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Chapter 3

Deformation theory

In this chapter, we review the Kodaira-Spencer map, which parametrizes the infinitesimal deformations of a variety by the first cohomology group of its tangent sheaf. Using the Kodaira-Spencer map, we define the infinitesimal period map which compares the deformation of a variety to the deformation of its Hodge structure. Finally, we show that the infinitesimal Torelli theorem holds for nodal surfaces.

3.1 The Kodaira-Spencer map

In this section, we define the Kodaira-Spencer map. Much of the contents of this section can be found in [Ser06] or [CMP03].

In their papers [KS58], Kodaira and Spencer showed that infinitesimal deformations of a smooth projective manifold $M$ can be expressed entirely in terms of the cohomology group $H^1(M, T_M)$. They gave an analytic construction (cf. [Man05]) but we shall give an algebraic definition of the Kodaira-Spencer map.

Let $\mathbb{C}[\varepsilon] = \mathbb{C}[x]/(x^2)$ be the square-zero extension of $\mathbb{C}$. A first order infinitesimal deformation is a pullback square

$$
\begin{array}{ccc}
M & \overset{i}{\rightarrow} & \mathcal{M}_\varepsilon \\
\downarrow & & \downarrow f \\
\text{Spec } \mathbb{C} & \rightarrow & \text{Spec } \mathbb{C}[\varepsilon]
\end{array}
$$

in which $f$ is a flat morphism. A morphism of infinitesimal deformations is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_\varepsilon & \rightarrow & \mathcal{M}_{\varepsilon'} \\
\downarrow f & & \downarrow f' \\
\text{Spec } \mathbb{C}[\varepsilon] & \rightarrow & \text{Spec } \mathbb{C}[\varepsilon']
\end{array}
$$
which restricts to the identity on the central fibre $M \to \text{Spec } \mathbb{C}$. Let $\text{Def}_M$ denote the set of isomorphism classes of first order infinitesimal deformations.

Given any representative $\mathcal{M}_\varepsilon \to \text{Spec } \mathbb{C}[\varepsilon]$ of an isomorphism class, there exists an open affine cover $\{U_i\}$ of $M$ such that the family is trivial on each $U_i$, i.e. there is an isomorphism $\theta_i : U_i \times \text{Spec } \mathbb{C}[\varepsilon] \iso \mathcal{M}_\varepsilon|_{U_i} = \mathcal{M}_\varepsilon \times_M U_i$. The infinitesimal deformation is uniquely determined by the set of transition maps

$$\theta_{ij} = \theta_i^{-1}\theta_j : U_{ij} \times \text{Spec } \mathbb{C}[\varepsilon] \to U_{ij} \times \text{Spec } \mathbb{C}[\varepsilon] \quad \text{on } U_{ij} = U_i \cap U_j.$$

The maps $\theta_{ij}$ define derivations $\mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$. The tangent sheaf is defined as

$$T_M = \mathcal{H}\text{om}(\Omega^1_M, \mathcal{O}_M) = \text{Der}(\mathcal{O}_M, \mathcal{O}_M),$$

so each $\theta_{ij}$ defines an element $\eta_{ij} \in \Gamma(U_{ij}, T_M)$. The Čech cocycle condition $\eta_{ij} + \eta_{jk} + \eta_{ki} = 0$ holds on the intersection, and $\{\eta_{ij}\}$ gives a well defined class in $H^1(M, T_M)$. This gives us a well-defined map

$$\kappa : \text{Def}_M \to H^1(M, T_M)$$

called the Kodaira-Spencer correspondence. Indeed it is a bijection when $M$ is smooth [Ser06, Prop. 1.2.9], and it gives $\text{Def}_M$ a vector space structure.

Let $f : \mathcal{M} \to B$ be a smooth family of smooth projective complex varieties. Let $0 \in B$ and $M = \mathcal{M}_0 = f^{-1}(0)$. A deformation family of $M$ is a commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{i} & \mathcal{M} \\
\downarrow & & \downarrow f \\
\{\text{pt}\} & \longrightarrow & B.
\end{array}$$

Where there is no risk of confusion, we shall just refer to a deformation family by the map $f$.

For a smooth variety $B$, the algebraic tangent space at 0 is given by

$$T_{B,0} = \mathcal{H}\text{om}_0(\text{Spec } \mathbb{C}[\varepsilon], B) = \{\phi \in \mathcal{H}\text{om}(\text{Spec } \mathbb{C}[\varepsilon], B) \mid f((\varepsilon)) = 0\}.$$ 

There is a well-defined map $T_{B,0} \to \text{Def}_M$ taking $\phi \in \mathcal{H}\text{om}_0(\text{Spec } \mathbb{C}[\varepsilon], B)$ to the infinitesimal deformation $\mathcal{M}_\varepsilon \to \text{Spec } \mathbb{C}[\varepsilon]$ which is the pullback of the deformation family $f : \mathcal{M} \to B$ along $\phi$. Combining the two maps gives the Kodaira-Spencer map

$$KS_f : T_{B,0} \to \text{Def}_M \xrightarrow{\kappa} H^1(M, T_M). \quad (3.1)$$

A family $f : \mathcal{M} \to B$ is said to be versal if $KS_f$ is surjective and universal if it is an isomorphism. There exists a versal deformation only if, for all classes $\xi \in H^1(M, T_M)$, the Lie bracket $[\xi, \xi] = 0 \in H^2(M, T_M)$ [KS58, §6]. A smooth
manifold $M$ admits a universal family if $H^2(M, T_M) = 0$ [KNS58, Theorem, p. 452].

We can extend the Kodaira-Spencer map in two different ways: the first is to consider deformations of pairs $(M, D)$ where $M$ is a smooth variety and $D \subset M$ is a smooth effective divisor, while the second is to consider deformations of singular varieties. In the second case, we shall only consider the simple situation of a quotient variety of the form $X = M/G$ where $M$ is a smooth manifold and $G$ is a finite group.

