mass would be entirely fatal to its stability this does not need to hold true for a stellar system. Considerations from an entirely different side give, indeed, direct indications that the galactic system has a flattening of the order required. From the data given by Kapteyn and Van Rhijn (*) we compute that the average distance of the stars from the galactic plane is, roughly, $250$ parsecs; half the thickness of the system near the sun being thus about $25$ times smaller than the estimated distance from the sun to the centre.

From a combination of the observed density distribution with the frequency law of velocities perpendicular to the galactic plane we may eventually compute the corresponding component of the actual force with which the galactic system attracts stars at various distances from the galactic plane (compare equation 25). From this we may obtain an estimate of the oblateness of the main attracting mass. A preliminary investigation has shown that the oblateness must certainly be very great. However, it would be necessary to discuss very critically, and with this purpose specially in view, all the data about the density distribution in high galactic latitudes before one could make a numerical estimate of any value.

4. Entirely apart from the question concerning the velocity of escape the motions of the high velocity stars may be expressed in terms of the density distribution of these stars in the surrounding of the sun, as will be shown in the following.

It is not impossible that the galactic system has made some approach to a dynamically steady state (**). We shall assume as a preliminary starting point that the galactic system has really attained such a state and that it is approximately symmetrical around its axis of rotation.

A few words in defence of this working hypothesis may not be out of place here. Many astronomers may think it to be in too flagrant contradiction with what we know about the distribution of the stars to be of any use. Personally I do not believe so. It must be granted, of course, that the B stars are distributed

(*) B. A. N. No. 120.

(**) Which we may define by requiring that shape and extension of the system do not vary by a great proportion during the time of one revolution of the system.
Figure 2.
Relation of the distribution of high velocities to the rotation of the galactic system.

Abcissae are components of the velocities parallel to the rotation, ordinates are the components parallel to the direction of the centre. The cross represents the origin, which corresponds with the velocity of the centre of the galaxy. The distance between this cross and the black square represents the velocity of rotation for the bright stars. The full drawn circle with the square as centre has a radius corresponding to 65 km/sec and contains practically all velocities of bright stars.

The dots and small circles outside this circle show the motions of all high velocity stars for which reliable space velocities are known. The dots are supposed to represent a homogeneous sample, complete down to 19.5 km/sec from the solar velocity (indicated by a small circle with a dot). The large broken circle has a radius of 365 km/sec and is supposed to indicate the limit beyond which velocities would surpass the velocity of escape.

The plane of drawing co-incides with the galactic plane; the numbers near the edges indicate galactic longitudes.
through space in a very irregular manner: they appear to cluster in both large and small condensations, so that it would seem decidedly wrong to start an investigation of \( B \) stars on the assumption that they are evenly distributed over these regions. The same must probably be admitted for some others among the absolutely brightest stars, whereas the existence of such moving clusters as the Ursa Major cluster and the Hyades prove that also for the more quickly moving types of stars the conditions are still very different from perfect smoothness. Yet, considering the distribution of stars and velocities as a whole, I do not believe that observations show so general a tendency to gather into local streams and clouds that all statistical discussions would a priori be made impossible.

If we define a dynamically steady configuration in the sense of the last footnote on the preceding page it is reasonable to suppose that after a considerable number of revolutions some approximation to such a state will be reached. Now it is generally admitted that the age of the sun will not be less than \( 10^9 \) years, the time of revolution is estimated to be \( 2 \times 10^8 \) years, so that stars of the sun’s age must have made at least 50 revolutions and must be pretty well mixed.

It is held that great part of the stars observed are linked together in one very extensive local system. If this were so the following discussion, in which the stars in our neighbourhood are treated as just an arbitrary section from the great galactic system, would have no meaning. However, the direct observational evidence in support of the hypothesis of a local system comprising all types of stars is not so very strong. On the contrary, it seems to me that the regularity of the rotation effect in radial velocities, in particular the derived position of the centre, which remains the same when passing from relatively near \( B \) stars to the very distant faint \( \delta \) Cephei variables and planetary nebulae, furnishes good arguments to dispute the significance of a local system.

Finally we may take the view that the value of a working hypothesis should be judged from the work it does. In this respect it seems to do sufficiently well to be retained as a first guess, as it affords an approximate explanation of all the more striking peculiarities in the motions of the stars.

5. For the discussion of the dynamics of the stellar system we shall use similar methods as have been in use for the kinetic theory of gases and as also, in the hands of EDDINGTON, JEANS, KAPTEYN and others, have been applied to the dynamics of stellar systems.

Let us take rectangular co-ordinates \( x, y, z \) and let us denote the corresponding velocities by \( \xi, \eta, \zeta \). Denote the number of stars lying between \( x \) and \( x + dx \), \( y \) and \( y + dy \), \( z \) and \( z + dz \) and possessing velocities between \( \xi \) and \( \xi + d \xi \), \( \eta \) and \( \eta + d \eta \), \( \zeta \) and \( \zeta + d \zeta \) at a time between \( t \) and \( t + dt \) by \( f (x, y, z, \xi, \eta, \zeta) \) \( dx \) \( dy \) \( dz \) \( d \xi \) \( d \eta \) \( d \zeta \) \( dt \). Consider the change which will have taken place in this number after a short time \( \Delta t \). On one hand this change will be equal to \( \frac{df}{dt} \Delta t \) \( dx \) \( dy \) \( dz \) \( d \xi \) \( d \eta \) \( d \zeta \) \( dt \). On the other hand we may compute the change by subtracting from the number of stars which during \( \Delta t \) move into the element of space considered the number which moves out of it. By an elementary computation it is then found that the change must also be equal to

\[
-\left( \xi \frac{df}{dx} + \eta \frac{df}{dy} + \zeta \frac{df}{dz} + K_x \frac{df}{d \xi} + K_\eta \frac{df}{d \eta} + K_\zeta \frac{df}{d \zeta} \right) \Delta t \frac{dx}{d \xi} \frac{dy}{d \eta} \frac{dz}{d \zeta} \frac{d \xi}{d \xi} \frac{d \eta}{d \eta} \frac{d \zeta}{d \zeta} \frac{dt}{dt}
\]

where \( K_x, K_\eta, \) and \( K_\zeta \) denote the components of the force exerted on a unit of mass.

A comparison of the two results gives the fundamental equation

\[
-\frac{df}{dt} = \xi \frac{df}{dx} + \eta \frac{df}{dy} + \zeta \frac{df}{dz} + K_x \frac{df}{d \xi} + K_\eta \frac{df}{d \eta} + K_\zeta \frac{df}{d \zeta}
\]

(1)

I have tacitly assumed that encounters between individual stars have no sensible effect during the interval of time considered.

In the case of a dynamically steady state we have

\[
\frac{df}{dt} = 0
\]

and, therefore, also

\[
\xi \frac{df}{dx} + \eta \frac{df}{dy} + \zeta \frac{df}{dz} + K_x \frac{df}{d \xi} + K_\eta \frac{df}{d \eta} + K_\zeta \frac{df}{d \zeta} = 0
\]

(3)

\(^1\) For a rigorous derivation compare BOLTZMANN, Vorlesungen über Gastheorie, I, pp. 100—104.

