CORRECTIONS OF 21-CM LINE PROFILES

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A number of methods for correcting an observed intensity distribution in one dimension for smoothing by a gaussian function (dissolving function) are tested. Each correction method may be represented by a resolving function. Eddington’s second-order correction gives a poor approximation but it is found empirically that a more satisfactory correction is obtained if a coefficient 2 or 3 times the theoretical value is used. A graph of the Fourier transforms of the resolving functions furnishes the explanation of this fact and permits the choice of coefficient, dependent on practical needs.

Observations of the 21-cm line from the Galactic System require intensity scanning in three co-ordinates: galactic longitude, latitude and frequency. The image is blurred in the first two co-ordinates by the limited angular resolving power of the telescope and in the third co-ordinate by the finite bandwidth of the receiver. If the last co-ordinate is to be translated into the distance from the sun by the method used in B.A.N. No. 452, additional distortion in this co-ordinate is caused by the deviations of the motion from perfectly circular orbits around the galactic centre.

The problem of correcting for such blurring effects has a long standing in spectroscopy, statistical astronomy, radio astronomy, and other fields 1). Nevertheless, in reducing the observations reported in the preceding paper it proved fairly hard to reach a decision on the most practical method and parameters to be used for the correction. It may be useful, therefore, to report on the various tests and considerations that were made before a reduction method was adopted. In the present paper only the correction of the distribution in frequency is considered.

1. General properties.

If \( f(x) \) represents the correct intensity of the spectral line as a function of frequency and if the blurring effect is represented by the function \( a(x) \), which we shall call the dissolving function, the observed line profile is given by the integral

\[
g(x) = \int_{-\infty}^{\infty} a(x-y) f(y) \, dy.
\]  

(1)

The problem is to solve for \( f(x) \) from the integral equation (1) if \( g(x) \) and \( a(x) \) are known functions. As is well known, any complete solution of this problem tends to increase the random inaccuracies in the observed profile, \( g(x) \), very greatly. Any practical solution consists, therefore, of employing another function \( b(x) \), which we shall call the resolving function, such that \( h(x) \), defined by

\[
h(x) = \int_{-\infty}^{\infty} b(x-y) g(y) \, dy,
\]  

(2)

approximates, but is not identical to the undistorted function \( f(x) \). It is also known that the operations (1) and (2) have the associative and commutative properties so that, when they are written for brevity as multiplications, we have:

\[
h = bg = b(a f) = (ba) f = cf.
\]  

(3)

While throughout our discussion the line profiles \( f(x) \), \( g(x) \) and \( h(x) \) will be continuous functions of \( x \), it is convenient to extend the definitions in such a manner that \( a(x) \), \( b(x) \), and \( c(x) \) can also be discontinuous functions (consisting of infinitely narrow peaks) or differential operators.

An example of a discontinuous function is Dirac’s delta function, which is the unit operator, \( 1 \). If \( ba = 1 \) in eq. 3, the resolving function gives the precise solution of the problem and \( b \) and \( a \) may be called inverse operators. Two examples of special interest may be cited.

a. If the kernel of the integral equation (1) has a gaussian form,

\[
a(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2\sigma^2},
\]  

(4)

the solution replacing eq. 2 is

\[
f(x) = g(x) - \frac{\sigma^2}{2} \frac{d^2 g}{dx^2} + \frac{1}{2!} \left( \frac{\sigma^2}{2} \right)^2 \frac{d^4 g}{dx^4} - \frac{1}{3!} \left( \frac{\sigma^2}{2} \right)^3 \frac{d^6 g}{dx^6} + \ldots
\]  

(5)

b. If the kernel has the form

\[
a(x) = \frac{1}{2\eta} e^{-|x|/\eta},
\]  

(6)

the exact solution is

\[
f(x) = g(x) - \eta^2 \frac{d^2 g(x)}{dx^2}.
\]  

(7)

In either case the resolving function is a differential operator. The form (6) assumed in the earlier reductions of the 21-cm line profiles 2) happens to be the only form that can be exactly resolved by using the second derivative only.


2) A. S. EDDINGTON, M.N. 100, 354, 1940.