3.1.1 Kodaira-Spencer map for divisors on varieties

Let $M$ be a smooth algebraic variety and $D \subset M$ an effective divisor. Let $\mathcal{L} = \mathcal{O}_M(D)$ be the line bundle associated to $D$ and $\Sigma_{\mathcal{L}}$ be the sheaf of differential operators of degree $\leq 1$ on $M$. Let $s \in H^0(\mathcal{L})$ be the section defining $D$, then $s$ defines a morphism $d_1 s : \Sigma_{\mathcal{L}} \to \mathcal{L} : \partial \mapsto \partial s$.

An infinitesimal deformation of the triple $(M, \mathcal{L}, s)$ is defined to be a triple $(M_\varepsilon, \mathcal{L}_\varepsilon, s_\varepsilon)$ where $M_\varepsilon$ is a flat $\mathbb{C}[\varepsilon]$-scheme ($\varepsilon^2 = 0$), $\mathcal{L}_\varepsilon$ is a line bundle on $M_\varepsilon$ and $s_\varepsilon \in H^0(M_\varepsilon, \mathcal{L}_\varepsilon)$, satisfying isomorphisms $M_\varepsilon \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C} \cong M$ and $\mathcal{L}_\varepsilon \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C} \cong \mathcal{L}$ which send $s_\varepsilon \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C}$ to $s$. Two infinitesimal deformations $(M_\varepsilon, \mathcal{L}_\varepsilon, s_\varepsilon)$ and $(M'_\varepsilon, \mathcal{L}'_\varepsilon, s'_\varepsilon)$ are isomorphic if there are $\mathbb{C}[\varepsilon]$-isomorphisms $M_\varepsilon \cong M'_\varepsilon$ and $\mathcal{L}_\varepsilon \cong \mathcal{L}'_\varepsilon$ sending $s_\varepsilon$ to $s'_\varepsilon$, restricting to the identity on $(M, \mathcal{L}, s)$ (cf. [Wel83, Section 1]).

We shall call an infinitesimal deformation of a triple $(M, \mathcal{L}, s)$ satisfying the assumptions in the first paragraph an infinitesimal deformation of the pair $(M, D)$, and denote the vector space of isomorphism classes of infinitesimal deformations of $(M, D)$ by $\text{Def}_{M, D}$.

Welters [Wel83, Prop. 1.2] proved that the set of isomorphism classes of infinitesimal deformations of the triple $(M, \mathcal{L}, s)$ is given by the first hypercohomology group $\mathbb{H}^1(M, d_1 s)$ of the complex

$$0 \to \Sigma_{\mathcal{L}} \xrightarrow{d_1 s} \mathcal{L} \to 0.$$ 

A simple manipulation gives us the following proposition.

**Proposition 3.1.1.** Let $M$ be a smooth algebraic variety and $D \subset M$ an effective divisor. Then, $\text{Def}_{M, D} = \mathbb{H}^1(M, \overline{d}_1 s)$ where $\overline{d}_1 s$ is the complex

$$0 \to T_M \xrightarrow{\overline{d}_1 s} \mathcal{O}_D(D) \to 0.$$ 

In particular, if $D$ is smooth, $\text{Def}_{M, D} = H^1(M, T_M(-\log D))$. 

31
Proof. The sheaf \( \Sigma_L \) lies in a short exact sequence \([Wel83, p. 178, (1.10)]\)

\[
0 \to \mathcal{O}_M \to \Sigma_L \to T_M \to 0.
\]

Since the composition \( \mathcal{O}_M \to \Sigma_L \xrightarrow{d_1 s} \mathcal{L} \) given by \( f \mapsto fs \) is injective, there is a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_M & \to & \Sigma_L & \to & T_M & \to & 0 \\
& & \downarrow & \downarrow & d_1 s & \downarrow & \bar{d}_1 s & & \\
0 & \to & \mathcal{O}_M & \to & \mathcal{L} = \mathcal{O}_M(D) & \to & \mathcal{O}_D(D) & \to & 0.
\end{array}
\]

This gives a quasi-isomorphism of the latter two vertical complexes in the derived category \( D^b(\mathcal{O}_M) \), hence \( \text{Def}_{M,D} = \mathbb{H}^1(M, d_1 s) = \mathbb{H}^1(M, \bar{d}_1 s) \).

If \( D \) is a smooth, the short exact sequence (Corollary 2.1.5)

\[
0 \to T_M(\log D) \to T_M \xrightarrow{d_1 s} \mathcal{O}_D(D) \to 0
\]

implies that there a quasi-isomorphism between \( T_M(\log D) \) seen as a complex concentrated in degree 0 and \( T_M \xrightarrow{\bar{d}_1 s} \mathcal{O}_D(D) \). This gives an isomorphism of cohomology groups

\[
H^1(M, T_M(\log D)) = \mathbb{H}^1(M, \bar{d}_1 s).
\]

\[\square\]

If \( D \) is not smooth, but is a V-manifold instead, the situation is more complicated. By Corollary 2.2.18, the sequence

\[
0 \to T_M(\log D) \to T_M \to \mathcal{O}_D(D)
\]

is usually only left exact. Let \( \mathcal{C} \) be the cokernel of the map \( T_M(\log D) \to T_M \) and \( \mathcal{C}' \) be the cokernel of the inclusion \( \mathcal{C} \to \mathcal{O}_D(D) \). We then get a diagram of short exact sequences

\[
\begin{array}{cccccc}
0 & \to & T_M & \to & T_M & \to & 0 \\
& & \downarrow & \downarrow & \bar{d}_1 s & & \\
0 & \to & \mathcal{C} & \to & \mathcal{O}_D(D) & \to & \mathcal{C}' & \to & 0.
\end{array}
\]

This induces a long exact sequence in hypercohomology

\[
0 \to H^1(M, T_M(\log D)) \to \mathbb{H}^1(M, \bar{d}_1 s) \to H^0(M, \mathcal{C}').
\]

Hence, we can conclude:
Corollary 3.1.2. Let $M$ be a smooth algebraic variety and $D \subseteq M$ an effective divisor. Suppose that $D$ is a V-manifold. Then, $H^1(M, T_M(-\log D)) \subseteq \text{Def}_{M,D}$.