As we want to consider a star system with rotational symmetry around an axis it will be profitable to introduce cylindrical co-ordinates. Let us call the distance to the axis \( \pi \), the position angle in the plane of the galaxy \( \theta \), the distance from the galactic plane \( z \). Let further the \( z \) co-ordinate used above co-incide with the one just introduced, and let the direction of the axis of \( x \) correspond with \( \theta = 0 \). The linear velocities in the three directions will be denoted by the corresponding capitals \( \Pi \), \( \Theta \) and \( Z \). If, from now on, \( f \) is considered as a function of these cylindrical variables, we have

\[
\frac{df}{d \pi} = 0
\]

(4)

and also

\[
K_\theta = 0
\]

(5)
In equation (3) all the differential quotients may be directly replaced by the corresponding quotients in the new co-ordinates, except \( \frac{\partial f}{\partial y} \) which changes into \( \frac{\partial f}{\partial y} - \frac{\partial f}{\partial y} \) so that in combination with (4) and (5) we find:

\[
\Pi \frac{\partial f}{\partial \sigma} + \Theta \left( \frac{\partial f}{\partial \Pi} - \frac{\partial f}{\partial \sigma} \right) + Z \frac{\partial f}{\partial \sigma} + K_{\sigma} \frac{\partial f}{\partial \Pi} + K_{\Pi} \frac{\partial f}{\partial Z} = 0 \quad (6)
\]

This equation is generally solvable \(^*\), but at present I shall only consider some particular solutions, which take account of the fact that the distribution of the peculiar motions of the stars has been found to approximate very closely to a function of the following type:

\[
f = f_0 e^{-l^2 + m^2 - l^2 (\Theta - \Theta_o)^2 - m Z} - \frac{m \Pi (\Theta - \Theta_o) - n Z - \rho (\Theta - \Theta_o) Z}{\Pi} \quad (8)
\]

In which \( h, k, l, m, n, \rho, f_0 \) and \( \Theta, \Theta_o \) are functions of \( \sigma \) and \( z \). Inserting (8) in equation (6) we get after dividing by \( -f \) and arranging according to powers of \( \Pi, \Theta, Z \):

\[
\Pi \frac{\partial h^2}{\partial \sigma} + \Pi \Theta \frac{\partial m}{\partial \sigma} - m \frac{\partial f_0}{\partial \sigma} + \Pi^2 Z \left( \frac{\partial h^2}{\partial \sigma} + \frac{\partial n}{\partial \sigma} \right) + \Pi \Theta \left( k^2 \frac{\partial h^2}{\partial \sigma} + 2 h^2 - 2 k^2 \right) + \Pi \Theta Z \left( \frac{\partial m}{\partial \sigma} + \frac{\partial \rho}{\partial \sigma} - \rho \frac{\partial f_0}{\partial \sigma} \right) + \Pi Z^2 \left( \frac{\partial m}{\partial \sigma} + \frac{\partial \rho}{\partial \sigma} - \rho \frac{\partial f_0}{\partial \sigma} \right) + 2 \Pi \Theta \left( \frac{\partial (k^2 \Theta_o)}{\partial \sigma} - \frac{k^2 \Theta_o}{\sigma} \right) - \Theta \left( \frac{\partial m}{\partial \sigma} + \frac{\partial \rho}{\partial \sigma} - \rho \frac{\partial f_0}{\partial \sigma} \right)
\]

As this equation must hold for all values of \( \Pi, \Theta \) and \( Z \), the co-efficients of the different powers must vanish separately. We thus get the following conditions:

\[
m = \rho = 0 \quad (10)
\]

\[
\frac{\partial h^2}{\partial \sigma} = \frac{\partial n}{\partial \sigma} = 0 \quad (11)
\]

\[
\frac{\partial h^2}{\partial \sigma} + \frac{\partial n}{\partial \sigma} = 0; \quad \frac{\partial h^2}{\partial \sigma} + \frac{\partial n}{\partial \sigma} = 0; \quad \frac{\partial h^2}{\partial \sigma} + \frac{n}{\sigma} = 0 \quad (12)
\]

\[
\frac{\partial h^2}{\partial \sigma} = 2 (k^2 - l^2) \quad (13)
\]

\[
\frac{\partial (k^2 \Theta_o)}{\partial \sigma} = \frac{k^2 \Theta_o}{\sigma} \quad (14)
\]

\[
\frac{\partial (k^2 \Theta_o)}{\partial \sigma} = 0 \quad (15)
\]

\[
\frac{\partial f_0}{\partial \sigma} = \frac{\partial (k^2 \Theta_o)}{\partial \sigma} + n K_\sigma + 2 h^2 K_{\sigma} \quad (16)
\]

\[
\frac{\partial f_0}{\partial \sigma} = \frac{\partial (k^2 \Theta_o)}{\partial \sigma} + n K_\sigma + 2 l^2 K_\sigma \quad (17)
\]

In the present paper I shall only discuss the motions and densities for regions practically in the galactic plane, which we suppose to be a plane of symmetry. It is clear that in this plane the differentials of \( h^2, k^2 \) and \( \Theta, \Theta_o \) with respect to \( z \) must be zero, so that beside \( m \) and \( \rho \) also \( n = 0 \). The distribution function takes the following form:

\[
f = f_0 e^{-l^2 + m^2 - l^2 (\Theta - \Theta_o)^2 - m \rho Z} \quad (18)
\]

The principal axes of the velocity ellipsoid are thus parallel to the axes of the cylindrical co-ordinates; \( h, k \) and \( l \) represent the moduli along the principal axes. Equations (11) show that \( h \) is independent of \( \sigma \); \( k \) varies in the manner indicated by (13) but the variation is so small that we cannot expect to verify it from observations. Assuming \( k/l = 0.6 \) (section 7) and \( \sigma = 6300 \) parsecs we find that the average transverse velocity should decrease by a factor of 0.90 for an increase of 1000 parsecs in \( \sigma \). The variation of \( l \) with \( \sigma \) must in general be zero (see the next section).

It will appear convenient for the following to rewrite some of the above equations in a somewhat

\(^*\) It was first introduced into astronomy by SCHWARZSCHILD, who showed that it could fairly represent the velocity law of the stars.
different form. Eliminating $\partial k^3/\partial \sigma$ between (13) and
(14) we obtain

$$
\frac{h^2}{k^2} = \frac{1}{2} \left( 1 + \frac{\text{c} \partial \Theta_0}{\partial \sigma} \right)
$$

(19)

Let us, further, introduce the number of stars per unit volume, \( \nu \), instead of \( f_0 \) in the left hand member of equation (16). \( \nu \) is found by integrating (8) over all values of \( \Pi, \Theta \) and \( Z \):

$$
\nu = \frac{\pi V}{h k I} f_0.
$$

Eliminating the differentials of \( k \) and \( \Theta_0 \) by means of
(13) and (14) and putting \( \partial f/\partial \sigma = 0 \) we thus find

$$
\frac{1}{\text{Mod}} \frac{\partial \log \nu}{\partial \sigma} = 2 \frac{h^2 \Theta_0^2}{\sigma} + \frac{\sigma h - k^2}{k^2 \sigma} + 2 h^2 K_{\sigma} (20)
$$

