RADIATIVE TRANSFER IN THICK ATMOSPHERES
WITH AN ARBITRARY SCATTERING FUNCTION

H. C. VAN DE HULST
Received 11 April 1968

1. The problem

McCormick (1967a) shows in a short note that four different approaches lead to the same asymptotic form of Chandrasekhar’s X and Y functions for very thick layers. The assumptions are: homogeneous layers, isotropic scattering, conservative or non-conservative scattering, neglected terms $O(e^{-d})$. At least two other derivations of the same formulae are available: Sobolev (1964) and Van de Hulst and Terhoeve (1966).

The equivalent formulae for anisotropic scattering with an arbitrary phase function are contained in, or easily derived from, the work of Germogenova (1961), Mullikin (1964a, b) and others. The present paper rederives some of those results but is original in presenting an all-out physical derivation, based on visualization of the problem with a minimum of formal mathematics. This has been supplemented elsewhere (Van de Hulst and Grossman, 1968; Van de Hulst, 1968) by accurate numerical results based on fitting numbers obtained by the doubling method to the asymptotic formulae.

2. Definitions and notations

We use mostly the definitions and notations conventional in astrophysics. Scattering in any volume element follows the unpolarized phase function $\Phi(\cos \alpha)$ with the first moments

$$\frac{1}{2} \int_{-1}^{1} \Phi(\cos \alpha) d(\cos \alpha) = a = \omega_0$$

$$\frac{1}{2} \int_{-1}^{1} \Phi(\cos \alpha) \cos \alpha d(\cos \alpha) = ag = \frac{\omega_1}{3}$$

where $a$ is the single scattering albedo, $g$ the asymmetry factor, and $\omega_0$ the coefficient in the expansion of the phase function in terms of Legendre polynomials (which we shall not use!).

Optical depth $\tau$ is measured from 0 at the top to $b$ at the bottom of the atmosphere. The cosine of the angle of any direction with the positive $\tau$ direction is written as $u(-1 \leq u \leq 1)$. We reserve $\mu$ for a cosine in the range $0 \leq \mu \leq 1$. Hence, the intensity at any point, or any other function of $u$ over all directions, can be represented by two functions of $\mu$ corresponding to the directions with an upward and those with a downward component.

The transmission and reflection functions $R(\mu, \mu_0)$ and $T(\mu, \mu_0)$ are symmetric functions of the direction of incidence $\mu_0$ and the direction of emergence $\mu$. The normalization is such that they are $(4\mu_0\mu)^{-1}$ times the corresponding $R$ and $S$ functions employed by Chandrasekhar. A perfectly white surface following Lambert’s law has $R = 1$. Azimuth angles $\varphi$ and $\varphi_0$ are absent because we consider only the azimuth-independent terms of the radiation field. Most easily we can

© Astronomical Institutes of The Netherlands • Provided by the NASA Astrophysics Data System
think of the radiation as conically incident from all
directions which have the same \( \mu_0 \). This restriction is
not essential (see section 6). The transmission contains
besides scattered radiation the singular term arising
from radiation penetrating the atmosphere without
scattering.

For purposes of a short terminology we call a function
of one variable \( \mu \) or \( \mu_0 \) defined on the interval
(0, 1) a vector; a function of two such variables is called
a matrix. Both vectors and matrices are denoted by
capitals. Quantities not dependent on \( \mu \) are scalars,
written by small letters. All of these functions may also
depend on \( a, g, b \) and other parameters.

The product of two vectors is a scalar defined by

\[
FG = \int_0^1 F(\mu)G(\mu)2\mu \, d\mu.
\]

Products of a vector and a matrix, or of two matrices,
are similarly defined. The insertion of the factor 2\( \mu \) (or
2\( \mu_0 \) if the integration is over \( \mu_0 \)) in all these definitions
is a device to arrive at simple formulae. The physical
interpretation of these multiplications is read from
right to left. For instance, from the definitions that
follow, we interpret:

\[
UR = (U \text{ operating on } R).
\]

### Table 1

**Definitions of vectors and matrices**

<table>
<thead>
<tr>
<th>Vector notation</th>
<th>Full notation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Matrices</strong></td>
<td></td>
</tr>
<tr>
<td>( R )</td>
<td>( R(\mu, \mu_0) )</td>
</tr>
<tr>
<td>( R_\infty )</td>
<td>( R_\infty(\mu, \mu_0) )</td>
</tr>
<tr>
<td>( T )</td>
<td>( T(\mu, \mu_0) )</td>
</tr>
<tr>
<td>( I )</td>
<td>( \delta(\mu-\mu_0)/2\mu )</td>
</tr>
<tr>
<td>( Z )</td>
<td>( \delta(\mu-\mu_0)/2\mu^2 )</td>
</tr>
<tr>
<td>( H_f )</td>
<td>( h(\mu, \mu') = \frac{h(-\mu, -\mu')}{4\mu \mu'} )</td>
</tr>
<tr>
<td>( H_b )</td>
<td>( h(-\mu, \mu') = \frac{h(\mu, -\mu')}{4\mu \mu'} )</td>
</tr>
<tr>
<td>( A )</td>
<td>( I(\tau, -\mu, \mu_0) )</td>
</tr>
<tr>
<td>( D )</td>
<td>( I(\tau, +\mu, \mu_0) )</td>
</tr>
</tbody>
</table>