Example 3.1.3. Suppose $M \cong \mathbb{P}^{n+1}$ and $D$ is a hypersurface defined by a homogeneous polynomial $F$ of degree $d$. Let $J = (\frac{\partial F}{\partial X_i}) \subset S = \mathbb{C}[X_0, \ldots, X_{n+1}]$ be the Jacobian ideal and $I = \sqrt{J}$ be the radical of $J$. In this case, we can evaluate $\text{Def}_{M,D}$: there is a diagram

$$
\begin{array}{c}
0 \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}} \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(1)^{\oplus(n+2)} \rightarrow T_{\mathbb{P}^{n+1}} \rightarrow 0 \\
0 \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}} \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(d) \rightarrow \mathcal{O}_D(D) \rightarrow 0
\end{array}
$$

where $h$ is given by $(G_i) \mapsto \sum_{i=0}^{n+1} G_i \frac{\partial F}{\partial X_i}$. Furthermore, $\mathcal{O}(1)^{n+2}$ and $\mathcal{O}(d)$ are $\Gamma(\mathbb{P}^{n+1}, -)$-acyclic, so $h$ is an acyclic resolution $\overline{d_1}s$. Hence,

$$
\text{Def}_{\mathbb{P}^{n+1},D} = \mathbb{H}^1(\mathbb{P}^{n+1}, \overline{d_1}s) = \mathbb{H}^1(\mathbb{P}^{n+1}, h) = \text{coker} (H^0(h)) = (S/J)_d.
$$

Recall from Remark 2.3.8 that $H^1(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(-\log D)) \cong (I/J)_d$. There is a short exact sequence

$$
0 \rightarrow H^1(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(-\log D)) \cong (I/J)_d \rightarrow \text{Def}_{\mathbb{P}^{n+1},D} \cong (S/J)_d \rightarrow (S/I)_d \rightarrow 0.
$$

Since $\mathbb{P}^{n+1}$ has no non-trivial deformations, $\text{Def}_{\mathbb{P}^{n+1},D}$ parametrizes the deformations of $D$ in $\mathbb{P}^{n+1}$. The kernel of the map $(S/J)_d \rightarrow (S/I)_d$ is precisely the deformations whose defining polynomials remain in $I$, thus fixing the singular locus. Hence, $H^1(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(-\log D))$ parametrizes the deformations of $D$ in $\mathbb{P}^{n+1}$ that preserve the singular locus.

### 3.1.2 Kodaira-Spencer map for quotient varieties

Let $M$ be a smooth projective complex algebraic variety, $G$ a finite group that acts on $M$ and $X = M/G$ be the quotient variety.

Definition 3.1.4. Let $X$ be a quotient variety. A deformation of $X$ as a quotient variety over a smooth base $B$ is defined to be a deformation $f : M \rightarrow B$ of $M$ such that the action of $G$ on $M$ extends to a (holomorphic or algebraic) action on $M$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\sigma} & \mathcal{M} \\
\downarrow f & & \downarrow f \\
B & & B
\end{array}
$$
commutes for all $\sigma \in G$. Such a deformation of $M$ is also called a $G$-equivariant deformation. Let the space of isomorphism classes of $G$-equivariant first order infinitesimal deformations of $M$ be denoted by $\text{Def}^G_M$ or $\text{Def}_X$ where $X = M/G$.

We shall construct the Kodaira-Spencer map for $G$-equivariant deformations.

There is a natural $G$-action on the category of deformation families of $M$ [Rim80] which acts by sending a deformation family $f : M \to B$ to the deformation family

$$
\begin{array}{ccc}
M & \xrightarrow{\sigma^{-1}} & M \\
\downarrow f & & \downarrow f \\
\text{Spec } \mathbb{C} & \longrightarrow & B
\end{array}
$$

for each $\sigma \in G$. This action preserves isomorphism classes, so it induces an action on $\text{Def}_M$. It is clear that $\text{Def}^G_M = (\text{Def}_M)^G$ is the $G$-invariant subspace of first order infinitesimal deformations.

**Remark 3.1.5.** There is a natural induced $G$-action on the tangent bundle $T_M$. It is defined as follows. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of $M$ by open balls such that $G$ acts as a permutation on the indices in $I$, i.e., for each $\sigma \in G$, there is an isomorphism $\sigma_{U_i} : U_i \xrightarrow{\sim} U_{\sigma(i)}$. Let $\{y_k\}$ and $\{x_k\}$ be local coordinates of $U_{\sigma^{-1}(i)}$ and $U_i$ respectively such that $\sigma \in G$ sends $y_k$ to $x_k$. The induced $G$-action on $T_M$ is by sending the basis $\{\frac{\partial}{\partial y_k}\}$ to $\{\frac{\partial}{\partial x_k}\}$.

**Proposition 3.1.6.** The Kodaira-Spencer correspondence

$$\kappa : \text{Def}_M \to H^1(M, T_M)$$

is an isomorphism of $G$-modules. Hence, it induces an isomorphism

$$\kappa : \text{Def}^G_M \to H^1(M, T_M)^G.$$

**Proof.** Let $\mathcal{U}$ be an open covering of $M$ and $f : M_\varepsilon \to B$ be a representative of a class $[f] \in \text{Def}_M$. The Kodaira-Spencer correspondence $\kappa$ is defined by sending the transition functions $U_{ij} \times \text{Spec } \mathbb{C}[\varepsilon] \xrightarrow{\theta_{ij}} U_{ij} \times \text{Spec } \mathbb{C}[\varepsilon]$ to the Čech 1-cycles $\{\eta_{ij}\}$, which defines a class in $H^1(M, T_M)$.

Choose the open covering $\mathcal{U}$ defined in Remark 3.1.5. Then, $\sigma \in G$ acts by sending

$$(\theta_{ij}) \mapsto (\sigma \theta_{\sigma^{-1}(i)\sigma^{-1}(j)}\sigma^{-1}) \quad \text{and} \quad (\eta_{ij}) \mapsto (\sigma \eta_{\sigma^{-1}(i)\sigma^{-1}(j)}\sigma^{-1}),$$

hence the action of $G$ commutes with the Kodaira-Spencer correspondence $\kappa$.  \qed
Combining Corollary 2.2.13 and Proposition 3.1.6, we get

**Corollary 3.1.7.** Let $M$ be a smooth manifold endowed with the action of a finite group $G$. Let $B$ be the union of the codimension 1 components of the branch locus of $X = M/G$. Then, the vector space $\text{Def}_X$ of isomorphism classes of the infinitesimal deformations of $X$ as a quotient variety is parametrized by $H^1(M, T_M)^G = H^1(X, T_X(- \log B))$.

