COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

On a method based on a property of central projection for reducing the rectangular co-ordinates measured on an arbitrary plate to those of any other plate,

by H. van Gent.

1. The problem of deriving the "astrophographic" position of an object from measures of this object and surrounding stars on an arbitrary plate is a problem of central projection. Any plate may be considered as a central projection of the sky; so when two plates with different centres cover a common part of the sky, this part on one plate is a central projection of the corresponding part on the other plate, with the centre of the objective as centre of projection. This holds also when the two plates have been taken with different instruments and different focal lengths (Paloqué, J. O. 12, p. 105).

The relation between the rectangular co-ordinates of corresponding points on both plates is fixed as soon as the co-ordinates of four corresponding points are known. When a fifth point has been measured on one plate, its corresponding position on the other plate can be computed. A way in which this computation can be done is given below. It is based on the following property of central projection: the anharmonic ratio of four lines through a point in one plane is not affected by central projection on another plane.

Suppose a plate has been taken on which an object (comet, meteor-trail, asteroid, etc.) occurs, the astrophographic position of which is required. Then the procedure is as follows. Measure the rectangular co-ordinates of the object and of four preferentially surrounding stars $a$, $b$, $c$ and $d$. These co-ordinates on the "object" plate may be denoted by $x_a, y_a, x_b, y_b, x_c, y_c$ and $x_d, y_d$. The co-ordinates of the four reference stars in the Carte du Ciel catalogue may be denoted by $x_a', y_a', x_b', y_b', x_c', y_c'$ and $x_d', y_d'$. The required quantities are $x_a'$ and $y_a'$, the "astrophographic" co-ordinates of the object.

The anharmonic ratio of the four lines from star $a$ to stars $b, c, d$ and the object $o$ on the "object" plate is computed by the following expression:

\[
\frac{y_d - y_o}{x_d - x_o} : \frac{y_o - y_a}{x_o - x_a} : \frac{y_b - y_o}{x_b - x_o} : \frac{y_o - y_a}{x_o - x_a},
\]

In the same way:

\[
\frac{y_d - y_o}{x_d - x_o} : \frac{y_o - y_b}{x_o - x_b} : \frac{y_c - y_o}{x_c - x_o} : \frac{y_o - y_b}{x_o - x_b}.
\]

The computation of these two quantities is fairly quick because the same quotients are repeated several times.

Now, from the property of central projection mentioned above, it follows:

\[
a' (b' c' d' o') = a (b c d o)
\]
or:

\[
\frac{y'_d - y'_a}{x'_d - x'_a} : \frac{y'_b - y'_a}{x'_b - x'_a} : \frac{y'_o - y'_a}{x'_o - x'_a} : \frac{y'_d - y'_o}{x'_d - x'_o} = a (b c d o)
\]

and

\[
\frac{y'_d - y'_b}{x'_d - x'_b} : \frac{y'_o - y'_b}{x'_o - x'_b} : \frac{y'_c - y'_b}{x'_c - x'_b} : \frac{y'_o - y'_b}{x'_o - x'_b} = b (a c d o)
\]
from which the two unknown quantities $x_1'$ and $y_1'$ are found.

This solution of the problem is exact so long as the plates are true central projections of the sky. The object may occur on any part of the plate, even at a great distance from the centre, so that this method overcomes the difficulty mentioned by GONNESSIAT (J. O. 12, p. 184): "S'il arrivait que l'astre à déterminer — mettons qu'il s'agisse d'une planète — se trouvât à une grande distance du centre, 4 ou 5° par exemple, dans une lunette à grand champ (genre Cook), on ne voit pas qu'on puisse faire autrement que d'employer les formules rigoureuses pour le rattachement du centre d'observation à un centre voisin de cet astre; et ce sont les coordonnées ainsi transformées qu'il faudrait comparer à celles du Catalogue."

