COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

Numerical values of the damping constant for radial oscillations of stars,
by Miss H. A. Kluyver.

§ 1. The time of decay for radial pulsations has been computed by various authors, e.g. Eddington\(^1\),
Edgar\(^2\) and Steensholt\(^3\); the results found by
Eddington and Edgar correspond to a time of decay
of the order of \(10^6\) times the period of oscillation,
whereas Steensholt finds the first mode of vibration
to be unstable.

In the present work special attention has been paid
to the outer region of the star, because there the
dissipation of energy is much stronger than in the interior,
especially for the higher modes of vibration.
Due account has been taken of the fact that near the
outer boundary the departure from purely adiabatic
conditions becomes appreciable, so that the adiabatic
approximation can no longer be used; this circumstance
tends to make the damping constant smaller.

First that part of the damping constant has been
computed which has no relation to the generation
of energy. For a star with a period of about \(5.4\) days
and \(3 - 4/\gamma = 0.4\) (\(\gamma\) is the ratio of specific heats
for matter and radiation jointly) the time of decay
resulting from the present work is about \(380\) years for
the first mode of vibration, and about \(1\) year for
the third mode, whereas Edgar found \(1660\) years for
the first mode. Moreover, the dissipation proves to
decay rapidly with decreasing value of \(\gamma\), as opposed
to the results reached by Edgar.

In order to investigate the effect of the generation of
energy on the dissipation the energy has been assumed
to be supplied by protons being captured by nuclei.
The results can only be rough, because this assumption
is not in accordance with the star model used,
viz. the polytropic model of index \(3\). The term thus
added to the damping constant decreases it; for small
values of \(\gamma\) the sign may even be reversed, at least for
the first mode of vibration.

Dr Wolter has shown\(^4\) that in a star where the
first and second modes have respectively a negative
and positive damping constant oscillations once set
up may continue indefinitely, if these two modes
have closely commensurable frequencies. Since comm-
mensurability probably occurs for a value of \(3 - 4/\gamma\)
in the neighbourhood of \(\sigma^2\) it appears possible that
both conditions are fulfilled at the same time.

§ 2. The damping constant may be found from the
well known equation\(^5\)

\[
\frac{x}{n} \int_{z}^{\infty} z u_{5} s_{3} dz = -\frac{4 - 3\bar{\beta}}{2\gamma\omega_{3}^2} \int_{0}^{z} z^{3} s_{3} d\frac{V_{\sin} u_{4}}{dz} dz;
\]

\(x\) is the damping constant, \(u\) and \(z\) are Emden
variables, \(Z\) being the star's radius; \(s\), a function of \(z\),
is a solution of the adiabatic equation for \(\partial r/\partial r\),
corresponding to the characteristic value of \(n\) (or in the
appropriate unit \(\omega\)); \(V_{\sin}\) is defined by the equations

\[
V = \frac{3T}{T} - \frac{\gamma - \beta}{4 - 3\bar{\beta}}; \quad V = e^{-\alpha t} \left( V_{\sin} \sin nt + V_{\cos} \cos nt \right);
\]

\(V\) is zero when the changes of state are strictly
adiabatic, and thus measures the departure from
adiabatic conditions. The other symbols have their
usual meaning \(^6\).

\(^1\) The internal constitution of the stars, 200. \(^2\) M.N. 93, 422, 1933. \(^3\) Oslo Publ. No. 8, 1933. \(^4\) B.A.N. No. 393, 1937.
\(^5\) Cf. Eddington, M.N. 79, 177, 1919, and The internal constitution of the stars, § 134; there the time of decay of the energy is
considered. \(^6\) Cf. Wolter, B.A.N. No. 282, 1936.
absorption coefficient having been taken proportional to $\rho / T^\alpha$.

Dr Woltjer has kindly informed me in advance of publication that the equation from which $V$ may be found by successive approximations is somewhat different from the one given in B.A.N. No. 282. Written in such a form that substitution of adiabatic values for the quantities in the right-hand member gives a correct way of approximating, the equation needed reads asymptotically

$$
-4\left(1 + \frac{1}{\beta}\right) \frac{d}{dz} \left(\frac{\beta}{\gamma} z^2 u^5 + \frac{4}{\beta} \frac{d V_\cos}{dz}\right) - 2A \left(\frac{5}{2} u'_0\right)^2 z^2 u^4 \left(\frac{dV_{\sin}}{dz}\right) =
$$

$$
= -\frac{d}{dz} \left[ z^2 \frac{du}{dz} \left(4 \frac{\partial r}{r} \right) + 7r - 4\frac{\beta}{4 - 3\beta} \left(3 \frac{\partial r}{r} \right) + z \frac{d}{dz} \left(\frac{\partial r}{r} \right) \cos \right] +
$$

$$
+ \frac{\gamma}{4 - 3\beta} u \frac{d}{dz} \left(3 \frac{\partial r}{r} \right) - z \frac{d}{dz} \left(\frac{\partial r}{r} \right) \cos \right] \right].
$$

$u'_0$ is the value of $du/dz$ for $z = Z$.

$$
A = \frac{1}{2} \left(\frac{5}{2} u'_0\right)^2 \left\{ T \frac{\partial (U_r + U_i)}{\partial T} \right\} M \frac{L}{n},
$$

where $U_r$ and $U_i$ are the energy of radiation and the thermal energy per unit mass, the index $e$ indicating central values, $(\partial r/r)_{\cos}$ is a coefficient analogous to $V_{\cos}$.