The equations derived above are intended to represent
the density and velocity distribution of an arbitrary class
of stars, provided its velocity law is of the
general ellipsoidal type. From evidence which cannot
be discussed at this place it is probable that only a
minor part of the forces \( K_{\sigma} \) and \( K_s \) introduced above
comes from the observed stars themselves, so that
it is not necessary to introduce conditions arising
from POISSON's equation into our present problem.

$$
K_{\sigma} \left( 2 \frac{\partial h^3}{\partial \sigma} - \frac{\partial h}{\partial \sigma} + \frac{\partial h}{\partial \sigma} \right) + K_s \left( -2 \frac{\partial h^3}{\partial \sigma} + \frac{\partial h}{\partial \sigma} + \frac{\partial h}{\partial \sigma} \right) + 2 h^2 K_{\sigma} - 2 h^2 K_s + n \left( \frac{\partial K_s}{\partial \sigma} - \frac{\partial K_{\sigma}}{\partial \sigma} \right) = 0
$$

or, inserting the expressions (21) and (22), putting \( c_6 = 0 \):

$$
\epsilon_s \left( 3 \left( s K_{\sigma} - \sigma K_s \right) + \left( s^2 - \sigma^2 \right) \frac{\partial K_{\sigma}}{\partial \sigma} + \sigma s \left( \frac{\partial K_s}{\partial \sigma} - \frac{\partial K_{\sigma}}{\partial \sigma} \right) \right) + 2 \left( c_1 - c_\sigma \right) \frac{\partial K_{\sigma}}{\partial \sigma} = 0
$$

(23)

where \( \partial K_{\sigma}/\partial \sigma \) has been replaced by the evidently
identical expression \( \partial K_{\sigma}/\partial \sigma \).

In general neither the coefficient of \( \epsilon_s \) nor that of
\( (c_1 - c_\sigma) \) will be identically zero so that the above
condition requires that \( c_6 = 0 \) and \( c_\sigma = c_\sigma \). We infer
from the first of these conditions that \( n \) must be
identically zero: the velocity ellipsoid must always be
so orientated that its principal axes are parallel
to the directions of \( \sigma, \theta \) and \( z \) respectively. It follows
from (12) that, for a definite category of stars considered,
both \( h \) and \( I \) are constant throughout the system.

The second condition would then imply \( h = I \).**

This is certainly far from being fulfilled in the galactic system,
for the data available about star streaming show that for the stars
in general \( h \) is approximately
only 0.6 \( I \), and a still smaller ratio is found for stars
of high velocity (compare page 283).

Though the condition \( h = I \) seems to be a necessary one for strict dynamical equilibrium it is not certain

\[ \left( c_1 - c_\sigma \right) \]

that the deviations caused by the observed inequality
would be perceptible after a time covering only a
few revolutions of the system. It is to be noted that
the present case, in which we are considering an
extremely flattened system of stars, the numerical
value of \( \partial K_{\sigma}/\partial \sigma \) must be very small *) so that a large
value of \( (c_1 - c_\sigma) \) may be admissible before the left
hand member of (23) becomes sensible. It is perhaps
easier to consider the motion of one star: it is probable
that only after many revolutions the average II motion
will become to a considerable degree converted into
motions in the \( s \) co-ordinate, for in the practical case
considered where the extent of the relative orbits is
very small compared to the dimensions of the entire
system **), the motion in the \( s \) co-ordinate must be
very nearly independent of the motion in the galactic
plane.

Assuming, for the moment, that the deviation from

\[ (c_1 - c_\sigma) \]

*) Inside an attracting ellipsoid of uniform density \( \partial K_{\sigma}/\partial \sigma \) would be zero. For the model considered in B. A. N, No. 133
(\( \theta \), the second example) at \( s = 200 \) parsecs \( \partial K_{\sigma}/\partial \sigma = 0.00024 \),
if the unit of distance is 1 parsec and the unit of velocity 1 km/sec,
the corresponding unit of time being 0.976 \( \times 10^6 \) years.

**) Compare the first footnote on page 278.

© Astronomical Institutes of The Netherlands • Provided by the NASA Astrophysics Data System
dynamical equilibrium following from the inequality of \( h \) and \( l \) is insignificant, we shall in the following sections compare the two equations (19) and (20) with the observed velocity distributions.

Inserting the expressions (21) and (22) for \( k^s, k^l \) and \( \Theta \), and putting \( c_6 = c_5 = 0 \) we find that equation (16) or (20) becomes:

\[
\frac{1}{\text{Mod}} \frac{\delta \log \nu}{\delta \sigma} = \frac{c_6 \omega^3 + (2 c_1 c_2^3 - c_2 \omega^2)}{(c_2 \omega^2 + \omega^2)} + 2 c_4 K_\sigma \tag{24}
\]

and, similarly, equation (17) becomes:

\[
\frac{1}{\text{Mod}} \frac{\delta \log \nu}{\delta \sigma} = 2 c_4 K_\sigma \tag{25}
\]

After the present paper had been practically written my attention was drawn to two papers, one by EDDINGTON *) and one by JEANS **), dealing with very much the same problem as discussed in this and the preceding section. This is especially true of EDDINGTON's article in which the most general case of a stellar system with ellipsoidal velocity law has been exhaustively treated. Instead of adopting a definite system of co-ordinates and introducing the orientation of the axes of the velocity ellipsoid (contained in \( m, n \) and \( p \) above) EDDINGTON follows the more logical way of considering general curvilinear co-ordinates, \( \lambda, \mu, \nu \) which are at each point tangent to the principal axes of the velocity ellipsoid. The surfaces \( \lambda = \text{const.}, \mu = \text{const.} \) and \( \nu = \text{const.} \) are called the principal velocity surfaces. From the discussion in the present section it is inferred that in the most general case of a steady system with symmetry around an axis these surfaces must be cylinders and planes; thus the cylindrical co-ordinates chosen are just those which reduce the equations to their simplest forms. The above equations are a special case of EDDINGTON's general formulae and his results have been used in checking these equations.

At this place I also want to express my indebtedness to Dr. WOLTER for a helpful discussion on the subject of the present section.

7. Equation (19) determines the ratio between the two axes of the velocity ellipsoid that are parallel to the galactic plane. As the right hand member of (19) must always be smaller than 1 it is seen that we must have star streaming with a preferential axis directed toward the centre. The quantities which occur in the right hand member are just those which can be determined, for each kind of stars separately, from the rotation effects in radial velocities and proper motions. We have for the semi-amplitude of the rotational term in the radial velocities:

\[
A = \frac{1}{2} \left( \frac{V}{R} \right) \left( \frac{\delta V}{\delta R} \right)
\]

in which, strictly speaking, \( V \) is not the circular velocity, but the velocity of rotation for the type of stars considered. In our present notation we have thus:

\[
A = \frac{1}{2} \left( \frac{\Theta_\sigma}{\sigma} \frac{\delta \Theta_\sigma}{\delta \sigma} \right)
\]

and similarly for the quantity derived from proper motions:

\[
B = \frac{1}{2} \left( \frac{\Theta_\sigma}{\sigma} \frac{\delta \Theta_\sigma}{\delta \sigma} \right)
\]

Thus, inserting these in (19):

\[
k^s/k^l = -B(A-B) \tag{26}
\]

As \( B \) is negative, and \( A \) positive, the ratio is seen to be smaller than 1. The present theory thus gives a qualitative explanation of the general phenomenon of star streaming in the galactic plane, the principal vertex being situated not far from the direction of the centre of the galactic system. Though there are still a number of contradictory details it becomes more and more evident that any explanation to be offered must be like the present a very general one, as recent observations of preferential motion among distant faint stars and also the indications of the same tendency among the high velocities quite definitely seem to rule out the possibility of a purely local explanation.

