A METHOD FOR COMPUTING ACCURATELY THE EPOCH OF MINIMUM OF AN ECLIPSING VARIABLE

by K. K. Kwee and H. van Woerden

A method is given for the accurate computation of the minimum epoch for eclipsing variables, and of its mean error. The method is presented in section 2 and analysed mathematically in section 3. The importance of using well-defined methods is stressed.

1. Introduction.

The derivation of times of minimum for eclipsing variables is a problem which ought to have more attention from variable star observers. In papers communicating results of photometric investigations of eclipsing binaries, very little is usually said about the methods used for the derivation of the minimum epochs. However, this is a matter of not only methodical interest. The time of minimum observed brightness being unacceptable as a definition of the epoch, already on account of observational errors, one will use both the descending and ascending branches of the minimum for the determination. But with asymmetrical minima – and slight asymmetry is to be considered the most usual case – different methods of using the observations on the branches will lead to different results, and the epoch of minimum is only exactly defined by the way of determining it. These differences will often be not negligible, since present observational techniques enable us to get series of many accurate observations. Combination of epochs derived by different methods may lead to spurious discrepancies and erroneous conclusions, and should therefore be avoided.

Although it is realized that atmospheric and instrumental troubles will often reduce the effect of exact definitions, the authors feel that full profit of the accuracy obtainable with photoelectric equipment will only be drawn if unnecessary inexactness is avoided. With this purpose in mind, a method to compute accurately the epoch of a minimum observed photoelectrically and its mean error is proposed in this article.

Although the basic ideas of the method have already been introduced by Hertzsprung in 1928, and later on worked out by Oosterhoff 3), Hertzsprung 4) and de Kort 5), a new presentation of the method is given here. Firstly, because these earlier discussions have been rather hidden in papers giving results about a great number of variable stars, and so probably did not receive the attention they deserved. Secondly, because in this paper an extension of the method is given to cases where the equations of condition are not independent of each other.

2. The method.

The observations of only one minimum should be used to determine its epoch. Fitting the observations to a “normal light-curve” determined previously, and deriving the epoch from a “normal minimum” and the time interval found by this fitting process, is dangerous as soon as there are only slight variations of the light-curve.

Before starting the computations, one must first decide about the total phase interval to be used. Preferably all phases of the eclipse should be included in the determination, but if, for any reason, this is impossible, care should be taken that on both branches corresponding phase intervals are used. In the case of asymmetric minima, it is of great importance that for the same star always the same phase interval is used, for the epoch is only exactly defined by this interval.

Let $N$ be the total number of observations in the

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phase interval. Form \((2n + 1)\) magnitudes spaced by equal time intervals \(\Delta t\), by linear interpolation between consecutive observations. To prevent a too unequal use of weights of the observed magnitudes, it is recommended to take \((2n + 1)\) about equal to \(N\). One of the equidistant times, \(T_1\) (say), should represent a preliminary time of minimum. A simple estimate made from a plot of the observations will usually suffice to determine \(T_1\). The computer will find it convenient to choose the equidistant times at rounded-off fractions of minutes or of Julian Days.

Take the time \(T_1\) as reflection axis and reflect the interpolated magnitudes of one branch upon the other, giving two magnitudes for every equidistant time on the latter branch. Take the differences of these magnitude pairs \(\Delta m_k\) \((k = 1, \ldots n)\), and compute the sum of their squares \(s(T_1) \equiv \sum_{k=1}^{n} (\Delta m_k)^2\). Then shift the symmetry axis to \((T_1 + \frac{1}{2} \Delta t)\) and \((T_1 - \frac{1}{2} \Delta t)\) successively, and proceed as above to compute the sums \(s(T_1 + \frac{1}{2} \Delta t)\) and \(s(T_1 - \frac{1}{2} \Delta t)\).

If \(T_1\) was properly chosen, \(s(T_1)\) will be smaller than both other sums. If this is not the case, e.g. if \(s(T_1 + \frac{1}{2} \Delta t) < s(T_1)\), the subsequent sum \(s(T_1 + \Delta t)\) must be computed; care should be taken that \(n\) is the same in all the summations.

The function \(s(T)\) is represented by a quadratic formula:

\[
s(T) = a T^2 + b T + c.
\]  

(1)

The constants \(a\), \(b\) and \(c\) can be computed using the three \(s\)-values derived above. The parabola represented by \(s(T)\) has a minimum value

\[
s(T_o) = c - \frac{b^2}{4a},
\]  

(2)

at

\[
T_o = -\frac{b}{2a};
\]  

(3)

\(T_o\) is the time of minimum sought.