2. Several checks on the method are possible. By using anharmonic ratios of the four lines meeting at star $c$ and the four lines meeting at star $d$, a fully independent computation can be made in the same way as described above. By using one of these and one of the pencils of rays radiating from $a$ and $b$, four other checks are possible. Care should be taken, in the computation as well as in the checks, to combine always anharmonic ratios of pencils, the lines of which to the object intersect at an angle not too far from 90°.

3. The present method gives the astrographic co-ordinates only.

When the right ascension and declination of the object are required, these are computed from the astrographic co-ordinates $x_1', y_1'$, found by the method described above, in the usual way by the aid of the formulae and the constants for the astrographic plate in question given in the astrographic catalogue.

4. The case may occur that no Carte du Ciel plate exists which overlaps the region of a plate on which an object is found. Then reference stars are used, the right ascension and declination of which are known. It can be shown that in this case the measures of three stars on the plate are sufficient to reduce the measures of the object to right ascension and declination. The measures of three reference stars determine a plane triangle; their known right ascensions and declinations determine a solid triangle. The problem of finding the relation between rectangular plate co-ordinates and right ascension and declination is identical with the problem of fitting the plane triangle into the solid triangle. The formulae connecting rectangular plate co-ordinates with right ascension and declination become complicated.

The method described under 1 may also be used in this case, though it uses one star more than strictly necessary. Four stars are selected, the known right ascensions and declinations of which may be denoted by $z_1, \delta_1, a_1, \beta_1$, and $a_2, \delta_2$. These four positions on the sphere are projected on a plane, with the centre of the sphere as centre of projection. For simplicity the plane tangent to the pole of the sphere may be taken; the rectangular co-ordinates in this plane become:

$$x' = \cot \delta \cos \alpha; \quad y' = \cot \delta \sin \alpha.$$

These quantities are computed for the four stars and correspond to the quantities $x', y'$ under 1. The same method is now used, which gives as result the co-ordinates $x_1', y_1'$ of the object on this plane tangent to the pole. The right ascension and declination of the object are found from these directly by the formulae:

$$\tan \alpha = \frac{y_1'}{x_1'}$$

$$\cot \delta = x_1'^2 + y_1'^2$$

5. When the astrographic position of an object on a photograph is derived by the "dependence" method (SCHLESINGER, A. J. 874) or variations of this method (WOOD, J. B. A. A. 39, p. 196, COMRIE id., p. 203), the measures of the rectangular co-ordinates of only three reference stars and of the object are required to compute the object's astrographic position. The dependence method however is based on the supposition that the triangles (or polygons) of the reference stars on the "object" plate and the Carte du Ciel plate are similar. This condition is only satisfied when the two plate centres are the same. If the astrographic position of an object is derived by the dependence method from a plate which has a centre differing appreciably from that of the Carte du Ciel plate on which the reference stars occur, a sensible error is introduced. This error depends mainly upon the angular distance between the plate centres, the size of the triangle (polygon) of the reference stars and the position of the object relative to these stars. In cases of plates covering a large portion of the sky (photographs by Steavenson covering 58° × 44° are mentioned in J. B. A. A. 41, p. 111), on which an object (comet, meteor, etc.) occurs, this error may be considerably greater than the error of a star place, especially when the choice of the reference stars is limited and a large triangle has to be taken. In this case the dependence method becomes untrustworthy. TURNER's method (M. N. 54, p. 11) and the method proposed in this paper will give a better result.

6. To get an idea of the error which may be introduced by uncritical application of the dependence...
method the following simplified case, shown in fig. 1, may be considered. Let $B$ be a plate from which the position of an object $P$ at a great angle from the plate centre is required. Two reference stars $S_1$ and $S_2$ may be found on either side of the object at the same angular distance $\epsilon$. A Carte du Ciel plate $A$ exists on which the stars $S_1$ and $S_2$ occur; the axis of this plate was $O.S_1$. The angle between the two plate centres is $\varphi$; the plate centres, reference stars and the object are on a great circle in the sky. Fig. 1, in which $O$ represents the centre of the objective, is in the plane of this great circle.