LINDBLAD who was the first to work out this explanation for a rapidly rotating system, *) has given a very direct derivation by following the motion of one star through its orbit. **) Consider a star moving exactly in the galactic plane but whose velocity deviates a little from the circular velocity. Its motion can be conveniently described by considering the relative motion of the star around a point moving in a circle in the great galactic system. We can choose the radius of this circle in such a way that the point is at the centre of the star's relative orbit. The shape of the relative orbit will be determined by the field

*) The dynamics of a stellar system. Third paper: Oblate and other distributions. (M. N. 76, 37, 1915).


© Astronomical Institutes of The Netherlands • Provided by the NASA Astrophysics Data System
of force of the galactic system. If the motions are small it will be an ellipse whose axes can easily be expressed in the force, $K_\sigma$, and its derivative $\frac{\partial K_\sigma}{\partial \sigma}$.)

The ratio of the two galactic axes of the velocity ellipsoid is then found by averaging the "peculiar" velocity of the star in radial as well as in transverse direction over the entire relative orbit. As we are in practice concerned with the average peculiar velocity of many different stars in a certain element of volume and not with the average for one star over its orbit, a little consideration shows that in order to get the right ratio we must define the peculiar velocity in transverse direction as the difference of the star's actual velocity from the circular velocity at a point near the star. If the star density is constant or changes proportional with $\sigma$, and if the average peculiar velocity is approximately the same at different points, then the ratio of the average peculiar velocities found for one star will be the same as that found in an element of volume; it is easily computed to be equal to:

$$\frac{\Theta - \Theta_c}{\Pi} = \sqrt{\frac{1}{2} \left( 1 + \frac{\Theta_c \sigma}{\Theta_\sigma \sigma} \right)}$$

where $\Theta_c$ is the circular velocity, $\Theta - \Theta_c$ and $\Pi$ are the average peculiar velocities in transverse and radial direction.

For small peculiar velocities, if the velocity of rotation is practically equal to the circular velocity, this expression coincides with the expression (19) found above. LINDBLAD's computation has been more general in so far as it applies to any kind of velocity law and not only to the exponential law used above; on the other hand it is only rigorous for small peculiar velocities which are entirely in the galactic plane.

It is interesting to compare the rotating stellar system with an ideal rotating gas as treated by BOLTZMANN. **) The equations governing the two cases are quite similar. The collisions between the molecules only act as enforcers of the Maxwellian velocity law at each point of the gas, otherwise they do not enter into the equations. The theorem that in BOLTZMANN's gas the modulus of the Maxwellian law has the same value throughout the gas is derived in exactly the same way as we have derived the constancy of $k$ and $l$ in section 5. BOLTZMANN shows further that the angular velocity of rotation must be the same throughout the gas. This is a consequence of the spherical symmetry of the Maxwellian law; the same property should hold for the stellar system if the two moduli $h$ and $k$ were equal, as we see from equation (19) if we put the left hand member equal to 1. This gives

$$\sigma \frac{\partial \Theta_\sigma}{\Theta_\sigma \sigma} = 1$$

from which we deduce

$$\Theta_\sigma \sigma = \text{constant}.$$ 

The fundamental difference between the dynamics of BOLTZMANN's idealized gas and the stellar system is caused by the fact that in the gas the density is supposed to be so large that the molecules cannot describe a considerable portion of one revolution around the centre before a collision occurs.

The differential rotation of the galactic system is seen to be directly tied up with the ellipsoid of the distribution of peculiar velocities. In a steady system the one cannot exist without the other. If there were a type of stars which did not show any differential rotation its velocity distribution in the galactic plane should be circular. This is of interest because in the long period variables we have an example where this expectation can to some extent be verified by observations (see below).

When we come to a more accurate comparison of this theory about the peculiar motions with observations we are led into several still unsolvable difficulties. In the first place there is a discordance between the average direction of the major axis of the velocity ellipsoid and the probable direction of the centre. The most homogeneous and at the same time comprehensive investigation of star streaming is probably that by EDDINGTON from the BOSS stars *) in which the principal vertex is found at $346^\circ$ galactic longitude and $+1^\circ$ latitude. In a solution by the generalized ellipsoidal method CHARLIER finds $343^\circ$; $-10^\circ$ from the same material. */) A very satisfactory confirmation was obtained by EDDINGTON and HARTLEY from radial velocities, giving $345^\circ$ longitude and $0^\circ$ latitude. **)

On the other hand greatly deviating positions have been found from more limited groups of stars, and that notwithstanding a very considerable accuracy in many cases. We may mention DYSON's and BELJAWSKY's solutions from stars of large proper motions where the galactic longitude of the true vertex has decreased to $335^\circ$ and $332^\circ$ respectively. */) From space velocities computed with the aid of spectroscopic parallaxes STRÖMBERG has made an extensive study of the dis-

---

1) The dimensions of this relative orbit are proportional to the maximum deviation of the velocity from the circular velocity. With the data of B.A.N. No. 132 I find 230 parsecs for the semi major axis in radial direction if the maximum velocity in radial direction is 15 km/sec.

**) Vorlesungen über Gastheorie, I, pp. 138—139.

*) See EDDINGTON, Stellar movements and the structure of the universe, p. 124.

**) M.N. 75, 526, 1915.
tribution of velocities for various spectral types. *) A survey of the principal results (Table V of his first paper) shows that for the \( A \) and \( F \) stars and the giants up to \( K1 \) the principal vertex is found near 350° galactic longitude. For the still later type giants (groups IV and V) it is very poorly determined but appears to sink down to a value near 330°. For the dwarfs, on the other hand, the stream motion is very pronounced and the vertex is found at 329° longitude with a probable error of \( \pm 4°.6 \). If we consider the stars of high velocity separately there is a quite pronounced preferential motion, indicated for instance by the two maxima in the curve representing the distribution of longitudes of apices (figure 1 of the present paper). The maxima are found near 170° and 310°, after correcting for the systematic motion of the whole, indicates true vertices at 150° and 330° galactic longitude.

To complete this enumeration of some of the principal results for the direction of preferential motion I want to make reference to the results obtained from fainter stars in general, especially to those recently obtained by SMART. **) Taking all the evidence together the vertex for faint stars, between the 10th and 12th apparent magnitude say, may be put down at about 335° galactic longitude and 0° galactic latitude.