**Remark 3.1.8.** We contrast our result with that of Pardini [Par91] for big subgroups of $GL(n, \mathbb{C})$. In [Par91, Proposition 4.1], Pardini showed that for a quotient map $f : M \to X$ branched over a divisor $B$ on $X$, the infinitesimal $G$-equivariant deformations are parametrized by $H^1(M, T_M)^G = H^1(X, T_X(- \log B))$.

We also understand the deformation of a divisor on a quotient variety.

**Corollary 3.1.9.** Let $M$ be a smooth manifold endowed with the action of a finite group $G$ and $f : M \to X = M/G$ be the quotient map. Let $D \subset X$ be a divisor and $\tilde{D} = f^{-1}D$ be the preimage of $D$ on $M$. Denote the space of isomorphism classes of the infinitesimal deformations of the pair $(X, D)$ obtained as a quotient of $(M, \tilde{D})$ by $\text{Def}^G_{M, \tilde{D}} = \text{Def}_{X, D}$. Then, $\text{Def}_{X, D} = H^1(M, \tilde{D})^G$ where $\tilde{D}$ is the complex

$$0 \to T_M \xrightarrow{\tilde{a}_i} \mathcal{O}_{\tilde{D}}(\tilde{D}) \to 0$$

and $s \in \mathcal{O}_M(\tilde{D})$ is the section defining $D$. Furthermore, the $G$-invariant cohomology group $H^1(M, T_M(- \log \tilde{D}))^G$ is a subspace of $\text{Def}_{X, D}$ and equality holds if $\tilde{D}$ is smooth.

**Proof.** Let $\mathcal{L} = \mathcal{O}_M(\tilde{D})$ and $s \in H^0(M, \mathcal{L})$ be the section defining $\tilde{D}$. The section $s$ is invariant under $G$.

Fixing an open cover $\mathcal{U}$ of $M$, a deformation in $\text{Def}_{M, \tilde{D}}$ can be represented by a triple $(\theta_{ij}, l_{ij}, \tilde{s}_i = s_i + b_i \varepsilon)$ where $U_{ij} \times \text{Spec} \mathbb{C}[\varepsilon] \xrightarrow{\theta_{ij}} U_{ij} \times \text{Spec} \mathbb{C}[\varepsilon]$ are transition functions,

$$l_{ij} = \begin{pmatrix} 1 & 0 \\ \eta_{ij} & 1 \end{pmatrix} \in \text{End}(H^0(U_{ij}, \mathcal{L})[\varepsilon]) \quad \text{with } \eta_{ij} \in \text{End}(H^0(U_{ij}, \mathcal{L})),$$

and $\tilde{s}_i \in H^0(U_i, \mathcal{L})[\varepsilon]$ are infinitesimal deformations of $s_i = s|_{U_i}$. By the proof of [Wel83, Proposition 1.2], the Kodaira-Spencer correspondence for manifold-divisor pairs sends a deformation $(\theta_{ij}, l_{ij})$ to a pair $(b_i, \eta_{ij}) \in C^0(\mathcal{U}, \mathcal{L}) \oplus C^1(\mathcal{U}, \Sigma_{\mathcal{L}})$ which is a 1-cocycle in $H^1(d_1s)$.

As in the proof of Proposition 3.1.6, using the basis given in Remark 3.1.5, it is clear that the $G$-action commutes with the Kodaira-Spencer correspondence. The conclusion then follows from Proposition 3.1.1 and Corollary 3.1.2. \qed
3.2 Infinitesimal period map

In this section, we briefly recall the definition of a period map, and generalize it to define the infinitesimal period map for V-manifolds. The results for smooth projective varieties in this section are due to Griffiths [Gri68] and can be found in many standard texts, for example, [Voi02, Chapter 10] and [CMP03, Chapter 5].

Let $B \ni 0$ be an open ball, and $f : M \to B$ be a family of smooth projective varieties. We wish to understand how the Hodge structure of $M_b = f^{-1}(b)$ varies across the family. The period map is a holomorphic map

$$P^k : B \to D_k : [M_b] \mapsto (F^p H^k(M_b, \mathbb{C}))$$

where $D_k$ is the period domain, which is the moduli space of pure Hodge structures of weight $k$. Fix $M_0 = f^{-1}(0)$ and the vector space $V = H^k(M_0, \mathbb{Q})$. Since $B$ is contractible, Ehresmann’s lemma gives canonical isomorphisms $\phi_b : H^k(M_b, \mathbb{C}) \cong V$ for all $b \in B$, so $F^p H^k(M_b, \mathbb{C})$ can be canonically identified with subspaces of $V$. Furthermore, the Hodge numbers are constant in the family (cf. [Voi02, Proposition 9.20]), so $D_k$ is in fact a subspace of a product of Grassmannians

$$\prod_{p=1}^k \text{Gr}(b_{p,k}, V) \quad \text{where} \quad b_{p,k} = \dim (F^p H^k(M_0, \mathbb{C})).$$

By Lefschetz’s hyperplane theorem, the Hodge structure on $H^k(M_b, \mathbb{Q})$ is determined by that on a general hyperplane section of $M_b$ for all $k \neq n = \dim M_b$. Hence, the most interesting case of the period map is when $k = n$.

If the deformation is trivial, the Hodge structure is constant over the family and the period map is trivial, so we suppose that all deformations in the family $f : M \to B$ are non-trivial. If the period map $P^n$ is injective, then the family can be identified with a subspace of the period domain. A universal family $f : M \to B$ is said to satisfy the Torelli property if the period map $P^n$ is injective.

Checking whether a family satisfies the Torelli property is difficult. A significantly easier question is to ask if the period map $P^n$ is locally injective, that is, if, for any $[M] \in B$, its differential

$$dP^n : T_{[M]}B \to T_{P^n[M]}D$$

is injective.