The mean error of the epoch is given by:

\[
\sigma_{e o}^2 = \frac{4a c - b^2}{4a^2 (Z - 1)},
\]  

(4)

\(Z\) is the maximum number of independent magnitude pairs. In the case of linear interpolation recommended above, \(Z = \frac{1}{2} N\) (section 3). If the observed magnitudes were already equidistant in time, making interpolation unnecessary, \(Z = \frac{1}{2} N\).

3. Analysis of the method.

In this section we will give an analytical treatment of the method described in section 2.

Let \(l(t)\) be the true magnitude as a function of time, freed from observational errors. We assume an axis of symmetry at \(t = T_o\), and define \(T_o\) to be the minimum epoch. If \(u\) is a certain interval of time, we have:

\[
l(T_o + u) = l(T_o - u).
\]  

(5)

Suppose we have \(2Z\) measured magnitudes at constant time intervals \(\Delta t\), independent of each other; denote them by \(m(T \pm k\Delta t)\), \(k = 1, \ldots Z\), then we have the relation:

\[
l(T + k\Delta t) = m(T + k\Delta t) - \mu_+ k,
\]  

(6)

where \(\mu_+ k\) is the deviation in magnitude at the time \((T + k\Delta t)\), arising from observational errors. We now choose \(T\) so that \(T - T_o = \tau\) is a small quantity not larger than \(\Delta t\). Then by (5) and (6) we may write:

\[
\Delta m_k = m(T + k\Delta t) - m(T - k\Delta t) = \mu_+ k - \mu_- k + l(T_o + \tau + k\Delta t) - l(T_o - \tau + k\Delta t),
\]  

and expanding \(l(T_o + k\Delta t \pm \tau)\) in Taylor series, we get:

\[
\Delta m_k = (\mu_+ k - \mu_- k) + 2 \tau (\frac{\partial l}{\partial T})_T_o + k\Delta t + \frac{1}{2} \tau^2 (\frac{\partial^2 l}{\partial T^2})_T_o + k\Delta t + \ldots .
\]  

(7)

For \(T = T_o\) or \(\tau = 0\), \(\Delta m_k\) should vanish, except for observational errors, and we get \(Z\) equations of condition:

\[
\Delta m_k = m(T + k\Delta t) - m(T - k\Delta t) = 0,
\]  

(8)

from which \(T_o\) can be solved so as to be the value of \(T\), satisfying these equations as well as possible. This solution can be easily made by the method of least squares. Its principle states that \(S(T) \equiv \sum_{k=1}^{Z} (\Delta m_k)^2\) should be a minimum. With (7) we have, disregarding terms of higher order in \(\tau\) than the second:

\[
S(T) = \sum_{k=1}^{Z} (\mu_+ k - \mu_- k)^2 + 4 \tau \sum_{k=1}^{Z} \left[ (\mu_+ k - \mu_- k) \times (\frac{\partial l}{\partial T})_T_o + k\Delta t \right] + \frac{1}{2} \tau^2 \sum_{k=1}^{Z} (\frac{\partial^2 l}{\partial T^2})_T_o + k\Delta t.\]  

(9)

As the \(\mu_\pm k\)'s only depend on accidental errors of observation, the second term on the right-hand side in (9) will, for not too small values of \(Z\), be small compared with the sum of the other two terms, and for the first approximation this term may therefore be neglected. So, setting \(\tau = T - T_o\), (9) transforms into:

\[
S(T) = \sum_{k=1}^{Z} (\mu_+ k - \mu_- k)^2 + 4 T_o \sum_{k=1}^{Z} \left(\frac{\partial l}{\partial T}\right)_T_o + k\Delta t - 8 T_o \sum_{k=1}^{Z} (\frac{\partial l}{\partial T})_T_o + k\Delta t + 4 T_o^2 \sum_{k=1}^{Z} (\frac{\partial^2 l}{\partial T^2})_T_o + k\Delta t,\]  

(10)

which we may also write as a quadratic function of \(T\), with coefficients \(A\), \(B\) and \(C\), standing for:
\[
A = + \frac{Z}{4} \sum_{k=1}^{n} \left( \frac{\delta l}{\delta t} \right)^{2}_{T_{0} + k \Delta t}, \\
B = -8 T_{o} \sum_{k=1}^{n} \left( \frac{\delta l}{\delta t} \right)^{2}_{T_{0} + k \Delta t}, \\
C = \sum_{k=1}^{n} (\mu_{+k} - \mu_{-k})^2 + 4 T_{o}^2 \sum_{k=1}^{n} \left( \frac{\delta l}{\delta t} \right)^{2}_{T_{0} + k \Delta t}.
\]

(11)

The condition for minimum of \(S(T)\) leads to the equation (differentiating formula (10)):

\[
A T_{o} + \frac{1}{2} B = 0,
\]