From the foregoing it seems probable that the direction of the true vertices for the bright stars and, according to the space velocity results especially for the absolutely bright \( A \), \( F \) and \( G \) stars, deviates considerably from that found from dwarf stars, apparently faint stars in general, and high velocity stars. The galactic longitude expected from theory is much closer to that derived from the latter categories than to that of the bright stars: from the rotation effects in radial velocities a longitude of 325° was derived for the direction of the centre, and this value is supported by two independent pieces of information, viz. the distribution in the sky of globular clusters and the direction derived from the region avoided by high velocities. All the longitudes discussed above are somewhat higher, but except for the absolutely bright \( A \), \( F \) and \( G \) stars the differences are not alarming. ***

Unsatisfactory though the conclusion is, it would seem probable that the motion of these brighter groups of stars are made discordant by the existence of local streams of very considerable extent. A glimpse of light is thrown upon this subject by STRÖMBERG's studies of the space velocities of bright stars and by his investigation of stars moving parallel to the Hyades and the Ursa Major cluster *), but, as matters stand, I do not think that a quite satisfactory insight has been obtained.

Obviously connected with the above problem is the observation that, for the bright stars at least, the distribution of peculiar velocities is asymmetrical and does not fit the ellipsoidal law at all; in star stream language this has been stated in the following way: about 60% of the bright non-\( B \)-stars belong to stream I and only 40% to stream II. This, of course, is also in contradiction with the hypothesis of a steady state. It may be noted that the same phenomenon is not shown by the apparently faint stars, which seem to be divided equally over the two drifts, and this again seems to support the view that the unequal distribution for the bright stars is due to local irregularities.

The proportion between the two galactic axes of the ellipsoidal velocity distribution appears to vary with the type of stars considered, it is approximately 0.5 for the \( A \) and \( F \) stars and for the dwarfs; it has a somewhat higher value, around 0.7, for the later type giants. For the faint stars treated by SMART a value \( h/k = 0.63 \) is indicated (average between results from Cambridge and Groningen proper motions).

The proportion derived theoretically from the rotation effects, according to formulae (19) or (26), is 0.75, if we use the results derived in B. A. N. No. 132 and if we assume that the peculiar velocities are so small that the velocity of rotation does not differ appreciably from the circular velocity. It must be emphasized, however, that the value of the constant \( B \) is rather uncertain and that it is not impossible that this theoretical value may have to be brought down to 0.60.

We may turn the argument the other way and estimate \( B \) from the observed stream motion. Adopting \( h/k = 0.60 \) we find from (26): \( B = -0.011 \), or in angular measure \( B = -5.0023 \) annually, which cannot be definitely said to contradict the direct observations. It is interesting to remark that with a value of \( B \) as found above the computed distance to the axis of rotation would be increased to 10000 parsecs, if we may point to HALM's analysis of Boss' proper motions by means of three drifts, where the galactic longitude of the apex of drift I is found 15° lower than in EDDINGTON's analysis by the two stream theory.


**) M. N. 87, 122, 1926 and 88, 144, 1927. Compare also EDDINGTON M. N. 87, 140.

***) Too high galactic longitudes for the vertex may arise from the fact that there will always be a considerable percentage of stars with small peculiar motions which do not participate in the main motions of the two streams (HALM's drift \( O; B \) stars). In an analysis by the two stream theory most of such stars will be grouped with stream I and will probably shift the apex of this stream to a position nearer the antapex of the sun's motion, i.e. to a higher galactic longitude. For an illustration
assume $\Theta_c = 300 \text{ km/sec}$. Considering that the adopted value of $\Theta_c$ is uncertain by at least 100 km/sec we see that the distance to the centre computed from the rotation effects is no longer in contradiction with the distance to the centre of the cluster system as estimated by Shapley.

In cases where considerable differences between the rotational velocity and the circular velocity are to be expected on account of large peculiar motions the ratio may differ considerably from the above value. Such a case seems to be presented by the long period variables where, so far as the data go, the peculiar velocities show no indication of an ellipsoidal distribution. The case is of special interest because the stars are so distant that they ought to afford a pretty good determination of the differential rotation. However, this was found to be practically zero for these stars *) which, if our theory is correct, should be accompanied by a circular velocity distribution of peculiar velocities in the galactic plane. The data about the velocity distribution are rather meagre, however. They are shown in Table 1. The radial velocities measured by Merrill **) were used in combination with southern observations by Miss Allen. ***)

### Table 1.

<table>
<thead>
<tr>
<th>Region</th>
<th>$M_{0}\text{e to } M_{5}\text{e}$</th>
<th>$M_{0}\text{e to } M_{8}\text{e}$; $Se$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gal. long.</td>
<td>Gal. lat.</td>
</tr>
<tr>
<td>115° to 175°</td>
<td>295° to 355°</td>
<td>$-30° + 30°$</td>
</tr>
<tr>
<td>25° to 85°</td>
<td>205° to 265°</td>
<td>$-30° + 30°$</td>
</tr>
<tr>
<td>o° to 360°</td>
<td>$-90° - 50°$</td>
<td>$+ 50° + 90°$</td>
</tr>
</tbody>
</table>

The first region is around the direction of the circle and around the point opposite. There is no indication that the average velocities in this region are larger than in the next region. The average velocities in the third line show the mean peculiar velocity in a direction perpendicular to the galactic plane. The fact that the early type variables this velocity seems to be lower is entirely due to two excessive velocities in the first two lines: excluding two peculiar velocities larger than 150 km/sec the first two average velocities are changed into ± 32 km/sec and ± 41 km/sec respectively. For the computation of the peculiar velocities I assumed a solar velocity of 69 km/sec towards $18^h 09^m; + 34^* +$ for the $M_{0}\text{e} - M_{5}\text{e}$ stars and one of 33 km/sec towards the same apex for the other group. Moreover a constant correction of $- 7$ km/sec was applied to the stars of the former group. In both cases the velocities reduced to the system of the absorption lines were used.

Some difficulty in the way of a general explanation of the star streams along the lines sketched in this section is presented by the radial velocities of $B$ stars. It was shown by Campbell **) as early as 1911 that the $B$ stars did not show any sign of ellipsoidal in their motions. He considered three zones at different distances from the two vertices and showed that the average peculiar radial velocities were practically equal in the three cases. I recently looked into this matter anew, including the Lick and Yerkes velocities recently published, and the outcome was essentially the same. However, these results are not entitled to a very large weight, for the number of radial velocities of $B$ stars sufficiently reliable for an investigation of the slow peculiar motions of these stars is quite small. Also we must consider the possibility that the real velocities may be much smaller than the measured Doppler shifts.

Supposing that the result is real I do not think that it forms a very serious objection against our general theory. We know that the nearer $B$ stars are arranged in a few extended groups; the average internal motion probably varies from group to group and this effect may easily have veiled the ellipsoidal character of the motions. To follow up this question would take us into a detailed study of the $B$ stars, which does not fall within the scope of the present paper.

In the present section I have only considered the moduli along the axes parallel to the galactic plane. The problem of the modulus in the direction perpendicular to this plane has been fully discussed on page 276.

8. The equations (16) and 17), or rather the equations (20) and (25) deduced from them, determine the distribution of star density in the idealized stellar system.

In order to see more clearly the meaning of equation (20) it may be written as follows:

$$\frac{\nu \Theta_c^2}{\sigma} = - \nu K_{\nu} + \frac{1}{2h^2} \left( \frac{\delta \nu}{\delta \sigma} + \nu \frac{h^2 - h^4}{\sigma^3} \right) \tag{28}$$

*) The apex as computed by Merrill from the velocities of these stars differs about 15° from the standard apex adopted here, but this cannot have had an appreciable effect on the average velocities.