By the Kodaira-Spencer isomorphism for universal families, we know that $T_{[M]}B \cong H^1(M, T_M)$. The codomain of $dP^k$ can be expressed in terms of the Hodge structure of $M$: 36
Proposition/Definition 3.2.1 ([Voi02, Theorem 10.21]). Let $M$ be a smooth projective variety. The infinitesimal period map is the morphism

$$dP^k : H^1(M, T_M) \to \bigoplus_{p=1}^k \text{Hom}(H^{k-p}(M, \Omega_p^M), H^{k-p+1}(M, \Omega_{M}^{k-1})) \quad (3.2)$$

defined by sending $\eta \in H^1(M, T_M)$ to the map $\eta \cup -$ : $\omega \mapsto \eta \cup \omega$. In local coordinates $z_1, \ldots, z_n$, we can write $\eta$ as $\sum f_i \frac{\partial}{\partial z_i}$, so $\eta \cup -$ is given by contracting the differential forms, with the action on each one form given by $\frac{\partial}{\partial z_i}(fdz_i) = \frac{\partial f}{\partial z_i}$.

Definition 3.2.2. Let $M$ be a smooth projective variety of dimension $n$. $M$ is said to satisfy the infinitesimal Torelli property if the infinitesimal period map $dP^n$ is injective.

Let $X = M/G$ be a quotient variety where $G$ is an abelian group. Taking the $G$-invariant components on both sides of the map (3.2) gives a map

$$dP^k : H^1(M, T_M)^G \to \bigoplus_{p=1}^k \text{Hom}(H^{p,k-p}(M), H^{p-1,k-p+1}(M))^G$$

$$= \bigoplus_{p=1}^k \bigoplus_{\chi \in \hat{G}} \text{Hom}(H^{p,k-p}(M)_\chi, H^{p-1,k-p+1}(M)_{\chi^{-1}})$$

where $\hat{G}$ is the character group of $G$ and $H^{p,q}(M)_\chi$ is the eigenspace of $H^{p,q}(M)$ corresponding to the character $\chi$.

Recall from Theorem 2.2.5(i) that the eigenspace corresponding to the trivial character $H^{p,q}(M)_1 = H^{p,q}(M)^G$ is precisely equal to $H^{p,q}(X)$, which defines a pure Hodge structure on $H^{p+q}(X, \mathbb{Q})$ by Theorem 2.2.7. Thus, one can define the infinitesimal period map for $X$ by projecting onto the components with trivial characters.

However, from a geometrical perspective, the period map $P^k$ is only well-defined if the filtration $(F^p H^k(-, \mathbb{C}))$ is constant dimensional, at least in an open neighbourhood of $[X]$. We thus need to impose an additional condition in the definition.

Definition 3.2.3. Let $X = M/G$ be a quotient variety of dimension $n$ and $f : \mathcal{M} \to B$ a versal $G$-equivariant deformation family with $M = M_0 = f^{-1}(0)$. Let $M_b = f^{-1}(b)$ for any $b \in B$. Suppose $H^{p,k-p}(M_b)^G$ is constant dimensional for all $b$ in an open neighbourhood of 0. Then, the infinitesimal period map is
defined to be
\[ dP^k : H^1(M, T_M)^G = H^1(X, \tilde{T}_X(-\log D)) \rightarrow \bigoplus_{p=1}^{k} \text{Hom}(H^{k-p}(X, \tilde{\Omega}_X^p), H^{k-p+1}(X, \tilde{\Omega}_X^{p-1})) \]

where \( D \) is the union of the codimension 1 components of the branch divisor. 
\( X \) is said to satisfy the infinitesimal Torelli property if the \( dP^k \) is injective.

Remark 3.2.4. The domain of the infinitesimal period map in Definition 3.2.3 is restricted to locally-trivial or \( G \)-equivariant infinitesimal deformations of \( X \) (cf. Section 3.1.2). If we embed a general singular variety \( X \) as a divisor in a smooth projective variety \( Y \), we see from Corollary 3.1.2 that a general infinitesimal deformation of \( X \) in \( Y \) is not equisingular. A general deformation will cause a jump in Hodge numbers, and as such the period map is not well-defined.

In this thesis, we will only be consider \( G \)-equivariant deformations of V-manifolds (cf. Chapter 4). The infinitesimal Torelli property determines if the Hodge structure on the middle cohomology distinguishes all non-trivial deformations of the V-manifold \( X \).

3.3 Infinitesimal Torelli theorem for nodal surfaces

After the proof of the original Torelli theorem for smooth projective curves, the next major result is Griffiths’ proof of the Torelli theorem for most smooth projective hypersurfaces [Gri69a; Gri69b]. The first step in Griffiths’ proof is to prove the infinitesimal Torelli theorem.

Theorem 3.3.1 (Infinitesimal Torelli theorem for smooth hypersurfaces [Gri69a, Theorem 9.8(b)]). Let \( X \subset \mathbb{P}^{n+1} \) be a smooth hypersurface of degree \( d \) and suppose \( n \geq 3 \) and \( d \geq 3 \) or \( n = 2 \) and \( d > 3 \). Then, the infinitesimal period map

\[ dP^n : H^1(X, T_X) \rightarrow \bigoplus_{p=1}^{n} \text{Hom}(H^{p,n-p}(X), H^{p-1,n-p+1}(X)) \]

is injective.

The proof uses the description of cohomology groups using polynomial rings given in Section 2.3.
Let $X \subset \mathbb{P}^{n+1}$ be a projective hypersurface defined by a homogeneous polynomial equation $F(z_0, \ldots, z_{n+1}) = 0$ of degree $d$. Let $S = \mathbb{C}[z_0, \ldots, z_{n+1}]$ be the ring of polynomials and $J = \langle \frac{\partial F}{\partial z_i} \rangle$ be the Jacobian ideal. We denote by $S_d$, $J_d$ and $(S/J)_d$ the homogeneous parts of degree $d$.