(12)

from which \(T_{o}\) can be solved. The minimum value of \(S(T)\) itself may be derived by substituting \(T = T_{o}\) in (10), resulting in:

\[
S(T_{o}) = \frac{Z}{2} \sum_{k=1}^{n} (\mu_{+k} - \mu_{-k})^2.
\]

(13)

The mean error of \(T_{o}\), \(\sigma_{T_{o}}\), can be derived as follows: we note that our procedure is equal to a least-squares solution with one unknown, formulae (8) serving as equations of condition, and expression (12) as normal equation. The weight of the determination of \(T_{o}\) will then be \(A\), and the mean error of a determination with unit weight \(\sigma = \sqrt{\frac{S(T_{o})}{Z - 1}}\). So we have:

\[
\sigma_{T_{o}}^{2} = \frac{S(T_{o})}{A(Z - 1)},
\]

or with (11) and (13),

\[
\sigma_{T_{o}}^{2} = \frac{Z}{4} \sum_{k=1}^{n} \left( \frac{\delta l}{\delta t} \right)^{2}_{T_{0} + k \Delta t} + \frac{1}{Z - 1},
\]

(14)

or

\[
\sigma_{T_{o}}^{2} = \frac{4AC - B^2}{4A^2} \times \frac{1}{Z - 1},
\]

In deriving the formulae above, we have assumed that \(Z\) is the number of independent equations of condition (8). Only in this case formula (14) holds. We see, however, that the first factor on the right-hand side of (14) is independent of \(Z\), except for statistical fluctuations and provided that the summation take place over the same interval on the branches. This means that if we had made \(2n\) equidistant magnitudes dependent of each other, e.g. by interpolation, out of our \(N\) observations – which may be either equidistant or not – and if we define, similar to formulae (11):

\[
a = + \frac{Z}{4} \sum_{k=1}^{n} \left( \frac{\delta l}{\delta t} \right)^{2}_{T_{0} + k \Delta t}, \\
b = -8 T_{o} \sum_{k=1}^{n} \left( \frac{\delta l}{\delta t} \right)^{2}_{T_{0} + k \Delta t}, \\
c = \sum_{k=1}^{n} (\mu_{+k} - \mu_{-k})^2 + 4 T_{o}^2 \sum_{k=1}^{n} \left( \frac{\delta l}{\delta t} \right)^{2}_{T_{0} + k \Delta t},
\]

(15)

we have, except for statistical fluctuations:

\[
\frac{\sum_{k=1}^{n} (\mu_{+k} - \mu_{-k})^2}{Z} \sum_{k=1}^{n} \left( \frac{\delta l}{\delta t} \right)^{2}_{T_{0} + k \Delta t},
\]

or

\[
\frac{Z}{4} \sum_{k=1}^{n} \left( \frac{\delta l}{\delta t} \right)^{2}_{T_{0} + k \Delta t},
\]

and we may write for (12) and (14) respectively:

\[
T_{o} = -\frac{B}{2A} = -\frac{b}{2a},
\]

\[
\sigma_{T_{o}}^{2} = \frac{4AC - B^2}{4A^2} \times \frac{1}{Z - 1} = \frac{4ac - b^2}{4a^2} \times \frac{1}{Z - 1},
\]

giving (3) and (4) respectively, and in which \(Z\) remains the number of independent equations of condition (8) which can be formed out of the \(N\) observed magnitudes.

In principle it is useful to make \(n\) large, since then the statistical fluctuations of \(\frac{4ac - b^2}{4a^2}\) will be smaller.

On the other hand, a large \(n\) will make the computations long and tiresome. The authors have found a compromise by taking \(n \approx 2Z\).

When the original observations are not equidistant in time, as is generally the case, interpolation is necessary for making equidistant magnitudes. Any interpolation, however, is then coupled by a loss of total weight. When the interpolation is made linearly, as the authors have proposed in section 2, this loss of total weight can be computed statistically, and it is found that the total weight is reduced by a factor \(\frac{1}{4}\).

Although other methods of interpolation might conserve more of the total weight, the simplicity of linear interpolation is considered to be a decisive advantage.

4. Discussion.

The methods have been analysed in section 3 on the assumption that the true light-curve is symmetric. Analytical treatment of the case of asymmetry is difficult. There is, however, no objection at all against application of the method described in section 2 to asymmetric minima. In that case, \(S(T_{o})\) will consist of observational errors and of asymmetry terms. Consequently, \(S(T_{o})\) will be larger than in the case of symmetry, and the mean error of the resulting epoch higher; both increase with the amount of asymmetry. Moreover, the resulting \(T_{o}\) will depend on the total phase interval on the branches used; it is therefore necessary always to use the same phase interval for one star. In case of serious asymmetry, the computed epoch may be sensibly shifted from the