2. Tests of practical resolving functions.

A resolving function \( b(x) \) may be subjected to several tests of its effectiveness:

A. To apply it to the observed profile by eq. 2 and see if the “sharpened” profile looks like the true profile \( f(x) \).

B. To apply it to the kernel \( a(x) \) and see if the resulting function \( c(x) \) looks sufficiently like a delta function.

C. To find the exact inverse function of this operator and see if this function resembles sufficiently the dissolving function \( a(x) \). This method was the more useful in the present case as we were not at all certain of the precise form of \( a(x) \).

The following tests are made.

a. Eddington's approximation. In correcting for a gaussian dissolving function, we may call "Eddington's approximation of the \( n \)-th order" the resolving method that results by breaking off series (5) after the term with the \( n \)-th differential quotient. Figure 1 shows the resulting functions \( ba \) if these approximations are applied to the gaussian profile itself (test \( B \)). It appears that increasingly deep valleys are introduced. They form a practical difficulty as similar valleys will occur in the resolved profile \( b \), when this method is used and the original profile \( f \) has sharp details. We also tested a case in which the three functions \( a, f \) and \( g \) (in eq. 1) were gaussian with dispersions in the ratios \( 3 : 4 : 5 \), respectively. Again, by applying Eddington's approximation to the function \( g \) (test \( A \)), spurious minima occurred and the convergence to the true profile \( f \) was slow. It appeared that little could be gained by going beyond the first approximation. However, somewhat to our surprise we noticed that a better approximation was obtained when the coefficient of the first correction term was doubled. This fact was confirmed in a sample reduction of the observed hydrogen line profile at \( \lambda = 190^{\circ}, b = 0^{\circ} \).

b. The \( \varepsilon-\delta \) method. When working with experimental data we have to replace differentials by differences. Thus it is strictly impossible to use differential operators. The only practical choice is a resolving function \( b(x) \) consisting of discrete peaks spaced by multiples of the tabular interval (20 kc/sec) used in the observations. We considered the use of a resolving function of the following type

\[
\text{peak height } +\varepsilon + \delta -2\varepsilon - \delta - \varepsilon + \delta \\
\text{at abscissa } -n\Delta - \Delta - 0 + \Delta + n\Delta \tag{8}
\]

in which the parameters \( \varepsilon, \delta, \Delta \) and \( n \) may still be chosen. In the special case of \( \delta = 0 \) there are only three peaks with heights

\[
- \varepsilon + \varepsilon + 2\varepsilon - \varepsilon \tag{9}
\]

The tests were made with a gaussian dissolving function (4) with \( \sigma = 30 \) kc/sec and an interval \( \Delta = 20 \) kc/sec. The first term of Eddington's approximation, translated into a second difference, gives a resolving function of type (9) with \( \varepsilon = \sigma^2/2\Delta^2 = 1.125 \). Tests made on an assumed observed profile \( g \) with \( \sigma = 50 \) kc/s showed again that \( \varepsilon \) should be taken higher for better results. The result that appeared to come closest to the true profile (with \( \sigma = 40 \) kc/s) was obtained with a function of type (8) with

\[
\varepsilon = 2.0, n = 3, \text{ and } \delta = 0.1. \tag{10}
\]

As a correction method of type (9) with small intervals is virtually identical with the solution (7) for a kernel of the form (6), we were curious to see if the addition of the terms with \( \delta \) made the method more suitable to correct for a kernel of gaussian form. This was checked by computing the inverse operators (test C). The inverse function of (9) has equidistant peaks with heights

\[
a_p = \frac{1}{\sqrt{1 + 4\varepsilon}} \left( \frac{1 + 2\varepsilon - \sqrt{1 + 4\varepsilon}}{2\varepsilon} \right) |p| \tag{11}
\]

for \( x = p\Delta \). Except for the fact that it consists of
the simultaneous correction with the combined coefficient. This may be explained as follows. Let two resolving functions of type (9) have parameters $\varepsilon_1$ and $\varepsilon_2$. Their successive application is identical to the use of the resolving function obtained by applying one to the other. This function has the form (8) with