- **Reflection function**, defined in the text.
- Same, for semi-infinite atmosphere.
- Transmission function.
- Unit matrix.
- Singular matrix effecting division by \( \mu \) or \( \mu_0 \). In particular:
  \[ ZR = \mu^{-1}R(\mu, \mu_0), \quad RZ = \mu_0^{-1}R(\mu, \mu_0). \]
- Forwards redistribution function.
- Backwards redistribution function.
- The matrices \( H_f \) and \( H_b \) together represent the function
  \[ h(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} \Phi[\mu \mu' + (1-\mu^2)(1-\mu'^2) \cos(\varphi - \varphi')] \, d\varphi, \]
  arising in the azimuth-independent part of the transfer equation
  (Chandrasekhar, 1950, p. 149).
- Intensity at depth \( \tau \) in an upward direction.
- Intensity at depth \( \tau \) in a downward direction.
- The matrices \( A \) and \( D \) are not symmetric.

<table>
<thead>
<tr>
<th>Vectors</th>
<th>Full notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I )</td>
<td>( I(\mu) ) or ( I(\mu_0) )</td>
</tr>
<tr>
<td>( P )</td>
<td>( P(\mu) )</td>
</tr>
<tr>
<td>( Q )</td>
<td>( P(-\mu) )</td>
</tr>
<tr>
<td>( K )</td>
<td>( K(\mu) )</td>
</tr>
<tr>
<td>( N )</td>
<td>( (2\mu_0)^{-1} )</td>
</tr>
<tr>
<td>( U )</td>
<td>( (2\mu)^{-1} )</td>
</tr>
<tr>
<td>( W )</td>
<td>( \mu_0 ) or ( \mu )</td>
</tr>
<tr>
<td>( V )</td>
<td>( \mu_0^3 ) or ( \mu_0^2 )</td>
</tr>
</tbody>
</table>

- Any intensity distribution in one hemisphere. Some specification
  (e.g. up, down, incident, emergent) is required to specify which
  hemisphere.
- Forward part of normalized diffusion pattern.
- Backward part of normalized diffusion pattern.
- Injection function.
- As right factor: incident radiation as emitted from a narrow layer
  of isotropic sources.
- As left factor: operator defining hemispherical average.
- As right factor: incident radiation from standard white (Lambert)
  surface. As left factor: operator defining flux through a horizontal
  unit area.
- Operators defining higher moments; occur only in the transition
to conservative scattering (section 5).
This is a function of $\mu_0$ expressing the fraction of the incident flux reflected from the atmosphere if incidence occurs from direction $\mu_0$. In many papers this is simply called the albedo of the atmosphere.

$$RU = (R \text{ operating on } U).$$

This is a function of $\mu$ expressing the intensity reflected in direction $\mu$ by an atmosphere exposed to incident radiation with uniform intensity from all directions in the hemisphere.

We shall use the vectors and matrices defined in table 1.

3. Basic relations

We shall discuss the radiation field established in a thick atmosphere by radiation (conically) incident from one direction $\mu_0$ in the absence of internal sources. We shall look only at the radiation outside (i.e. incident, reflected and transmitted) and the radiation deep inside the atmosphere in what we shall presently define as the diffusion domain. This means that we choose to (and can afford to!) omit a discussion of the radiation field inside the atmosphere near the top and bottom boundaries. These imposed restrictions are not essential to the method (see section 6) but they are helpful in permitting a clear presentation.

The diffusion domain is physically defined as the range of optical depths far enough from the boundaries (or from internal source layers) to let the radiation be transported by simple diffusion as if the medium were unbounded. Let the medium be non-conservative, $a < 1$. The physically plausible form for the intensity in the diffusion domain is

$$I(\tau, u) = s_1 e^{-kr} P(u) + s_2 e^{kr} P(-u), \quad (5)$$

where $k$ is the diffusion exponent ($k^{-1}$ is the diffusion length), $P(u)$ the diffusion pattern, $s_1$ the strength of the diffusion stream in the positive $\tau$ direction, and $s_2$ the strength of the diffusion stream in the negative $\tau$ direction. Evidently, the values $s_1$ and $s_2$ are transformed if $\tau$ is counted from a different zero-level.

