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2 Notations and conventions

2.1 General notions

(1) As mentioned earlier, for a prime number \( p \) in this paper, we assume that \( p \geq 3 \).

(2) All Dieudonné crystals and Diudonné modules are contravariant.

(3) If \( X \) is a scheme over a ring \( R \) (resp. over a scheme \( S \)) and \( R' \) is an \( R \)-algebra (resp. \( S' \) is an \( S \)-scheme), we often denote by \( X_{R'} \) (resp. \( X_{S'} \)) the pullback of \( X \) along the ring homomorphism \( R \to R' \) (resp. along the structure morphism \( S' \to S \)). Sometimes we suppress the subscripts \( R' \) (resp. \( S' \)) if the base ring (resp. base scheme) is clear from the context.

(4) Similar convention as in (3) applies for modules, \( p \)-divisible groups, stacks, etc.

(5) If \( R \) is a ring of characteristic \( p \) or a \( \mathbb{Z}_p \)-flat \( p \)-adic ring and \( \phi \) a Frobenius lift of \( R \) (i.e., the reduction mod \( p \) of \( \phi \) is the absolute Frobenius of \( R/pR \)), we often suppress the subscript \( R \) when it is clear from the context. Here and everywhere in the sequel a \( p \)-adic ring is \( p \)-adically complete and separated.

(6) If \( \phi : R \to R \) is an endomorphism of rings and \( M \) is an \( R \)-module, we use the following notations interchangeably

\[
M^{(\phi)} := \phi^* M := M \otimes_{R, \phi} R.
\]

(7) If \( R \) is a ring with \( r \in R \), and \( M \) an \( R \)-module, we sometimes simply denote by \( M[\frac{1}{r}] \) the module \( M \otimes_R R[\frac{1}{r}] \) over \( R[\frac{1}{r}] \) and \( f[\frac{1}{r}] : M[\frac{1}{r}] \to N[\frac{1}{r}] \) the induced homomorphism of a homomorphism \( f : M \to N \) of \( R \)-modules.

(8) By a linear algebraic group over a ring \( R \) we mean an affine group scheme of finite presentation over \( R \). By a reductive group over \( R \) we mean a connected smooth affine group scheme over \( R \) such that for each geometric point \( s \) of \( \text{Spec} R \), the pullback \( G_s \) is a connected reductive group over \( s \).

(9) Let \( k \) be a field of characteristic \( p \). For any \( k \)-scheme \( S \), we denote by \( \text{Frob}_S : S \to S \) the absolute Frobenius endomorphism of \( S \), and by \( (\_)^{(p)} \) the pull-back along \( \text{Frob}_S \) of a scheme or a sheaf or a morphism over \( S \). For example, if \( G \) is a reductive group over \( k \), we denote by \( G^{(p)} \) the pull-back of \( G \) along \( \text{Frob}_{\text{Spec} k} \); if \( P \) is a \( G \)-torsor over \( S \), then \( P^{(p)} \) is a \( G^{(p)} \)-torsor over \( S \); the pull back \( \lambda^{(p)} : \mathbb{G}_{m,k} \cong \mathbb{G}_{m,k}^{(p)} \to G^{(p)} \) along \( \text{Frob}_{\text{Spec} k} \) of a cocharacter \( \lambda : \mathbb{G}_{m,k} \to G \) (over \( k \)) is a cocharacter of \( G^{(p)} \) (over \( k \)). In particular, if \( G \) is defined over \( \mathbb{F}_p \), then \( \lambda^{(p)} \) is again a cocharacter of \( G \).
since we then have canonical isomorphism $G^{(p)} \cong G$. When $k$ is perfect, we have an automorphism of $X_*(G) := \text{Hom}_k(\mathbb{G}_{m,k}, G)$, given by
\[ \sigma : X_*(G) \longrightarrow X_*(G), \quad \lambda \longmapsto \lambda^{(p)}. \quad (2.1.1) \]

Let $G$ be reductive model over $\mathbb{F}_p$ of $G$, i.e., $G \otimes k = G$, and $\sigma : k \to k(x \mapsto x^p)$ the Frobenius element in the Galois group $\text{Gal}(k/\mathbb{F}_p)$. Then the action $\sigma$ in (2.1.1) is the same as the Galois action of $\sigma \in \text{Gal}(k/\mathbb{F}_p)$ on $X_*(G)$, i.e., we have
\[ \lambda^{(p)} = (\mathbb{G}_{m,k} \xrightarrow{id \otimes \sigma} \mathbb{G}_{m,k} \xrightarrow{\lambda} G \otimes k \xrightarrow{id \otimes \sigma^{-1}} G \otimes k = G). \quad (2.1.2) \]

(10) If $G$ is a group scheme over a field of characteristic $p$, and $R$ is a $k$-algebra, then for any $x \in G(R)$, by $\sigma(x)$ we mean the image of $x$ in $G^{(p)}(R)$ under the relative Frobenius $G \to G^{(p)}$. Note that there are two Frobenii $\sigma$ and $\varphi$ on $\mathcal{L}G$ and on $\mathcal{L}^+G$ (Section 3). The difference between them is discussed in Section 3.3.

### 2.2 Tensors and contragredient representations

We introduce below the “contragredient representations” following Wortmann’s thesis ([Wor13](#)) but note that it is used slightly differently in this thesis.

Let $R$ be a ring and $M$ a finite locally free $R$-module. Denote by $M^*$ the dual $R$-module of $M$ and by $M^\otimes$ the direct sum of all $R$-modules that arise from $M$ by applying the operation of taking duals, tensor products, symmetric powers and exterior powers a finite number of times. An element of $M^\otimes$ is called a **tensor** over $M$. For an isomorphism $f : M_1 \cong M_2$ of finite locally free $R$-modules, we have an induced isomorphism $(f^{-1})^* : M_1^* \to M_2^*$, and hence $f^\otimes : M_1^\otimes \to M_2^\otimes$.

For any $M$ as above, there is a canonical isomorphism of group schemes, called the **contragredient representation** of $\text{GL}(M)$, defined as
\[ (\cdot)^\vee : \text{GL}(M) \cong \text{GL}(M^*), \quad g \longmapsto g^\vee := (g^{-1})^*. \]

Through the contragredient representation of $M$, $M^\otimes$ is naturally identified with $(M^*)^\otimes$. If $s \subset M^\otimes$ is a set of tensors over $M$, which defines a subgroup $G \subset \text{GL}(M)$, then they also defines a subgroup $\{g^\vee | g \in G \} \subset \text{GL}(M^*)$, when considered as tensors over $M^*$. Since $M$ is canonically identified with $(M^*)^*$, we also have the contragredient of $\text{GL}(M^*)$,
\[ (\cdot)^\vee : \text{GL}(M^*) \to \text{GL}(M), \]

which we shall use frequently in the thesis.