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + 4\varepsilon_1\varepsilon_2, \quad n = 2, \quad \delta = \varepsilon_1\varepsilon_2. \quad (12)$$

The effect is identical to that of an operator of type (9) with $\varepsilon = \varepsilon_1 + \varepsilon_2$, if higher derivatives than the third are negligible in the observed function $g(x)$; this corresponds to addition of the values of $\sigma^2$ in the second-order Eddington approximation. For profiles with pronounced ups and downs the difference with this simple addition is great. For instance, with $\varepsilon_1 = 2.0$ and $\varepsilon_2 = 0.3$ (corresponding approximately to the assumed cloud velocities and to the bandwidth used if the unit is 20 kc/sec), the peaks are by eq. 12:

$$+0.6 -4.7 +9.2 -4.7 +0.6$$

Barring the complication of the conversion of intensities into optical depth, these are the actual values used in the reduction of the line profiles, as described in detail in the following paper. Random errors in the tabulated values are multiplied by 11.4 with this reduction method. If, instead, eq. 9 with the combined value $\varepsilon = 2.3$ were used, the corresponding coefficients would be

$$0 -2.3 +5.6 -2.3 0$$

and random errors would be multiplied by 6.8.

3. Fourier analysis.

Any integral equation of the type (1) or (2) may be transformed into a multiplication by taking the Fourier transforms of each of the functions involved. Let

$$A(\varphi) = \int_{-\infty}^{\infty} a(x) e^{i\varphi x} dx \quad (13)$$

and let $F(\varphi), G(\varphi), B(\varphi), H(\varphi)$, be similarly derived from $f(x), g(x), b(x), h(x)$. Equations 1 and 2 then reduce to

$$G(\varphi) = A(\varphi) F(\varphi) \quad (14)$$
$$H(\varphi) = B(\varphi) G(\varphi) \quad (15)$$

and $a(x)$ and $b(x)$ are inverse operators if

$$A(\varphi) B(\varphi) = 1. \quad (16)$$

The same equations hold if one or several functions are changed into collections of discrete peaks. A function of equidistant peaks gives

$$A(\varphi) = \sum_{n=-\infty}^{\infty} a_n e^{i\Delta \varphi}, \quad (17)$$

where $a_n$ is the height (in the transition from a continuous function it equals the area) of the peak.
at abscissa \( x = n\Delta \). Fourier transforms of the continuous and discrete functions here considered all are continuous and may be displayed in the same graph. The only difference is that the latter are periodic functions of \( \varphi \), with period \( 2\pi/\Delta \).

A convenient comparison of the various methods may be made by plotting the Fourier transforms \( B(\varphi) \) of the resolving functions. The useful region in these graphs is limited: (a) At the top side, for very large values of \( B(\varphi) \) cannot be tolerated as they would amplify a spurious wiggle in the observed result too strongly. (b) At the right side, for high values of \( \varphi \) correspond to very sharp details in the initial profile, which cannot be resolved with practical certainty if the width of the details is too far below the width of the dissolving function.

These limitations are related as, in general, the problem mentioned in (a) does not occur, unless the frequency limit in (b) is pushed too far. In the practical examples (Figures 3 and 4) we have put the limits at \( B = 8 \) and \( \varphi = 1.6 \times (20 \text{ kc/sec})^{-1} \). This value corresponds to a period of

\[
\frac{2\pi}{1.6} \times 20 \text{ kc/sec} = 79 \text{ kc/sec}
\]

in the wiggles, or to a peak-to-valley distance of 40 kc/sec.

Figure 3 shows a comparison of a number of resolving methods that are identical in the quadratic approximation for small \( \varphi \), namely

\[
B(\varphi) = 1 + 1.125 \varphi^2 + \ldots
\]

This means that all methods would give the same results when applied to observed intensity curves with very gradual changes. The following curves are shown.