The integral equation from which $P(u)$ and $k$ follow will appear in the course of the derivation [eq. (13)]. In the context of our intuitive method we feel excused from proving even the existence of the solution (5). For it seems clear that the energy pumped in at one end must somehow be transported and that, given enough depth, the transport process will settle to a fixed angular pattern and to a damping more slowly than that for a single beam (i.e., $k < 1$). Also the pattern must have a forward peak, i.e., $P(1) > P(-1)$, and probably is a monotonic function of $u$.

From the work of other authors it follows that a number of discrete solutions with $k < 1$ may exist. This does not affect our further derivation because the least damped solution, which is the one with smallest $k$, prevails in thick layers. The errors made in the asymptotic theory for layers with large thickness $b$ then are $O(e^{-k_2b})$ where $k_2$ is the next higher root (Kuščer, 1955). In the examples studied so far this occurs only for very strongly forward or very strongly backward directed phase functions. If only one root exists, as in the case of isotropic scattering, the errors are $O(e^{-b})$.

We now turn to the derivation proper, which can be read from the seven boxes of figure 1.

Box 1 shows the effect of a thin scattering and absorbing layer, depth $d\tau$, on an arbitrary incident radiation field from below. The unscattered radiation is weakened to $(1 - Z d\tau)I$ but part of the energy reappears as radiation scattered into the forward hemisphere ($H_I d\tau$) or into the backward hemisphere ($H_B d\tau$). The emerging intensity in any forward direction thus is $I + (-Z + H_I)I d\tau$.

Box 2 shows an unbounded medium carrying a diffusion stream of strength 1 in the negative $\tau$ direction and no stream in the positive $\tau$ direction. The intensity at the level $\tau = 0$, for clarity drawn as a gap, is by definition $P$ up and $Q$ down.

Box 3 shows an arbitrary intensity distribution $I$ incident upon a semi-infinite atmosphere. Part of the energy is reflected, giving the emergent intensity $R_o I$. Another part of the energy trickles down into deep layers and thus injects an inward diffusion stream with strength $K I$, where $K$, the injection function, is a vector yet unknown.

Box 4 shows a diffusion stream of strength 1 approaching the free surface. Part of the radiation escapes with a certain angular intensity pattern which we call $m K$. The coefficient $m$ is unknown but the vector $K$ must be the same as in box 3 for reciprocity reasons. In addition, the absence of backscatter from layers outside the atmosphere upsets the standard pattern of the diffusion stream, causing the intensity to be less than in an unbounded medium. This negative correction is
strong near the surface and makes its influence felt further in as an inward diffusion stream of negative strength. See Van de Hulst and Terhoeve (1966) for a more complete description. The net effect, shown in box 4, is that the upward diffusion stream suffers at the surface an internal reflection with the reflection factor $-l$, where $l$ is a positive constant still to be determined. A fully equivalent description is that reflection occurs at a virtual level at optical depth $q$ outside the atmosphere with a reflection factor $-1$. Then

$$l = \exp(-2kq).$$

We may call $q$ the extrapolation length.

Boxes 5, 6 and 7 repeat the situations of boxes 2, 3 and 4 with a thin layer $d\tau$ added. The strengths of the diffusion streams are defined with respect to the $\tau = 0$ level indicated. Their values follow from the simple invariance principle that, by relabeling $\tau$, it must be possible to restore the situation in the box above. For instance, if in box 6 the level $\tau = 0$ is chosen on top of the added layer, the injected stream is $KI$. The same stream referred to the $\tau = 0$ level shown has the strength $KI \exp(-k d\tau) = KI(1 - k d\tau)$.

The following equations result.

Consider box 2. Apply definitions from boxes 3 and 4. We can write at once

$$Q = R_\infty P$$

$$P = R_\infty Q + mK$$

$$1 = KP$$

$$0 = KQ - l.$$  \hspace{1cm} (7)  \hspace{1cm} (8)  \hspace{1cm} (9)  \hspace{1cm} (10)

Consider box 5. Apply definitions from box 1 and write

$$P = (1 - Z d\tau + H_t d\tau)P(1 + k d\tau) + H_b d\tau Q$$

$$Q(1 + k d\tau) = (1 - Z d\tau + H_t d\tau)Q + H_b d\tau P(1 + k d\tau).$$

Hence, keeping terms proportional to $d\tau$,

$$(Z - k)P = H_t P + H_b Q$$

$$(Z + k)Q = H_t Q + H_b P.$$  \hspace{1cm} (11)  \hspace{1cm} (12)

Together, these two equations represent the integral equation

$$(1 - ku)P(u) = \frac{1}{2} \int_{-1}^{1} h(u', u')P(u') \, du'$$

which is the familiar equation from which the eigenvalue(s) $k$ and eigenfunction(s) $P(u)$ may be solved.