**) Lick Bull. 6, 117, 1911.

---

© Astronomical Institutes of The Netherlands • Provided by the NASA Astrophysics Data System
When considering stars whose peculiar velocities remain very small compared with \( \Theta_o \) and if we limit ourselves to reasonable values of \( \frac{\partial \log \nu}{\partial \sigma} \), the second term of the right-hand member will be small compared to the first; for small peculiar velocities (28) approaches to the ordinary equation for the circular motion of a point of mass (taking the mass of one star of the type considered as unit of mass). As the peculiar motions increase and the individual stars describe orbits sensibly deviating from circles, the group motion, \( \Theta_o \), becomes different from the circular motion of a mass point. It is as though the group of stars were describing a circular orbit under the influence of a force differing somewhat from \( K_o \). According to the above formula the main term \(^1\) of this difference is equal to the product of the average square of the peculiar velocity (1/2 \( h^2 \)) and the gradient of the density in space of the group of stars considered (\( \frac{\partial \nu}{\partial \sigma} \)); it is the gradient of what we are accustomed to call "pressure" in the case of a gas. Though we cannot here realize the force exerted by the pressure gradient by putting in a screen, as we could in a gas, it seems quite legitimate to keep on using the same term, as the two forces are in all respects similar phenomena.

It is of interest to add here the more general equations determining the group motion of a number of individuals in case we do not restrict ourselves to an ellipsoidal distribution of velocities. Suppose there be cylindrical symmetry as before, but beyond this let the velocity distribution be perfectly arbitrary. Denote the average motion of the stars in an element of volume by \( \Theta_o \), let further \( \Pi^a \) represent the average square velocity in radial direction, \( \Theta^a \) the same in transverse direction in the galactic plane, and \( \overline{Z}^a \) the average square velocity perpendicular to the galactic plane, then the general equations read as follows: \(^2\)

\[
\frac{\partial (\nu \Pi^a)}{\partial \sigma} + \frac{\nu \Pi^a - \nu \Theta^a}{\tau} = \nu K_o \tag{29}
\]

\[
\frac{\partial (\nu \overline{Z}^a)}{\partial \sigma} = \nu K_s \tag{30}
\]

It is easily seen that these are reduced to (20) and (25) in the case of an ellipsoidal distribution, for then we have

\[
\Pi^a = \frac{1}{2} h^a ; \quad \overline{Z}^a = \frac{1}{2} h^a ; \quad \Theta^a = \Theta_o^a + \frac{1}{2} k^a
\]

\( k \) is independent of \( \sigma \) and \( l \) is independent of \( x \).

For the present we shall only discuss the less general equation (28). Instead of \( K_o \) we may introduce the

\[
(\Theta_o - \Theta_e) \frac{\Theta_o + \Theta_e}{\tau} = \frac{1}{2 h^2} \left( \frac{1}{2} \theta \frac{\partial \log \nu}{\partial \sigma} \sin \frac{k^a - h^a}{k^a} \right) \tag{31}
\]

If \( |\Theta_o - \Theta_e| \) is small compared to \( \Theta_e \), we may replace \( \frac{\Theta_o + \Theta_e}{\tau} \) by \( 2 \Theta_o/\tau \), which fraction is known from the rotation effects in radial velocities and proper motions. According to the results obtained previously \(^3\)

\[
\Theta_o/\tau = A - B = 0.043 \text{ km/sec. parsec, so that we find approximately:}
\]

\[
\Theta_o - \Theta_e = \frac{11.6}{2 h^2} \left( \frac{1}{2} \theta \frac{\partial \log \nu}{\partial \sigma} + \frac{k^a - h^a}{k^a} \right) \tag{32}
\]

We see from these formulae that unless the density gradient varies from type to type in a very special manner, stars with high peculiar motions should possess an average systematic velocity with respect to stars with slow peculiar motions. If \( \frac{\partial \log \nu}{\partial \sigma} \) is negative (that is, if the density of the stars considered increases toward the centre) this systematic motion should be directed towards the negative \( \Theta \) axis.

Such a state of things seems actually to exist. We have seen that there is a systematic relative motion of the stars of high velocity with respect to stars of low velocity and that the direction of this motion agrees closely with that of the negative \( \Theta \) axis. If, for a certain type of stars, the percentage of high velocities is larger, this relative motion is also larger, as a direct consequence of the distribution of the directions of these large motions; or, as we may also put it: the systematic motion increases with the average peculiar velocity, and therefore with \( 1/2 h^2 \). \(^4\)

We are led to investigate this matter a little further and to inquire which would be the numerical value of \( \frac{\partial \log \nu}{\partial \sigma} \) required to explain the observed systematic motions. The numerical data appear to be very meagre, yet the matter seems to have such a general importance that it seems worth while to try to make a guess.

A brief summary of the mostly rather uncertain data available is given in tables 2 and 3. The third column of table 2 and the fifth column of table 3 give the group motion with respect to the sun decomposed into the direction of the \( \Theta \) axis. Indications about the real directions found for the group motions have been omitted because the discussion in section 2 has already shown that the relative motion of the objects with high velocities, with respect to objects of low velocity, is approximately parallel to the \( \Theta \) axis. The next columns show the value of \( a \), the mean square velocity along the \( \Pi \) axis.

---

\(^1\) B.A.N. No. 132.


\(^3\) For a derivation compare JEANS, M.N. 82, 123—125, 1922.
Table 2.

<table>
<thead>
<tr>
<th>Type</th>
<th>( n )</th>
<th>( \Theta_o - \Theta_{run} ) (km/sec)</th>
<th>( a = \sqrt{1/2h^2} ) (km/sec)</th>
<th>( h/k )</th>
<th>( \Theta_o - \Theta_c ) (km/sec)</th>
<th>( \log \Sigma ) ( \frac{\partial}{\partial \delta} )</th>
<th>Percent, m.e.</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>±400</td>
<td>-18</td>
<td>8</td>
<td>1</td>
<td>&gt;-3</td>
<td>----------------</td>
<td>----------------</td>
<td>(a)</td>
</tr>
<tr>
<td>( c ) stars</td>
<td>113</td>
<td>-18</td>
<td>14</td>
<td>0.8</td>
<td>&gt;-3</td>
<td>(b)</td>
<td>----------------</td>
<td>(b)</td>
</tr>
<tr>
<td>( B - M ), vel. &lt; 63 km/sec</td>
<td>2091</td>
<td>-16</td>
<td>24</td>
<td>0.6</td>
<td>&gt;-3</td>
<td>(b)</td>
<td>0.00024</td>
<td>(b)</td>
</tr>
<tr>
<td>( F - M ) (high velocities)</td>
<td>86</td>
<td>-36</td>
<td>60</td>
<td>0.5</td>
<td>&gt;-20</td>
<td>0.5</td>
<td>25</td>
<td>(d)</td>
</tr>
<tr>
<td>( M_o - M_{36} )</td>
<td>22</td>
<td>-106</td>
<td>100</td>
<td>0.5</td>
<td>&gt;-90</td>
<td>0.5</td>
<td>34</td>
<td>(d)</td>
</tr>
<tr>
<td>( M_{46} - M_{56} )</td>
<td>41</td>
<td>-65</td>
<td>46</td>
<td>1.0</td>
<td>&gt;-49</td>
<td>1.0</td>
<td>79</td>
<td>(e)</td>
</tr>
<tr>
<td>( M_{66} - M_{86}, Sx )</td>
<td>86</td>
<td>-35</td>
<td>30</td>
<td>1.0</td>
<td>&gt;-19</td>
<td>1.0</td>
<td>76</td>
<td>34°f</td>
</tr>
<tr>
<td>RR Lyrae variables</td>
<td>26</td>
<td>-108</td>
<td>74</td>
<td>1</td>
<td>&gt;-92</td>
<td>1</td>
<td>52</td>
<td>48°f</td>
</tr>
<tr>
<td>Planetary nebulae</td>
<td>110</td>
<td>-34</td>
<td>40</td>
<td>1</td>
<td>&gt;-18</td>
<td>1</td>
<td>41</td>
<td>39°f</td>
</tr>
<tr>
<td>Globular clusters</td>
<td>18</td>
<td>-272</td>
<td>114</td>
<td>1</td>
<td>&gt;-256</td>
<td>1</td>
<td>57</td>
<td>(g)</td>
</tr>
</tbody>
</table>

