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Let $n \geq 2$ be an integer and let $K$ be a number field. In the following statements, all varieties and morphisms are defined over $K$.

1) Let $C$ be a smooth curve of genus two with Jacobian $\text{Jac}(C)$, let $E_1$ be an elliptic curve, and let $\varphi_1: C \to E_1$ be a covering of degree $n$ that is optimal, i.e. a covering that does not factor through an isogeny. Then, possibly after extending $K$, there exist another elliptic curve $E_2$, an optimal covering $\varphi_2: C \to E_2$ of degree $n$, and an isogeny $\text{Jac}(C) \to E_1 \times E_2$ whose kernel is $\varepsilon_1(E_1[n]) = \varepsilon_2(E_2[n]) \subset \text{Jac}(C)[n]$, where $\varepsilon_i: E_i \hookrightarrow \text{Jac}(C)$ are the embeddings induced by $\varphi_i$.

2) Let $(E_1, O_1)$ and $(E_2, O_2)$ be elliptic curves and let $\alpha: E_1[n] \to E_2[n]$ be an isomorphism (of finite $K$-group schemes) that is anti-symplectic with respect to the Weil pairing and denote its graph by $\Gamma_\alpha$. Let $\Theta$ denote the divisor $E_1 \times \{O_2\} + \{O_1\} \times E_2$, that induces a principal polarization on $E_1 \times E_2$. Finally, let $\varphi: E_1 \times E_2 \to J$ denote the isogeny such that $\text{Ker}(\varphi) = \Gamma_\alpha$. Then there exists a divisor $C$ on $J$ with arithmetic genus two that induces a principal polarization on $J$ and satisfies $\varphi^*(C) \sim n\Theta$. If $C$ is irreducible then it is a curve of genus two and $J \cong \text{Jac}(C)$. If $C$ is reducible then it is a sum $F_1 + F_2$ of two elliptic curves that meet in a rational 2-torsion point, such that $J \cong F_1 \times F_2$. Moreover, the curves $E_1, E_2, F_1, F_2$ are all isogenous.

We say that the curves $E_1$ and $E_2$ are glued along their $n$-torsion. If $C$ is irreducible, we say that $\text{Jac}(C)$ is $(n,n)$-split.

3) With assumptions as in 1), if $n = 3$ and both $\varphi_1$ and $\varphi_2$ have a point of ramification index three, then $E_1$ and $E_2$ are isomorphic and their modular invariants are either $1728$ or $-873722816/59049$.

4) With assumptions as in 2), if $E_1$ and $E_2$ are such that the product of their (minimal) discriminants is a square in $K$ or such that they both have a rational point of order two, then they can be glued along their 2-torsion via a $K$-rational isomorphism $\alpha: E_1[2] \to E_2[2]$. 
5) With assumptions and notations as in 2), suppose that \( n \) is odd. Then the principally polarized abelian surface \( J \) is isomorphic to \( F_1 \times F_2 \) if and only if the divisor \( \varphi^*(C) \) contains a (necessarily \( K \)-rational) point of \((E_1 \times E_2)[2]\) that is not a point of order two on \( E_1 \times \{O_2\} \) or \( \{O_1\} \times E_2 \). If \( n = 3 \) and \( J \cong F_1 \times F_2 \), this point is not \( (O_1, O_2) \).

6) With notations as above, the Lang-Silverman conjecture holds for \((n, n)\)-split Jacobians \( \text{Jac}(C) \) if and only if it holds for elliptic curves that can be glued along their \( n \)-torsion with another elliptic curve to form \( \text{Jac}(C) \).

7) Let \( \text{Tr}_\infty \) denote the archimedean trace and let \( \Delta \) denote the minimal discriminant. The Lang-Silverman conjecture holds for Jacobians that are \((n, n)\)-isogenous to a product \( E_1 \times E_2 \) of elliptic curves such that at least one of the following is satisfied for \( i = 1, 2 \):
   
   i) \( \text{Tr}_\infty(E_i) > \frac{1}{7} \log N_{K/Q}(\Delta_{E_i}) \);
   
   ii) The Szpiro ratio \( \sigma_{E_i} \) is uniformly bounded.

8) Let \( n \geq 2 \) be an integer, let \( S \) be a finite set of \( m \geq 3 \) elements, and let \( \mathcal{T}(S) \) denote the set of total orderings of \( S \). Suppose that \( f : \mathcal{T}(S)^n \to \mathcal{T}(S) \) is a function that satisfies:
   
   i) For all \( a, b \in S \) and for all \( \mathcal{O} = (O_1, \ldots, O_n) \in \mathcal{T}(S)^n \), if \( a < b \) is in \( \cap_{i=1}^n O_i \) then \( a < b \) is in \( f(\mathcal{O}) \);
   
   ii) For all \( a, b \in S \), if \( a < b \) is in \( f(\mathcal{O}) \) for some \( \mathcal{O} = (O_1, \ldots, O_n) \) then \( a < b \) is in \( f(\mathcal{O}) \) for all \( \tilde{O} = (\tilde{O}_1, \ldots, \tilde{O}_n) \) such that \( \{a < b, b < a\} \cap O_i \cap \tilde{O}_i \neq \emptyset \) for all \( i \in \{1, \ldots, n\} \).

Then \( f \) is a projection \( (O_1, \ldots, O_n) \mapsto O_i \) for some \( i \in \{1, \ldots, n\} \). This statement is known as Arrow’s theorem. It is not a statement about all possible functions with codomain \( \mathcal{T}(S) \).

9) There exist reasonably well behaved functions \( f : I^{m \times n} \to \mathcal{T}(S) \), with \( S, m, n \) as in 8) and \( I = [0, 1] \) or \( I = \{0, 1\} \).

10) There are good practical reasons to introduce a notation other than \( 2\pi \) for the real number that is the length of the unit circle. The set of non-horrible choices has measure zero and it contains the symbol \( \pi \).

11) It is difficult, and perhaps foolish, to get rid of a notation established by Euler.

12) Theorems of the form “If \( A \) then \( B \)” have a very limited practical use if \( A \) happens to be false.