Consider box 6. Apply definitions from boxes 1 and 3 and write

$$R_\infty I = (1 - Z d\tau + H_t d\tau) (R_\infty + R_\infty H_b d\tau R_\infty) \times$$

$$(1 - Z d\tau + H_t d\tau)I + H_b d\tau I$$

$$KI(1 - k d\tau) = K(1 + H_b d\tau R_\infty)(1 - Z d\tau + H_t d\tau)I.$$
This must hold for any $I$. The terms linear in $d\tau$ give
\[ ZR_\infty + R_\infty Z = H_b + H_t R_\infty + R_\infty H_t + R_\infty H_b R_\infty \]
\[ K(Z-k) = KH_b + KH_t \cdot \] (14)

\[ \text{Consider box 7. Apply definitions from boxes 1, 3 and 4 and write} \]
\[ mK(1-k d\tau) = (1-Z d\tau + H_t d\tau) (R_\infty H_b d\tau + 1) mK \]
\[ -l(1-2k d\tau) = -l + KH_b d\tau mK \]
from which
\[ (Z-k)K = R_\infty H_b K + H_t K \]
\[ KH_b K = 2kl/m. \] (15)

The nine equations (7)-(12), (14)-(16) form a set of vector equations, the physical consistency of which is guaranteed by the derivation. One could pose the mathematical question whether the equations permit a unique solution of $R_\infty$, $P$, $Q$, $K$, $k$, $m$, $l$ if the phase function, and therefore $H_t$ and $H_b$, are given. We shall not try to answer this question in its full generality but note some points.

First, a normalization convention [eq. (17)] is necessary to define $P$ and $Q$; this convention affects also $K$ and $m$ but not $R_\infty$, $k$ and $l$. Secondly, the solution for $k$, $P$ and $Q$ from eq. (13) is known not always to be unique. Thirdly, at least some redundant exists, for eq. (16) can be derived by subtracting eq. $(15') \times Q$ from $K \times$ eq. (12) and simplifying the result by means of eqs. (8) and (10).

We adopt the normalization
\[ N(P + Q) = 2 \]
(17)
in agreement with the choice in the traditional method of deriving $P(u)$ by the coefficients of its expansion in terms of Legendre functions, e.g., Mullikin (1964b). Using simple identities
\[ UZ = 2N \]
\[ U(H_t + H_b) = 2aN, \] (18)
we then find
\[ U(P - Q) = 4k^{-1}(1-a). \] (20)

Similarly, using further identities
\[ WZ = U \]
\[ W(H_t - H_b) = agU, \] (21)
we find
\[ W(P + Q) = 4k^{-2}(1-a)(1-ag). \] (23)

Evidently, the integral (17) occurs in the radiation density, eq. (20) in the net-flux, and eq. (23) in Eddington's $K$ integral, all in a simple diffusion stream in an unbounded medium. Higher moments can be obtained by the recurrence relations between the coefficients of the expansions in terms of Legendre functions (Mullikin, 1964b).

An interesting property of the diffusion pattern follows by taking $P \times$ eq.(8) $- Q \times$ eq. (7) and employing the symmetry of $R_\infty$ and relation (9). The result, written both in vector form and in full notation, is
\[ m = PP - QQ = \int_{-1}^{1} [P(u)]^2 2u \; du. \] (24)

4. Very thick layers

We maintain the assumption $a < 1$ and consider a layer of finite total thickness $b$, large enough to permit a diffusion domain to exist between the boundary zones near top and bottom. No restriction is placed on the product $kb$: the losses of the diffusion stream in penetrating the entire layer may be large, intermediate or almost negligible.

We assume radiation with intensity $I(\mu_0)$ incident on the top surface and absence of internal sources. Figure 2 defines the two unknown diffusion streams. They have strengths $d$ and $-c$ when referred to the level $\tau = 0$ at

![Figure 2. Diffusion streams and external radiation fields set up by radiation incident on a very thick atmosphere.](image-url)
the top surface, but would have strengths $d \exp(-kb)$ and $-c \exp(kb)$ when referred to the bottom surface as $\tau = 0$ level. By the definitions from figure 1, boxes 3 and 4, we may write at once

$$RI = R_x I - cmK$$
$$TI = d \exp(-kb)mK$$
$$d = KI + ic$$
$$-c \exp(kb) = -ld \exp(-kb).$$