Table 3.

<table>
<thead>
<tr>
<th>STRÖMBERG group no.</th>
<th>Spectrum</th>
<th>Limits ( m + 5 \log \mu )</th>
<th>( n )</th>
<th>( \Theta_o - \Theta_{run} ) (km/sec)</th>
<th>( a )</th>
<th>( h/k )</th>
<th>( \Theta_o - \Theta_c ) (km/sec)</th>
<th>( \log \Sigma ) ( \frac{\partial}{\partial \delta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( K_4 ) to ( M_9 )</td>
<td>( \geq +3.1 ) to +12.0</td>
<td>79</td>
<td>-13.7</td>
<td>52.2</td>
<td>0.21</td>
<td>+2</td>
<td>-0.00004</td>
</tr>
<tr>
<td>12</td>
<td>( G_9 ) to ( K_3 )</td>
<td>+3.1 to +12.0</td>
<td>104</td>
<td>-24.5</td>
<td>33.7</td>
<td>1.02</td>
<td>-8</td>
<td>-26</td>
</tr>
<tr>
<td>15</td>
<td>( G_0 ) to ( G_8 )</td>
<td>+1.1 to +5.0</td>
<td>154</td>
<td>-19.7</td>
<td>44.0</td>
<td>0.49</td>
<td>-4</td>
<td>-14</td>
</tr>
<tr>
<td>16</td>
<td>( G_0 ) to ( G_8 )</td>
<td>+5.1 to +10.0</td>
<td>111</td>
<td>-31.2</td>
<td>46.2</td>
<td>0.70</td>
<td>-15</td>
<td>-29</td>
</tr>
<tr>
<td>20</td>
<td>( F_0 ) to ( F_9 )</td>
<td>+3.1 to +5.0</td>
<td>82</td>
<td>-16.8</td>
<td>48.4</td>
<td>0.31</td>
<td>-1</td>
<td>-8</td>
</tr>
<tr>
<td>21</td>
<td>( F_0 ) to ( F_9 )</td>
<td>+5.1 to +14.0</td>
<td>38</td>
<td>-105.0</td>
<td>122.0</td>
<td>0.62</td>
<td>-89</td>
<td>-31</td>
</tr>
<tr>
<td>Mean</td>
<td></td>
<td></td>
<td>568</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-0.00019</td>
</tr>
</tbody>
</table>

Notes to the tables.

(a) Different solutions appear to yield rather divergent systematic motions, so that the value given must be considered to have an uncertainty of about 3 km/sec. It is the mean of four determinations of \( \Theta_o - \Theta_{run} \) computed from solar motion results by GYLLENBERG and MÄLMQUIST \(^*\)), by STRÖMBERG \(^**\)), by FROST, BARRETT and STRÜVE \(^***\)), and by CAMPBELL and MOORE \(^****\)). The given value for \( a \) has been computed by the author from a combination of Lick and Yorkes results for 118 stars with the best measurable spectra.

(b) \( \Theta_o - \Theta_{run} \) derived from SCHILT's determination of the solar motion with respect to these stars. \(^****\) The peculiar velocities have been corrected for rotational effects before computing \( a \). The value of \( h/k \) has little weight as it was computed from a very limited material, in order to avoid effects of the differential rotation.

(c) Derived from GYLLENBERG and MÄLMQUIST's \(^*\)) results after exclusion of peculiar velocities higher than 63 km/sec as indicated in Groningen Publication No. 40 (p. 68).

(d) The numbers given in these lines do not claim to be more than a guess. They resulted from an attempt to depict the distribution of a representative sample of stars of types \( F \) to \( M \) (such as shown by the dots in figure 2, for example) as a combination of ellipsoidal distributions with different origins. The sample of observed velocities used was somewhat more extended than that shown by the dots in figure 2, especially so for the highest velocities. Both the distribution of high velocities in the direction of the \( \psi \) axis and the distribution of high velocities over the different galactic longitudes are tolerably well represented if the relative numbers in the three groups are taken as indicated in the second column. (The

\(^*\)) Lund Meddel. No. 108, 1925.
\(^***\)) Aph. J. 64, 75, 1926.
\(^****\)) Lick Publ. 16, Introd. p. XXXVIII, 1928.
\(^*****\)) B. A. N. No. 48, p. 50, 1924.
total number used for the estimates concerning group III was however about three times larger than indicated).

No data about the distribution of velocities in the direction of the galactic pole have been given in the table. The proportions $k/l$ found are roughly: group I 0.46, group II 0.32 and group III 0.40.

It is certainly doubtful whether a representation by a sum of ellipsoidal distributions is the most simple and plausible way to describe the known data about the distribution of high velocities. It was only made with the object of obtaining an estimate of the probable distribution in space of these stars, to be compared with the other data of table 2, and I hope the representation may be sufficient for this purpose. It is far from me to want to suggest that we have to do with four more or less separate systems; there is one system with a somewhat complicated velocity distribution; the dissection is made purely for the convenience of computation.

Strömbärg has made a dissection of the common type stars by grouping them according to different values of $H = m + 5 \log \mu$, and then determining the group motion and velocity distribution from the radial velocities. *) Though the separate velocity distributions obtained in this way are probably not quite symmetrical I believe that this would not affect his average results to a considerable extent. In table 3 I have collected Strömbärg's results for those groups of spectra $M_0 - M_9$ for which the average value of $H$ exceeds $2.0$. The differences between his axes and the ones used above being small we may practically identify Strömbärg's $\gamma$ with our $\Theta_0 - \Theta_m$, his $a$ with our $a$, whereas $k/l = b/a$ in Strömbärg's notation.

(a) These data were derived by the author from a direct solution of the $\Theta$ component of the systematic motion with respect to the sun. Radial velocities reduced to absorption lines were used as published by Merrill and by miss Allen. The groups were chosen so as to get as much difference in average velocity as possible. Though I have not succeeded by this division to get groups with symmetrical velocity distribution I am pretty sure that an additional splitting up into perfectly symmetrical groups would not greatly alter the average results. The mean errors of the three values of $\Theta_0 - \Theta_m$ are respectively $\pm 32$, $\pm 12$ and $\pm 6$ km/sec; in the same order the mean errors of $a$ are $\pm 15$, $\pm 5$ and $\pm 2$ km/sec.