In Section 2.3, we showed that there are isomorphisms $H^{p,n-p}(X)_{\text{prim}} = (S/J)_{(n-p+1)d-n-2}$ and $H^1(X, T_X) = (S/J)_d$. The infinitesimal period map factors through the map

$$\Pi : (S/J)_d \to \bigoplus_{p=1}^{n} \text{Hom}((S/J)_{(n-p+1)d-n-2}, (S/J)_{(n-p+2)d-n-2})$$

$$[P] \mapsto ([Q] \mapsto [P \cdot Q]).$$

The proof of the injectivity of $\Pi$ (and hence Theorem 3.3.1) relies on a key lemma.

**Lemma 3.3.2** (Macaulay’s theorem [Mac16, §86], cf. [CMP03, Theorem 7.4.1]). Let $(P_0, \ldots, P_{n+1})$ be a regular sequence of homogeneous polynomials of degrees $d_0, \ldots, d_{n+1}$ in $S$ and $\rho = \sum_{i=0}^{n+1} d_i - (n + 2)$. Then $(S/J)_l = 0$ for all $l > \rho$ and there is a perfect pairing

$$(S/J)_l \otimes (S/J)_{\rho-l} \to (S/J)_\rho \cong \mathbb{C}$$

induced by multiplication in $S$.

To prove Theorem 3.3.1, the lemma is applied to $P_i = \frac{\partial F}{\partial z_i}$. The sequence $(P_i)$ is regular if and only if the hypersurface $X \subset \mathbb{P}^n$ is smooth. The ideal $\langle P_0, \ldots, P_n \rangle$ is precisely the Jacobian ideal $J$ and $\rho = (n+2)(d-2)$.

However, for singular hypersurfaces, the radical ideal $I = \text{rad}(J)$ of the Jacobian is not the irrelevant ideal $m$, so the sequence $(\frac{\partial F}{\partial z_i})$ is not regular and Macaulay’s theorem cannot be applied. Indeed, the radical ideal $I = I(S\text{ing } X)$ is the ideal defining the singular locus of $X$.

To prove the infinitesimal Torelli theorem for certain singular hypersurfaces, we need an analogue of Macaulay’s theorem.

From now on, we shall assume that $n = 2$, so $X$ is a surface of degree $d$ in $\mathbb{P}^3$. We further assume that $X$ is a nodal surface.

Let $X \subset \mathbb{P}^3$ be a nodal surface defined by a homogeneous polynomial $F \in \mathbb{C}[z_0, \ldots, z_3]$ of degree $d$. Let $Z = \{p_1, \ldots, p_k\}$ be the set of nodes of $X$ and $F_i = \frac{\partial F}{\partial z_i}$ ($0 \leq i \leq 3$) be the partial derivatives of $F$. So, $Z$ is the zero locus of the set of partial derivatives. Let $J = \langle F_0, \ldots, F_3 \rangle$ be the Jacobian ideal of $F$ and $I = \sqrt{J}$ be its radical.
Let $E = \bigoplus_{i=0}^{3} \mathcal{O}_{\mathbb{P}^3}(d-1)$ and $s$ be the section of $E$ defined by $(F_0, \ldots, F_3)$. Consider the Koszul complex

$$
\mathcal{K}^\bullet = \left( 0 \rightarrow \bigwedge^4 E^\vee \rightarrow \bigwedge^3 E^\vee \rightarrow \bigwedge^2 E^\vee \rightarrow E^\vee \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow 0 \right)
$$

where the differential maps are contractions by the section $s$. We place $\mathcal{O}_{\mathbb{P}^3}$ in degree 0.

Since the zero locus of the section $s$ is the finite set $Z$, which is of codimension 3 in $\mathbb{P}^3$, by [CMP03, Problem 7.2.2(b)], the Koszul complex $\mathcal{K}^\bullet$ is exact in degrees $< -(4 - 3) = -1$. Note that the map $E^\vee \xrightarrow{(F_0, \ldots, F_3)} \mathcal{O}_{\mathbb{P}^3}$ is surjective away from $Z$, and the cokernel at each point of $Z$ is isomorphic to $\mathbb{C}$, so there is an exact sequence

$$
E^\vee \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_Z \rightarrow 0.
$$

Thus, the extended complex, which we shall also call $\mathcal{K}^\bullet$ by abuse of notation,

$$
\mathcal{K}^\bullet = \left( 0 \rightarrow \bigwedge^4 E^\vee \rightarrow \bigwedge^3 E^\vee \rightarrow \bigwedge^2 E^\vee \rightarrow E^\vee \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_Z \rightarrow 0 \right)
$$

is exact everywhere except in degree $-1$. Indeed, there is a quasi-isomorphism

$$
\mathcal{K}^\bullet \xrightarrow{\text{qis}} \ker(\text{im}(\bigwedge^2 E^\vee \rightarrow E^\vee))[1].
$$

The righthand side is supported on $Z$ since the complex $\mathcal{K}^\bullet$ is exact away from $Z$ by [CMP03, Problem 7.2.2(b) or Theorem 7.4.1]. We define $K$ to be the sheaf

$$
K := \frac{\ker(\mathcal{O}_{\mathbb{P}^3})}{\text{im}(\bigwedge^2 E^\vee \rightarrow E^\vee)}.
$$

Consider the twisted complex $\mathcal{K}^\bullet \otimes \mathcal{O}(l)$ for some integer $l$. We can associate to it the spectral sequence

$$
E_1^{p,q} = H^q(\mathbb{P}^3, K^p(l)) \Rightarrow H^{p+q}(\mathbb{P}^3, \mathcal{K}^\bullet \otimes \mathcal{O}(l)) = H^{p+q}(Z, K[1]),
$$

$$
d^r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}. \quad (3.3)
$$

Note that $H^{p+q}(\mathbb{P}^3, \mathcal{K}^\bullet \otimes \mathcal{O}(l)) = H^{p+q+1}(Z, K)$ is independent of the twist since it is supported on a finite set.