These four equations must be satisfied for arbitrary $I$. This permits the four unknowns to be expressed in terms of quantities pertaining to the semi-infinite atmosphere

$$R = R_x - mf(1-f^2)^{-1} \exp(-kb)K \cdot K$$
$$T = m(1-f^2)^{-1} \exp(-kb)K \cdot K$$
$$d = (1-f^2)^{-1} KI$$
$$c = f(1-f^2)^{-1} \exp(-kb)KI,$$

where $K \cdot K$ is the matrix notation for the product function $K(\mu)K(\mu_0)$ and

$$f = l e^{-kb} = e^{-k(b+2a)}.$$  

Upon substitution of $s_1 = d$, $s_2 = -c$ into eq. (5) we have the intensity at any depth and direction in the diffusion domain. Representing this by $DI$ for $u > 0$, $\mu = u$, and by $AI$ for $u < 0$, $\mu = -u$, we obtain the asymmetric matrices

$$D = (1-f^2)^{-1} [e^{-k}P - f e^{k(1-b)}Q] \cdot K$$
$$A = (1-f^2)^{-1} [e^{-k}Q - f e^{k(b-b)}P] \cdot K.$$

The average intensity at the mid-layer level, $\tau = b/2$, is

$$\frac{1}{2} N(A+D)I = (1+f)^{-1} e^{-kb/2}KI,$$

and the net-flux/$\pi$ flowing down through this level is

$$U(D-A)I = 4(1-a)k^{-1}(1-f)^{-1} e^{-kB/2}KI.$$

Heavy losses through the atmosphere correspond to $kb \gg 1$, and hence $f \ll 1$. The terms remaining when $f = 0$ correspond to a semi-infinite atmosphere. The extra terms due to $f \neq 0$ can be visualized as a consequence of repeated internal reflections of the diffusion stream, which lead to geometric series. The sums of these series, namely $(1-f^2)^{-1}$, $(1+f)^{-1}$ and $(1-f)^{-1}$, can be verified at once from figures 1 and 2.

5. Weak losses and conservative scattering

Conservative scattering, $a = 1$, $k = 0$, was excluded from the preceding derivations for obvious mathematical reasons. The physical reason for breakdown is that in this limit a positive diffusion stream down is indistinguishable from a negative diffusion stream up. Rather than treat this case separately (for which, of course, good methods are available), we shall present some expansions in $k$ and then make the transition $k \to 0$.

Consider a set of phase functions which all have the same form and differ in albedo. Thus, if $\omega_n$ are the coefficients in the expansion of the phase function in terms of Legendre functions, we consider the set of phase functions defined by

$$\omega_n = (2n+1)ab_n,$$

where $b_0 = 1$, $b_n$ is fixed and $a$ variable. All functions defined in the preceding sections then depend on $a$. Among them is the main diffusion exponent $k$, of which it is plausible to assume a steady increase with decreasing $a$. This makes it possible to consider $k$ instead of $a$ as the independent variable. It now turns out that all quantities defined above, which include matrices, vectors and scalars, can be written in the form of power series in $k$, where the coefficients are again matrices, vectors or scalars.

This is a statement without proof. It is made plausible a) by the solutions in the isotropic case, b) by the form of the recurrence relations for $g_n$ in Mullikin’s (1964b) theory, c) by the fact that similar expansions have proved useful in neutron scattering (Kuščer, 1967), and d) by numerical checks for $g = 0.5$ presented below. We shall not attempt a fuller discussion but simply state without derivation the first few terms of the expansions of each quantity. They can be verified by substitution into eqs. (7)–(17), (20), (23) and (24). Nor do we enter here into the modifications necessary if we wish to read for $k$ another (not the smallest) root of the characteristic equation.

The expansions of quantities pertaining to the unbounded medium can be derived with relative ease from the recurrence relations for $g_n$ in Mullikin’s formulae. The key simplification arises from the fact that the coefficient of $k^n$ in the pattern $P(u)$ is expressible in a series of (odd or even) Legendre functions up to order $n$ at most. We find
have a noticeable effect only in very deep layers where the product $k \tau$ has become large enough. Hence the radiation field near the surface very closely resembles that for fully conservative scattering, which explains the dominant term $R_0$. The losses by absorption form an energy sink deep in the atmosphere not unlike the losses by escape through the bottom surface of finite layers [eq. (25)]. The result must be similar, so again we expect a correction term proportional to $K(\mu)$, and for symmetry to $K(\mu_0)$. It is shown elsewhere (Danielson et al., 1968) how these two kinds of losses can be combined into simple and accurate formulae describing the reflection, transmission and absorption of extremely thick layers with albedos very close to 1, as are exemplified by terrestrial stratus clouds.

b) The dominant term in both eqs. (7) and (8) is

$$U = R_0 U$$

(46)

which represents the physical statement that a conservative, semi-infinite atmosphere reflects the entire incident flux, independently of the direction of incidence or of the form of the phase function.