In the right-hand member the adiabatic value $s$ must be substituted for $(\partial r/r)_{\cos}$.

The adiabatic approximation for $V_{\sin}$ is found by omitting the differential operator in this equation and putting $- 2A \left(\frac{5}{2} u'_0\right)^2 z^2 u^4 V_{\sin}^{ad}$ equal to the right-hand member. This approximation is valid for the greater part of the star, but it breaks down near the outer boundary. However, it has first been used for the whole range of $z$, the outer part being dealt with afterwards.

The integrals in the right-hand member of the equation for $z/\pi$ has been transformed by partial integration into

$$
z^2 s V_{\sin} u^4 \frac{dz}{dz} - \int_0^z V_{\sin} z^2 u^4 \left(3 s + z \frac{ds}{dz} \right) dz.
$$

The quantity $z^2 s V_{\sin} u^4$ is zero in the centre and also at the outer boundary, because for $z = Z$ the variable $u$ is zero and the other factors are finite.

After substitution of the adiabatic approximation the integral has been once more partially integrated; the integrated parts disappear for $z = 0$.

The actual contributions for the outer part have been estimated from the amounts found with $V_{\sin}^{ad}$ by multiplication with a factor smaller than unity; in order to determine this factor the integrals concerned have been expanded asymptotically.

The integral to be computed is

$$
\int_{6^4}^z V_{\sin} z^2 u^4 \left(3 s + z \frac{ds}{dz} \right) dz = \int_{6^4}^z V_{\sin}^{ad} z^2 u^4 \left(3 s + z \frac{ds}{dz} \right) dz + \int_{6^4}^z \left(V_{\sin} - V_{\sin}^{ad}\right) z^2 u^4 \left(3 s + z \frac{ds}{dz} \right) dz.
$$

The lower limit has been chosen equal to $6^4$ because for smaller values of $z$ there is practically no difference between $V_{\sin}$ and $V_{\sin}^{ad}$. With the aid of the differential equation the second integral in the right-hand member may be transformed into

$$
\frac{1}{2A} \left(\frac{5}{2} u'_0\right)^2 \gamma \int_{6^4}^Z \left(3 s + z \frac{ds}{dz} \right) u\frac{d}{dz} \left(z^2 u^5 + \frac{4}{\beta} \frac{d V_\cos}{dz}\right) dz.
$$

As only the first term in the asymptotic expansion is considered the factor $3 s + z \frac{ds}{dz}$ must be kept constant; then the result can easily be expressed in the boundary values of the various quantities. For

$$
\int_{6^4}^Z V_{\sin}^{ad} z^2 u^4 \left(3 s + z \frac{ds}{dz} \right) dz
$$

the first term in the expansion has been computed too.

The resulting value for the required ratio is

$$
1 - \frac{2^{1/2} A^{1/2}}{5 u_{z = 6^4}} \frac{\Gamma'(1 + 4/5\beta)}{\Gamma(4/5 + 4/5\beta)} \sin \frac{3\pi}{5}.
$$

1) Dr Woltjer has kindly communicated to me the analytical data needed to derive the boundary value of $V_{\cos}$ used here.
where the symbol $\Gamma$ represents the gamma-function. With the assumption that the same ratio holds for the further terms in the expansion the actual contributions for the outer part have been computed from those found with $V^{\infty}$ by multiplying with the factor found above.

§ 3. Details of the computation are given in Table 1; it should be noticed that the numbers in different parts of the table correspond to different central