The cohomology group $E_1^{p,q} = H^q(\mathbb{P}^3, K^p(l))$ is zero except where $q = 0$ or $q = 3$, so we have $E_2 = E_3 = E_4$ and $E_5 = E_6 = \cdots = E_\infty$. Indeed, $E_r^{p,q} \neq 0$ only if $q = 0,3$ and $-4 \leq p \leq 1$. 40
Consider the map $d_4 : E_4^{-4,3} \to E_4^{0,0}$. We have

$$E_4^{-4,3} = \frac{\ker(E_1^{-4,3} \to E_1^{-3,3})}{\im(E_1^{-5,3} = 0 \to E_1^{-4,3})}$$

$$= \ker \left( H^3(\mathbb{P}^3, \bigwedge^4 \mathcal{E}^\vee(l)) \xrightarrow{\alpha_l} H^3(\mathbb{P}^3, \bigwedge^3 \mathcal{E}^\vee(l)) \right)$$

$$= \ker \left( H^3(\mathbb{P}^3, \mathcal{O}(\rho - 4 + l)) \xrightarrow{\alpha_l} H^3(\mathbb{P}^3, \bigoplus_{i=0}^3 \mathcal{O}(\rho - 5 + l + d)) \right)$$

$$\cong \coker \left( H^0(\mathbb{P}^3, \bigoplus_{i=0}^3 \mathcal{O}(\rho - l - d + 1)) \xrightarrow{\alpha_l^*} H^0(\mathbb{P}^3, \mathcal{O}(\rho - l)) \right)^\vee$$

$$= \coker \left( \bigoplus_{i=0}^3 S_{\rho - l - d + 1} \xrightarrow{\alpha_l^*} S_{\rho - l} \right)^\vee = (S/J)_{\rho - l}^\vee$$

where $\rho = 4d - 8$ and the fourth isomorphism is given by Serre duality. $\alpha_l$ is multiplication by the polynomials $(F_0, \ldots, F_3)$ and $\alpha_l^*$ is its dual. Similarly,

$$E_4^{0,0} = \frac{\ker(E_1^{0,0} \to E_1^{1,0})}{\im(E_1^{-1,0} = 0 \to E_1^{0,0})}$$

$$= \frac{\ker \left( H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) \to H^0(\mathbb{P}^3, \mathcal{O}_Z(l)) \right)}{\im(H^0(\mathbb{P}^3, \mathcal{E}^\vee(l)) \to H^0(\Proj^3, \mathcal{O}_{\mathbb{P}^3}(l)))}$$

$$= \frac{\ker \left( H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) \to H^0(\Proj^3, \mathcal{O}_Z(l)) \right)}{\im(H^0(\mathbb{P}^3, \bigoplus_{i=0}^3 \mathcal{O}(l - d_i)) \xrightarrow{\beta_l} H^0(\Proj^3, \mathcal{O}(l)))}$$

$$= \frac{I_l}{\im \left( \bigoplus_{i=0}^3 S_{l - d_i} \xrightarrow{\beta_l} S_l \right)} = (I/J)_l$$

where $\beta_l$ is multiplication by $(F_0, \ldots, F_3)$. Note that $\alpha_l^* = \beta_{\rho - l}$.

Thus, $d_4$ induces a morphism

$$d_4 : (S/J)_{\rho - l}^{\vee} \to (I/J)_l.$$

(3.4)

**Lemma 3.3.3.** The morphism $d_4$, restricted to $(I/J)_{\rho - l}^{\vee}$ induces a duality $(I/J)_{\rho - l}^{\vee} \cong (I/J)_l$. In particular, the morphism $d_4 : (S/J)_{\rho - l}^{\vee} \to (I/J)_l$ is surjective.

**Proof.** The composition of the map (3.4) for $l$ and $\rho - l$ gives

$(S/J)_{\rho - l}^{\vee} \to (I/J)_l \to (I/J)_{\rho - l}^{\vee}$ and $(S/J)_l^{\vee} \to (I/J)_{\rho - l} \to (I/J)_l^{\vee}$
which are dual to the inclusion $I/J \to S/J$. Hence, there is a natural duality $(I/J)_l \cong (I/J)^\vee_{\rho - l}$.

We can also prove directly that $d_4$ is surjective. Since $E_4^{0,-3} = 0$, we have $E_5^{0,0} = \text{coker}(E_4^{-4,3} \xrightarrow{d_4} E_4^{0,0})$. The spectral sequence of (3.3) converges to $H^{p+q}(Z, K[1])$ and $H^0(Z, K[1]) = H^1(Z, K) = 0$ since $Z$ is a finite set, so $E_5^{0,0} = E_{\infty}^{0,0} = 0$ and $d_4$ is surjective. □

**Remark 3.3.4.** Unlike the regular case (Macaulay’s theorem, Lemma 3.3.2), the perfect pairing

$$(I/J)_{\rho - l} \otimes (I/J)_l \to \mathbb{C}$$

is not induced by multiplication of polynomials in $I$. This is clear since $(I/J)_{\rho} = 0$.

We can however show a weaker form of the multiplication property (Lemma 3.3.11). To do so we need to relate the algebraic independence of the partial derivatives $F_i$ and the independence of the set of nodes.

**Definition 3.3.5.** Let $Z = \{p_1, \ldots, p_k\}$ be a set of points on a hypersurface $X$ and $\{e_i\}$ be a basis for $S_l$, the vector space of homogeneous polynomials of degree $l$. We say that the set $Z$ imposes independent conditions in degree $l$ (or simply, is independent in degree $l$) if the rank of the matrix $(e_i(p_j))_{i,j}$ is $k$.

It is clear from the definition that if $Z$ is independent in degree $l$, then it is independent in all degrees $\geq l$.

**Remark 3.3.6.** The kernel of the matrix $(e_i(p_j))_{i,j}$ is the vector space $I_l$.

So, if $Z$ is a set of $k$ points, then $Z$ is independent in degree $l$ if and only if $\dim S_l - \dim I_l = k$.

There are numerous results regarding independence of nodes. The main result that we shall use in this thesis is that of Severi. He showed that the set of nodes $Z$ on a nodal surface $X \subset \mathbb{P}^3$ is independent in degree $2d - 5$ [Sev46, §14].