If now from the measures of $S_1$, $S_2$ and $P$ (in this case in one co-ordinate only) on plate $B$ the astrographic position on plate $A$ is derived by the dependence method, the dependences being the ratio's

$$\frac{S_1}{S_1}
\frac{P}{S_1}
\frac{S_2}{S_2}
\frac{P}{S_2},$$

the error introduced will be, expressed in radians:

$$\frac{1}{2} [\tan (\varphi + \epsilon) + \tan (\varphi - \epsilon)] - \tan \varphi.$$

For some values of $\varphi$ and $\epsilon$ the amount of this error was computed. Table 1 gives the results.

**Table 1.**

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$\epsilon$</th>
<th>$3^\circ$</th>
<th>$1^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5^\circ$</td>
<td>1'4</td>
<td>5'5</td>
<td></td>
</tr>
<tr>
<td>$10^\circ$</td>
<td>2'8</td>
<td>11'1</td>
<td></td>
</tr>
<tr>
<td>$20^\circ$</td>
<td>5'7</td>
<td>23'1</td>
<td></td>
</tr>
<tr>
<td>$30^\circ$</td>
<td>9'1</td>
<td>36'7</td>
<td></td>
</tr>
</tbody>
</table>

7. An example may be given of what the method exposed in this paper can do. On two plates, taken with the Franklin Adams-instrument ($a = 4'5, f = 112'3$ cm) which overlapped by a region of about $3'\times 3'$, a set of five stars was measured in two co-ordinates. The plates are of $20\times 20$ cm size and cover about $10^\circ \times 10^\circ$;

their guiding stars were AG Cam 6392, 18$^h$33$^m$8$-$12$^2$20$'$ (1900) and AG Alg 8087, 19$^h$01$^m$3$-$18$^5$3$'$ (1900) respectively. The guiding star corresponds rather closely to the plate centre. From the measures of the five stars on the first plate and the first four stars on the second plate, the co-ordinates of the fifth star on the second plate were computed by the method of this paper. The result is as follows:

<table>
<thead>
<tr>
<th>Measures Plate I</th>
<th>Plate II</th>
</tr>
</thead>
<tbody>
<tr>
<td>star x $\text{mm}$</td>
<td>y $\text{mm}$</td>
</tr>
<tr>
<td>$a$ 23'248</td>
<td>185'430</td>
</tr>
<tr>
<td>$b$ 20'117</td>
<td>132'250</td>
</tr>
<tr>
<td>$c$ 74'933</td>
<td>135'236</td>
</tr>
<tr>
<td>$d$ 77'010</td>
<td>184'970</td>
</tr>
<tr>
<td>$e$ 47'262</td>
<td>152'086</td>
</tr>
</tbody>
</table>

The two anharmonic ratio's $a (b c d o)$ and $b (a c d o)$ become:

$$a (b c d o) = -1'401023$$
$$b (a c d o) = +1'766025$$

(a decimal more than necessary was taken for convenience).

From these the required co-ordinates $x'_o, y'_o$ follow:

$$x'_o = 46'726; \quad y'_o = 165'090.$$  

For comparison the dependence method (COMRIE, *J. B. A. A.* 39, p. 203) was applied to the same example, using the stars $a, b, c$ as reference stars. From the measures of these three stars on both plates and the measures of star $o$ on plate I, the position of star $o$ on plate II was derived. This gave:

$$x'_o = 46'867; \quad y'_o = 164'950.$$  

The measures of star $o$ on plate II are a test on both methods; they were:

$$x'_o = 46'732; \quad y'_o = 165'075.$$  

Comparison gives:

$O - C$ in $x'_o$  in $y'_o$  Error in$''$  

Dependence method:  $-1'35$  $+1'25$  33'7  
Projective method:  $+0'06$  $-0'15$  2'9  

It shows the value of the projective method described above in cases which are unfavourable for the dependence method. As a lens does not make a true central projection on the plate, and refraction and bad quality of images in the corners of plates come in, the error of the projective method in this example is still considerably more than the error of measurement, which may be of the order of $\pm 0'02$. 