<table>
<thead>
<tr>
<th>$z$</th>
<th>$z^2 \frac{d u}{dz} \cdot \frac{d}{dz} (3z - \frac{d s}{dz})$</th>
<th>$\frac{d u}{dz} (3z - \frac{d s}{dz})$</th>
<th>$\frac{d u}{dz} (\frac{d}{dz} (3z - \frac{d s}{dz})^2)$</th>
</tr>
</thead>
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<tr>
<td>$1$</td>
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<td>$0.00$</td>
<td>$1.00$</td>
</tr>
<tr>
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<td>$1.10$</td>
<td>$1.01$</td>
<td>$1.12$</td>
</tr>
<tr>
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<td>$2.27$</td>
<td>$2.04$</td>
<td>$2.23$</td>
</tr>
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<td>$3.02$</td>
<td>$2.80$</td>
<td>$3.00$</td>
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<td>$3.90$</td>
</tr>
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<td>$4.78$</td>
</tr>
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<td>$5.38$</td>
<td>$5.55$</td>
</tr>
<tr>
<td>$0.7$</td>
<td>$6.52$</td>
<td>$6.04$</td>
<td>$6.20$</td>
</tr>
<tr>
<td>$0.8$</td>
<td>$7.32$</td>
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<td>$6.78$</td>
</tr>
<tr>
<td>$0.9$</td>
<td>$8.09$</td>
<td>$7.32$</td>
<td>$7.27$</td>
</tr>
<tr>
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<td>$8.83$</td>
<td>$7.83$</td>
<td>$7.64$</td>
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</tbody>
</table>

Table 1.

I: $3^{-4/7} = 0.4$, $\omega^2 = 1.0391$  
II: $3^{-4/7} = 0.4$, $\omega^2 = 3.9179$  
III: $3^{-4/7} = 0.2$, $\omega^2 = 0.58866$  
s (centre) = 1 throughout the table.

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values of the function $s$, as indicated at the foot of the table. The computations have been made for the first and third modes of vibration in the case $3-4/\gamma = 0.4$ \(^1\), and for the first mode for $3-4/\gamma = 0.2$ \(^2\). For $3-4/\gamma = 0.4$ the integrals have been computed numerically for the first mode from $0$ to $Z$, for the third mode graphically from $0$ to $5/3$ and numerically for the region outside $5/3$; for $3-4/\gamma = 0.2$ the integrals have been found graphically. In the last two lines of Table 1 the integrals for $z$ from $0$ to $6/4$ and from $6/4$ to $Z$ are given. Table 2 shows the integrated parts at $z = 6/4$ and $Z$. Table 2 and the last two lines of Table 1 correspond to normalized functions for the two cases with $3-4/\gamma = 0.4$, and to unit central value of $s$ for the case with $3-4/\gamma = 0.2$.

### Table 2.

<table>
<thead>
<tr>
<th>(z)</th>
<th>(d^2u/dz^2)</th>
<th>(-3s - z d^2s/dz^2)</th>
<th>(d^2u/dz^2)</th>
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<th>(d^2u/dz^2)</th>
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<td>(-3s - z d^2s/dz^2)</td>
<td>(d^2u/dz^2)</td>
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<td>6/4</td>
<td>+ 16'359</td>
<td>+ 181'174</td>
<td>- 72'62</td>
<td>- 105'715</td>
<td>- 38'03</td>
<td>- 35'87</td>
</tr>
<tr>
<td>6/4</td>
<td>+ 29'772</td>
<td>+ 189'38</td>
<td>+ 103'8</td>
<td>- 203'117</td>
<td>- 3'5110'69</td>
<td>- 53'90</td>
</tr>
<tr>
<td>6/4</td>
<td>+ 31'9318</td>
<td>+ 421'89</td>
<td>+ 66'95</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6/4</td>
<td>+ 30'298</td>
<td>+ 194'86</td>
<td>+ 89'08</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

I: $3-4/\gamma = 0.4$, \(s^2 = 10391\)

II: $3-4/\gamma = 0.4$, \(s^2 = 39179\)

III: $3-4/\gamma = 0.2$, \(s^2 = 0.58806\); \(s\) (centre) = 1.

For $s$ (centre) = 1 the normalizing integral has for $3-4/\gamma = 0.4$ the value $37'879$ for the first mode and $25'778$ for the third; for $3-4/\gamma = 0.2$ it is equal to 18'53 for the first mode.

The factor used for the computation of the contributions of the outer region depends on $\beta$. However, for a given value of $\gamma$, $\beta$ is restricted by the condition that the ratio of specific heats for matter only should not exceed $5/3$. For $3-4/\gamma = 0.4$ the extreme values of $\beta$ are $85913$ and $1$, for $3-4/\gamma = 0.2$ they are $51205$ and $1$. For values of $\beta$ in these intervals the range of the factor is only small; therefore it has been computed for a fixed, intermediate value of $\beta$, although at other places $\beta$ has been left undefined. $\beta$ has been taken equal to $8/9$ for $3-4/\gamma = 0.4$, and for $3-4/\gamma = 0.2$ equal to $0.8$, so that the factor is equal to respectively $1-46'736 A^{-1/5}$ and $1-47'490 A^{-1/5}$. For the four representative Cepheids given by Eddington \(^3\) $A^{1/5}$ ranges from about 160 to 40. It is to be borne in mind that the expansions used for deriving the above factor are only valid for large values of $A^{1/5}$. The values given in §1 for the times of decay have been computed with $A = 46.16^8$ and $92.16^8$ respectively.