c) Because $K_0 U = 1$ by eq. (45), the quantity $K_0 W$ is the average value of $\mu$ in the quanta escaping from a conservative atmosphere emitting a constant net-flux. The same quantity equals $(1 - g)q_0$ by eq. (39). By eq. (38) we can transform this expression into

$$K_0 W = \frac{1}{2}(1 + WR_0 W)$$

(47)

which permits a fairly accurate evaluation, even if we should have only a wild guess about $R_0$. For instance, a simple form satisfying eq. (46) would be

$$R(\mu, \mu_0) = 1 + c(1 - \frac{1}{2} \mu)(1 - \frac{1}{2} \mu_0)$$

giving

$$K_0 W = \frac{\mu_2}{4} + \frac{1}{4} c = 0.708 + 0.005 c$$

Here $c = 0$ corresponds to reflection by Lambert's law. The condition $-1 \leq c \leq 2$ is necessary to make $R(\mu, \mu_0)$ positive over the physical domain of $\mu$ and $\mu_0$. This crude estimate thus puts $K_0 W$ in the range 0.703 to 0.718. The actual values for various phase functions (Van de Hulst and Grossman, 1968) fall in the range 0.710 to 0.712; those for the phase functions with $N = 2$ considered by Horak and Janousek (1965, their constants c) fall in the range 0.710 to 0.714.

Further formulae for conservative scattering (Piotrowski, 1956; Sobolev, 1957) are obtained from the preceding equations by the transition $k \to 0$. First, as-

$$a = 1 - \frac{1}{3(1 - g)} k^2 - \frac{4 - 9g + 5gh}{45(1 - g)^2(1 - h)} k^4 + O(k^5)$$

(35)

$$m = \frac{8}{3(1 - g)} k + \frac{8(16 - 21g + 5gh)}{45(1 - g)^3(1 - h)} k^3 + O(k^4)$$

(36)

$$P(u) = 1 + \frac{1}{1 - g} ku + \frac{2}{3(1 - g)(1 - h)} k^2 P_2(u) + O(k^3)$$

(37)

where $g = b_1 = \text{asymmetry factor}$, $h = b_2$, and $P_2(u) = (-1 + 3u^2)/2$.

The expansions of $R_0$ and $K$ are not so obvious. Let quantities for the conservative case ($k = 0$) be represented by the index 0. Imagine that the reflection function $R_0(\mu, \mu_0) = R_0$ (we now omit the index $\infty$) has been found from the integral equation (14). Then by eqs. (8), (35) and (36)

$$K_0(\mu) = K_0 = \frac{3}{2}(W + R_0 W)$$

(38)

Further defining

$$q_0 = \frac{1}{1 - g} K_0 W$$

(39)

we find the expansions

$$R = R_0 - \frac{4k}{3(1 - g)} K_0 \cdot K_0 + O(k^2)$$

(40)

$$K = (1 - q_0 K_0) K_0 + O(k^2)$$

(41)

$$l = 1 - 2q_0 k + 2q_0^2 k^2 + O(k^3)$$

(42)

$$q = q_0 + O(k^2)$$

(43)

All of these results can be found from eqs. (6)–(24) by substituting power series with unknown coefficients. It seems likely that subsequent coefficients involve solutions of new integral equations, like (14) or (15), which may not be expressible in terms of the functions or constants already defined. However, some further terms of the moments and bimoments can be expressed without recourse to such new functions. For instance,

$$URU = 1 + \frac{4}{3(1 - g)} (-k + q_0 k^2) + O(k^3)$$

(44)

$$KU = 1 - q_0 k + (K_0 V) k^2 + O(k^3)$$

(45)

The following physical comments are in order:

a) Both terms of eq. (40) can be easily understood. Very weak losses accompanying each scattering process
suming an upward diffusion stream $c$ and keeping only terms linear in $k$ we find from eqs. (5) and (37) the intensity in the diffusion domain

$$I(\tau, u) = c(1 + k\tau) [1 - ku(1 - g)] - lc(1 - k\tau) [1 + ku(1 - g)].$$

(48)

This corresponds to the net upward flux

$$\pi F = \frac{8\pi kc}{3(1 - g)}. \quad (49)$$

The constant $c$ thus has to be large in order to compensate the near-cancellation of the radiation fields of the upward and downward diffusion stream, caused by the smallness of $k$. By eqs. (42) and (49), eq. (48) transforms in the limit $k = 0$ to the familiar intensity formula for the diffusion domain

$$I(\tau, u) = \frac{3}{4} F(1 - g) \left[ \tau + q_0 - \frac{u}{1 - g} \right]. \quad (50)$$