(f) From Strömbärg's results. **) (g) In the derivation of the result for $\Theta_0 - \Theta_m$ I omitted 11 nebulae which were more than 30° from the galactic circle and in order to avoid the influence of effects of the differential galactic rotation I also omitted all nebulae whose galactic longitudes differed more than 45° from the longitudes of the $\Theta$ axis (55° and 235° respectively). The loss of weight resulting from these limitations is not serious. Effects of differential rotation and of a possible $K$ term on the remaining areas were eliminated by combining opposite areas with equal weights before making the solution. The resulting value for $\Theta_0 - \Theta_m$ has a mean error of $\pm 4.6$ km/sec.

(h) From a computation by Strömbärg. *)

As they stand the data about the group motions are not yet suitable for estimates of $\delta \log \nu/\delta \sigma$. We must first transform the group motions relative to the sun into motions relative to the circular velocity $\Theta$.

In order to form an estimate of the difference between the circular velocity and the velocity of the sun we naturally look for stars with small peculiar motions, such as $B$ type stars or stars with the $c$ characteristics, because the group motion is then likely to agree closely with the circular velocity. For instance, if we take 0.003 as an upper limit of $|\delta \log \nu/\delta \sigma|$ for the $B$ stars (corresponding with the assumption that the average density does not increase by a factor larger than 2 as we proceed 100 parsecs in the direction of the centre) we find from formula (32) that $\Theta_0$ should differ less than 3 km/sec from the circular velocity.

A probably more trustworthy limit of about the same size can be set for the $c$ stars. The systematic motions of both kinds of stars are, however, subject to considerable uncertainty on account of limited data, and especially for the $B$ stars, on account of their tendency to be aggregated in large clusters.

It is rather improbable that the group velocity of the slow moving bright stars in the third line of table 2 would be algebraically larger than the circular velocity, as this would mean a thinning out of these stars in the direction of the centre. It therefore seemed better to adopt a value of $-16$ km/sec for the difference between the circular velocity and the velocity of the sun and to consider the results of the first two lines as accidentally too low. A further corroboration is obtained by referring to the determinations of the sun's motion with respect to stars of small proper motions **) or to stars of small transverse velocities ***) both from radial velocities. They yield results in close agreement with those derived from the bright stars in general.

However, for the present purpose differences such

*) ApJI, 61, 357, 1925; Mt Wilson Contr. No. 293.
Footnote 2, p. 282.

**) HERTZSPRUNG, A. N. 208, 183, 1918.

as discussed above are hardly significant. I have adopted \( \theta - \theta_{\text{sun}} = -16 \) km/sec and therewith formed the differences \( \theta_i - \theta \), in the sixth column of table 2 and the eighth column of table 3. In the next column are the estimates of \( \Delta \log v \sigma \) derived from formulae (31) or (32). The unit of \( \sigma \) is one parsec. I have adopted \( \sigma = 6300 \) parsecs; this constant has hardly any influence except on the result for the third group of high velocity stars.

The average of \( \Delta \log v \sigma \) as computed from table 2 is \(-0.00044\); table 3 gives \(-0.00019\). In order to explain the observed systematic motions of the objects of high velocity our theory would thus require an average increase by a factor of about 2 of the density in space of these stars when we proceed 1000 parsecs in the direction of the centre. We are still far from being able to confirm or reject this density gradient from evidence presented by direct observations. It is true that for planetary nebulae an increase of the density in the sky towards the general direction of the centre has been established and that the same has been indicated for the \( Md \) variables \( *) \), but a numerical estimate cannot be made. \( ** \) It may reasonably be hoped that in a not too distant future systematic surveys for long and short period variables may bring some evidence on this point.

The fact that the better determined gradients are all of about the same order of magnitude seems to point to some common cause controlling this density gradient. One partial cause might be sought in the restriction that the velocities may not exceed the velocity of escape.

In table 2 as well as in table 3 the gradient for the common type stars seems to come out smaller than the gradients computed for the long period variables, the RR Lyrae variables and the nebulae. Were they exactly the same, formula (31) would for small values of \( \theta_i - \theta \) take the approximate form:

\[
\theta_i - \theta_{\text{sun}} = c_1 a^2 + c_2,
\]

where \( c_1 \) and \( c_2 \) are constants. For stars where the axes of the velocity ellipsoid are equal this is the parabolic relation advocated by Strömbärg. \( ***) \)

Appendix.

The construction of figure 2.

The completeness of the dots in figure 2 was ascertained in the following manner: From the Yale parallax catalogue I have chosen the stars satisfying the following conditions:

\( *) \) Shapley, Harvard Circular No. 245, 1923.

\( ** \) The slight evidence presented by the brighter \( Md \) stars would indicate a rather smaller gradient than required; the matter must be left to a future investigation for decision.

\( ***) \) Compare, for instance: Ap. J. 61, 379; Mt Wilson Contr. No. 293.

1. they must be brighter than 9m.5.
2. the parallax must be larger than 0.050 and have a probable error not exceeding \( \pm 0.010 \) if the parallax is between 0.050 and 0.100, and not exceeding \( \pm 0.020 \) if the parallax is larger than 0.100.

These stars were separated in two groups:

(a) those with transverse velocities with respect to the sun larger than 19 km/sec \( \);

(b) those with transverse velocities below this limit.

To each of the stars of group (a) a completeness factor was attached to counterbalance the effect of selection of large proper motions. \( ** \) For these stars a plot was made of the projections of the space velocities on the galactic plane.

If in this plot we consider a certain annular zone around the point representing the velocity of the sun as centre, the number of velocities plotted in this zone will be too small for two reasons. Firstly because of the incompleteness of the stars of group (a) in the parallax catalogue; the amount of this incompleteness can be determined by averaging all the completeness factors attached to the velocities in this zone. Secondly the number of dots will be too small on account of the omission of all stars belonging to group (b). It is an elementary geometrical problem to compute the incompleteness accruing from this cause. This having been found we have now the total relative factors by which the number of plotted points in each zone must be multiplied in order to get a complete sample of velocities. The actual factors have been somewhat reduced by also putting in, afterwards, those stars of group (b) whose projected space velocities with respect to the sun are larger than 19 km/sec. The average factors still required are shown in table 4.

<table>
<thead>
<tr>
<th>Ring</th>
<th>Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 — 25 km/sec</td>
<td>1.65</td>
</tr>
<tr>
<td>25 — 40    &quot;</td>
<td>1.32</td>
</tr>
<tr>
<td>40 — 100 &quot;</td>
<td>1.21</td>
</tr>
<tr>
<td>( \geq 100 &quot; )</td>
<td>1.00</td>
</tr>
</tbody>
</table>

The radii of the dots in figure 2 have been made proportional to the square roots of these final completeness factors. Overlapping of different dots has been avoided so far as practicable by artificially shifting apart the velocity points which happened to fall too close to one another.

\( *) \) This limit was chosen because the most serious incompleteness of the parallax catalogue sets in at an annual proper motion of 0.200 which, at a parallax of 0.050, corresponds with a linear velocity of 19 km/sec.

\( ** \) Taken from table 9, Groningen Publ. No. 40.