Below, the formulae for $\alpha/n$ are given for each of the three oscillations considered; the expressions found when $\beta$ is put equal to its extreme values have been added.

For the first mode of vibration in the case $3-4/\gamma = 0.4$:

\[
\frac{\alpha}{n} = \frac{10^4}{A} \left( \frac{3.4702 + 3.9576 (1-\beta)}{1 - \frac{24.576}{A^{1/5}} \left(1 + 1.2208 (1-\beta) \right)} \right)
\]

\[
\frac{\alpha}{n} = \frac{4.0278 \times 10^4}{A} \left( \frac{24.814}{A^{1/5}} \right), \beta = 85913;
\]

\[
\frac{\alpha}{n} = \frac{3.4702 \times 10^4}{A} \left( \frac{24.576}{A^{1/5}} \right), \beta = 1.
\]

For the third mode in the case $3-4/\gamma = 0.4$:

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\(^1\) B.A.N. No. 276, 1936.  \(^2\) B.A.N. No. 268, 1935.  \(^3\) M.N. 92, 480, Table III, 1932.
\[
\frac{\alpha}{n} = \frac{10^7}{A} \left[ 1.8201 + 2.4714 (1 - \beta) \right] \left[ 1 - \frac{42.64}{A^{1/5}} \right] \left[ 1 + 1.369 (1 - \beta) \right];
\]
\[
\frac{\alpha}{n} = \frac{2.1682 \times 10^7}{A} \left[ 1 - \frac{42.64}{A^{1/5}} \right], \beta = .85913;
\]
\[
\frac{\alpha}{n} = \frac{1.8201 \times 10^7}{A} \left[ 1 - \frac{42.64}{A^{1/5}} \right], \beta = 1.
\]

For the first mode in the case \(3 - 4/\gamma = 0.2\):
\[
\frac{\alpha}{n} = \frac{10^3}{A} \left[ 5.78 + 8.96 (1 - \beta) \right] \left[ 1 - \frac{18.2}{A^{1/5}} \right] \left[ 1 + 1.73 (1 - \beta) \right];
\]
\[
\frac{\alpha}{n} = \frac{1.015 \times 10^4}{A} \left[ 1 - \frac{19.1}{A^{1/5}} \right], \beta = .51205;
\]
\[
\frac{\alpha}{n} = \frac{5.78 \times 10^3}{A} \left[ 1 - \frac{18.2}{A^{1/5}} \right], \beta = 1.
\]

For the first mode the dissipation increases with increasing \(\gamma\); this might be expected because for the larger value of \(\gamma\) the function \(\varepsilon\) increases more rapidly towards the outer boundary. The same holds for the third mode as compared with the first one.

\$\S\ 4$. The contribution of the changes in the generation of energy to the damping constant has been computed for the case where the generation of energy consists in capture of protons by nuclei (see \$\S\ 1$). For this process the dependence on density and temperature has been given by A. H. Wilson \(^1\); that on density is weaker than that on temperature, the latter acting chiefly through an exponential factor \(e^{-C/T^{1/5}}\). As an approximation this exponential factor only has been taken into account, so that

\[
\frac{\delta \varepsilon}{\varepsilon} = \frac{1}{3} \frac{C \delta T}{T^{1/5}} = K \frac{\delta T}{T},
\]

where \(\varepsilon\) is the energy generated per unit mass and time. The corresponding amount of energy dissipated per unit time is

\[
- \int_0^M \varepsilon \frac{\delta T}{T} \frac{\delta T}{T} dM = - \int_0^M K \varepsilon \left(\frac{\delta T}{T}\right)^2 dM,
\]

the bar indicating the mean value over a period. Since \(\varepsilon\) decreases rapidly from the centre outwards no great mistake is made if \(K\) is taken constant and equal to its central value; then this term is simply proportional to \(K\). In the computations the factor \(\left(\frac{\delta T}{T}\right)^2\) has also been taken constant and equal to its value at \(z = 1\).

The term found here would counterbalance that computed in the preceding section for the first mode in the case \(3 - 4/\gamma = 0.4\) if \(K\) were equal to \(780\); for the third mode the value of this fictitious \(K\) is \(12 \times 10^6\), and for the first mode in the case \(3 - 4/\gamma = 0.2\) it is respectively \(43\) and \(0\). From the investigations of Steensholt \(^2\) it appears that the actual value of \(K\) is of the order of \(10\). Hence it is probable that for small values of \(3 - 4/\gamma\) the damping constant is negative for the first mode of vibration.

I am indebted to Dr Woltjer for his continued help during this work.

\(^1\) M.N. 91, 283, 1931.
\(^2\) Oslo Publ. No. 4, 1932.