The net upward flux is $\pi F$ at any level. Applying the definition of figure 1, box 4, together with eqs. (36) and (49), we find that this flux emerges from the surface with the intensity distribution

$$I(0, -\mu) = FK_\mu(\mu). \quad (51)$$

The finite-layer formulae (25)–(33) are adapted to conservative scattering by writing $f = 1 - (b + 2q_0)k$ and then letting $k \to 0$. For instance,

$$R_\infty - R = T = \frac{4K \cdot K}{3(1 - g)(b + 2q_0)}, \quad (52)$$

again a familiar result. The mid-layer intensity for the case of normalized incident radiation from direction $\mu_0$ becomes

$$A = I \left( b^2, u \right) = \left[ \frac{1}{2} \tau \cdot \frac{u}{(1 - g)(b + 2q_0)} \right] K(\mu_0). \quad (53)$$

This formula is useful in the process of "asymptotic fitting" to the numerical data from the doubling method (Van de Hulst, 1968).

6. Extensions and comparisons

We shall now compare our results with those of other authors and at the same time indicate various ways in which the problem treated here can be or has been extended.

a) Linearly anisotropic scattering, the simplest practical example of anisotropic scattering, emerges from our formulae by putting $b_1 = 0$, $n \geq 2$. For detailed results see Ivanov and Leonov (1965), McCormick and Kuscher (1965). Isotropic scattering arises by also putting $b_1 = g = 0$; some references have been given in the first section of this paper.

b) Early derivations by methods more rigorous but less graphic than mine, but confined, like the results of this paper, to the dominant diffusion term, are contained in the work of many authors. To my knowledge Kuscher (1955) was the first to solve Milne's problem for a semi-infinite atmosphere with arbitrary anisotropic scattering and to provide a recipe for computing the coefficients in the asymptotic formulae. He makes the usual limitation to phase functions with a finite expansion in Legendre polynomials but this can hardly be considered essential. Maslenikov (1958) treated the same problem. His final forms for the intensity in the diffusion domain and $a < 1$ correspond to the description we have presented with figure 1, box 4, and his expression for the coefficient which we have called $l$ reads in our notation $KQ/KP$, which agrees with eqs. (9)–(10). The corresponding formula for $a = 1$ agrees with our eq. (49), and $q_0$ defined by eq. (39). Some years later Gromogenova (1961) extended this work to finite, very thick layers. Her results correspond precisely to eqs. (25)–(28) of this paper if I may assume that in the definition of her quantity $\phi_\alpha$, a factor $\exp (qk) = l^{-1}$ has by error been omitted. The appearance of the equation is rather different. For instance by our eq. (29) and the translations $v = k$, $h = b$, $x_i = g$, we have in her eq. (2.4): $\sinh \{v(b + 2x_i)] = (1 - f^2)/2f$. Also, her eq. (2.5), equivalent to my eq. (25), does not clearly show the symmetry of the reflection function.

c) The rigorous solution can be found by Case's method, which has been applied extensively in the field of neutron scattering. It may be regarded as the rigorous limit of the Wick-Chandrasekhar approximation by discrete ordinates. The principle is to consider all possible solutions of eq. (13) and not only the one with smallest $k$. Again, solutions for isotropic scattering and linearly anisotropic scattering have preceded those for arbitrary phase functions and solutions for a semi-infinite atmosphere (half-space) have preceded those for a finite layer (slab). The biggest job is to find the expansion coefficients from the boundary conditions.
Full formulae for a finite slab with arbitrary phase function were presented by Kaper (1966). An alternative method is to use bi-orthogonality relations (McCormick and Kuščer, 1966) in a manner analogous to the solution of the slab problem for isotropic scattering (McCormick and Mendelson, 1964). Further references are found in these papers.

Apparently independent, but along similar lines is the work of Mullikin (1964a, b) and Leonard and Mullikin (1964). In the notation of Carlstedt and Mullikin (1966) our I is written in the form

$$I = -N \left( -\frac{1}{k} \right) / N \left( \frac{1}{k} \right).$$

By their eq. (3.8) and our eq. (6) this gives the extrapolation length

$$q = \int_0^1 \frac{\Theta(t) dt}{1 - k^2 t^2}$$

which, incidentally, confirms our eq. (43). The function $\Theta(t)$ is defined by Carlstedt and Mullikin's eq. (3.7).

d) Internal source layers can be incorporated in the physical method employed in this paper. Figure 3 shows a (transparent) layer of isotropic sources, emitting to both sides the intensity $|u|^{-1} = 2N$, placed at $\tau = 0$ in an unbounded medium. This source layer creates by multiple reflection against both sides the intensity represented by the vector $S$ satisfying the equation

$$S - 2N = R_\infty S.$$  \hspace{1cm} (56)

At sufficient distance from the source layer this radiation sets up the diffusion stream $KS$. Write eq. (56) in the form

$$2N = S(1 - R_\infty)$$

and multiply both sides by the vector $P + Q$. Then by eqs. (7), (8) and (17) we find

$$KS = 4 m^{-1}$$

which is the required diffusion stream.