1. The exact inverse function for a gaussian dissolving function, eqs. 4 and 5, with \( \sigma = 1.5 \). Here

\[
A(\varphi) = e^{-\varphi^2/\sigma^2}, \quad B(\varphi) = e^{\varphi^2/\sigma^2}.
\]

2. The exact inverse function for an exponential dissolving function, eqs. 6 and 7, with \( \eta = 1.06 \). Here

\[
A(\varphi) = \frac{1}{1 + \eta^2 \varphi^2}, \quad B(\varphi) = 1 + \eta^2 \varphi^2.
\]

As eq. 7 is the second-order approximation of eq. 5, we may say that the quadratic approximation of \( B(\varphi) \) for small \( \varphi \) defines the EDDINGTON approximation to the precise resolving function.

3. The fourth-order approximation

\[
B(\varphi) = 1 + \frac{\sigma^2}{2} \varphi^2 + \frac{1}{2} \left( \frac{\sigma^2}{2} \right)^2 \varphi^4
\]

is the Fourier transform of the resolving function defined by breaking off eq. 5 after the third term.

4. The curve corresponding to the method of subtracting \( \varepsilon \) times the second differences, i.e., a resolving function of type (9), is given by

\[
B(\varphi) = 1 + 2\varepsilon (1 - \cos \varphi \Delta).
\]

This function changes into the one sub 6, if \( \varepsilon \to \infty \), \( \Delta \to 0 \), such that \( \varepsilon \Delta^2 = \eta^2 \). Curve d, drawn for \( \varepsilon = 1.125, \Delta = 1 \) shows that the interval \( 1 = 20 \text{ kc/sec} \) is small enough to make a good approximation to the method of second-order differentials (curve b).

5. The same equation with \( \varepsilon = 0.125, \Delta = 3 \). This falls very far from curve b in spite of the correct product \( \varepsilon \Delta^2 \).

6. The curve that is obtained if the correction \( B(\varphi) - 1 \) in d is increased by a factor 2 and the correction in \( \varepsilon \) is subtracted. So

\[
B(\varphi) = 1 + 4.5 (1 - \cos \varphi) - 0.25 (1 - \cos 3\varphi).
\]

The corresponding function \( b(x) \) is of the type (8) with \( \varepsilon = 2.25, n = 3, \delta = 0.125 \) and fairly well resembles the function with which some of the tests in the preceding section were made. The curve shows indeed that a correction of this type resembles the precise correction for a gaussian profile (curve a) much better than the EDDINGTON approximation (curve b).
Figure 4 shows in a similar manner the curves corresponding to a number of resolving functions that have been considered for practical use in the reduction of the 21-cm profiles. The unit of $\varphi$ is $\Delta^{-1}$, where $\Delta = 20$ kc/sec is the interval adopted for computing first and second differences from the tabulated values of the optical depth.

Curve $a$ is the same as curve $a$ from Figure 3, and represents the exact correction for a gaussian distribution of cloud velocities with $\sigma = 30$ kc/s = 6.3 km/sec. Curve $a'$ is obtained by quadratically adding the effect of the bandwidth with $\sigma = 16$ kc/s, so that the combined value is $\sigma' = 34$ kc/s.

Subtraction of $\varepsilon$ times the second difference corresponds to the curves $B(\varphi)$ shown for $\varepsilon = 0, 1, 2$ and 3 in Figure 4. The curve with $\varepsilon = 1.125$ (curve $b$ of Figure 3) would give the Eddington approximation, i.e., the correct amplification factor for very long waves. The shorter waves are more important, however. For instance, the observed line profile for $l = 190^\circ, b = 0^\circ$ roughly resembles a section of a sinusoidal wave with the period (= distance between peaks) of $100$ kc/sec. This means that we have to enter Figure 4 at the abscissa

$$\varphi = 2\pi \times \frac{20}{100} = 1.26.$$ 

We then find that the value $\varepsilon = 3$ would give the correct amplification factor ($\sim 5$ to 6) for the amplitudes of waves with this period. This explains why it is often practical to use the Eddington approximation with a coefficient that is several times larger than follows from Eddington's formula.

Of course, a completely correct reduction will not be obtained unless the given profile is a pure sine wave and even in that case the coefficient $\varepsilon$ should be adapted to the period of the wave in each case. The value of $\varepsilon$ to be adopted for general use has to be a compromise value.

**Figure 4**

Fourier transforms of various resolving functions.