Again, a larger effort is required to find the radiation field at smaller distances from the source layer. We must then apply Case's method with the higher modes included. An explicit solution based on a similar method was presented by Vanmassehnove and Grosjean (1967).

The same layer at sufficiently large depth $\tau_0$ in a semi-infinite medium gives by the definition of box 4 the emerging intensity

$$4 m^{-1} mK e^{-k\tau_0} = 4 K(\mu) e^{-k\tau_0}.$$  

The corresponding point-direction gain, as defined by Van de Hulst (1964), is

$$G(\tau_0, \mu) = 4\mu K(\mu) e^{-k\tau_0}.$$  \hspace{1cm} (58)

This result can also be verified in the reciprocal situation by computing from eq. (17) and box 3 the radiation density at depth $\tau$ caused by radiation incident from $\mu_0$ and dividing this by the radiation density in the incident beam.

Extension to an internal source layer in a finite atmosphere is obvious. So is the reduction of eq. (57) to the familiar form $-dk/da$ valid only for isotropic scattering (Case et al., 1953).

e) Azimuth-dependent terms and polarization can surprisingly be treated by the same formalism. We have reasons to expect that none of the derivations and none of the equations presented here in vector form need a change! Only the following changes are necessary. The definition of a vector now needs an added index $m = 0, 1, 2 \ldots$ to indicate the component proportional to exp [$im(\varphi - \varphi_0)$] in the azimuthal Fourier expansion and another index $j = 0, 1, 2, 3$ indicating which Stokes parameter is meant. Each of these components is a
complex function of \( \mu \). The vector multiplication, eq. (4), has to be newly defined for these vectors, and so has the function \( h(\mu, u') \) describing the elementary scattering process. This plan has not yet been executed; the work may be eased by referring to the detailed formulae given in Chandrasekhar's book.

f) The transition region between the diffusion domain and the surface has not been discussed above but can be included by similar physical reasoning. For instance, the radiation field at any depth corresponding to a diffusion stream of unit strength approaching the surface of a semi-infinite atmosphere (box 4 of figure 1) equals that of the undisturbed diffusion stream minus the radiation field that would be set up by exposing the surface to incident radiation equal to the back half of the diffusion pattern [\( P(-\mu) = \text{vector } Q \)]. In the preceding derivations we have considered only the asymptotic form \( le^{-kL}P(u) \) of this negative correction, but there is no reason why we should not with some further effort compute and use the entire correction. The values of these correction factors for isotropic scattering have been presented by Van de Hulst and Terhoeve (1966). Of course, complete results can also be based on the heavier formalism of Case's method (McCormick, 1967b; McCormick and Kuščer, 1966).

g) Expansion in powers of \( k \) has been used in section 5 to explain the transition from near-conservative scattering to conservative scattering. Full proofs and algorithms are lacking but probably such expansions are convergent in the full range \( 0 \leq k < 1 \). As a numerical example I find for Henyey-Greenstein functions with \( g = 0.5 \) the expansion \( a = 1 - 0.666667k^2 - 0.029630k^4 - 0.0620k^6 - 0.012k^8 \) which gives values correct to the sixth decimal down to \( a = 0.800000, k = 0.541836 \). The reflection of a semi-infinite layer, as exemplified by the bimoment \( UR_{\infty}U \), then has dropped to 0.232. This opens the rather attractive possibility of treating planetary absorption lines or bands, in which \( a \) varies continuously, by a systematic use of such expansions.

Acknowledgements

In the period of 20 months since the first version of this paper was ready I have profited from the oral and written advice of many colleagues and friends, of whom I should like to mention specifically Dr. K. Grossman, Dr. V. V. Ivanov and Dr. H. G. Kaper.

References

S. Chandrasekhar, 1950, Radiative Transfer (Oxford Univ. Press, London)
T. A. Germogenova, 1961, Zhur. Vych. Mat. 1 1001
V. V. Ivanov and V. V. Leonov, 1965, Izv. Akad. Nauk S.S.R. 1 803
S. Piotrowski, 1956, Acta Astr. 6 61
V. V. Sobolev, 1957, Astr. Zhur. 34 336 (Soviet Astr. 1 332)
F. R. vanMassefiove and C. C. Grosjean, 1967, Electromagnetic Scattering, ed. Powell and Stein (Gordon and Breach) 721