A recent work of Mustață and Popa uses mixed Hodge modules and Hodge ideals to give a vast generalization of this lemma. They showed that for any reduced hypersurface $D \subset \mathbb{P}^{n+1}$ of degree $d$, with isolated singularities of multiplicity $m$, the set $Z$ of singular points imposes independent conditions in degree $\left(\left\lfloor \frac{n+1}{m} \right\rfloor + 1 \right)d - n - 2$ [MP16, Corollary H]. However, in the case of nodal surfaces, their result is slightly weaker, it only yields independence in degree $2d - 4$, which is insufficient for proving the infinitesimal Torelli theorem for nodal surfaces.
To prove the infinitesimal Torelli theorem, we need conditions on the algebraic independence of the partial derivatives $F_i$.

**Lemma 3.3.7.** Let $X \subset \mathbb{P}^3$ be a nodal surface defined by a homogeneous polynomial $F$ of degree $d$. Then, the partial derivatives $F_i = \frac{\partial F}{\partial x_i}$ are algebraically independent in degrees $\leq 2d - 4$, that is, if $\sum_{i=0}^{3} H_i F_i = 0$ with $\deg H_i F_i \leq 2d - 4$, then $H_i = 0$ for all $i$.

**Proof.** Since the Jacobian ideal $J$ is generated by the four partial derivatives $F_i$, which are of degree $d - 1$, the homogeneous module $J_{2d-4}$ is generated by the products $F_i z_{j_1} \cdots z_{j_{2d-4}}$ where $0 \leq j_1 \leq \cdots \leq j_{2d-4} \leq 3$. Hence, $\dim J_{2d-4} \leq 4 \dim S_{d-3} = 4(d_3)$. The partial derivatives are algebraically independent in degrees $\leq 2d - 4$ if and only if the given set of generators is linearly independent, i.e. $\dim J_{2d-4} = 4(d_3)$.

Let $Z = \{p_1, \ldots, p_k\}$ be the set of nodes. Since $I = I(Z)$ is the ideal of polynomials that vanish on $Z$, the homogeneous part $I_{2d-4}$ is generated by the kernel of the matrix $(e_i(p_j))_{i,j}$, which has dimension at least $\dim \ker (\tilde{\omega}_X) = 4(d_3)$.

By Proposition 2.3.6, we have $\dim (I/J)_{2d-4} = h^1(\tilde{\omega}_X)^\text{prim} = h^1(\tilde{\omega}_X^1) = 1$. Let $\pi : \tilde{X} \to X$ be a resolution of singularities of $X$. By Proposition 2.2.21, there is an isomorphism $\tilde{\omega}_X^1 = R\pi_* \omega_X^1(\log E)$. The distinguished triangle

$$R\pi_* \omega_X^1 \to R\pi_* \omega_X^1(\log E) = \tilde{\omega}_X^1 \to R\pi_* E \xrightarrow{+1}$$

gives a long exact sequence

$$0 \to H^{1,0}(\tilde{X}) \to H^{1,0}(X) \to H^0(E, O_E) \cong \mathbb{C}^k \to H^{1,1}(\tilde{X}) \to H^{1,1}(X) \to 0.$$  

So, $h^{1,1}(\tilde{X}) = h^{1,1}(X) + k$, and by Noether’s formula, for a smooth hypersurface,

$$h^{1,1}(\tilde{X}) = 10\chi(O_{\tilde{X}}) - K_{\tilde{X}}^2 = 10 \left(1 + \binom{d-1}{3}\right) - d(d-4)^2.$$  

We can thus compute the dimension of the vector space $J_{2d-4}$ to be

$$4 \binom{d}{3} \geq \dim J_{2d-4} = \dim I_{2d-4} - \dim (I/J)_{2d-4} \geq \dim S_{2d-4} - k - h^1(\tilde{\omega}_X^1) + 1 = \binom{2d-1}{3} - k - h^{1,1}(\tilde{X}) + k + 1 = 4 \binom{d}{3}. \quad (3.5)$$

Hence, equality holds throughout, and we conclude that the partial derivatives are algebraically independent in degree $\leq 2d - 4$.  

\hfill $\square$
Note the fact that equality holds in (3.5) implies that \( \dim S_{2d-4} - \dim I_{2d-4} = k \), so by Remark 3.3.6, the nodes are independent in degree \( 2d - 4 \). Thus, we have given an alternate proof of the result.

We now want to further show that the partial derivatives are algebraically independent in degrees up to \( 2d - 3 \).

Recall the surjective map \( d_4 : (S/J)^\vee \rightarrow (I/J)_l \) (cf. (3.4)) induced from the spectral sequence (3.3). We try to understand the kernel of the map \( d_4 \). Note that \( E_5^{-1,0} = \ker(E_4^{-4,3} \xrightarrow{d_4} E_4^{0,0}) \) and

\[
H^0(Z, K) = H^{-1}(X, K[1]) = E_5^{-4,3} \oplus E_5^{-1,0}.
\]

We can compute

\[
E_5^{-1,0} = E_2^{-1,0} = \frac{\ker \left( H^0(\mathbb{P}^3, \mathcal{E}^\vee(l)) \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) \right)}{\text{im} \left( H^0(\mathbb{P}^3, \bigwedge^2 \mathcal{E}^\vee(l)) \rightarrow H^0(\mathbb{P}^3, \mathcal{E}^\vee(l)) \right)}
\]

\[
= \frac{\ker \left( \bigoplus_{i=0}^3 S_{l-d+1} \rightarrow S_l \right)}{\text{im} \left( \bigoplus_{0 \leq i < j \leq 3} S_{l-2d+2} \rightarrow \bigoplus_{i=0}^3 S_{l-d+1} \right)}.
\]

Note that \( J_l = \text{im} \left( \bigoplus_{i=0}^3 S_{l-d+1} \rightarrow S_l \right) \).

**Definition 3.3.8.** We define the defect \( \text{Dft}(J_l) \) of \( J_l \) be the kernel of the map \( \bigoplus_{i=0}^3 S_{l-d+1} \rightarrow S_l \) and let \( \text{dft}(J_l) = \dim \text{Dft}(J_l) \). Then \( \text{dft}(J_l) + \dim J_l = 4\dim S_{l-d+1} \).

**Proposition 3.3.9.** The partial derivatives \( F_i \) are algebraically independent in degree \( 2d - 3 \), i.e. \( \text{dft}(J_{2d-3}) = 0 \).

**Proof.** Note that for \( l < 2d - 2 \), we have \( S_{l-2d+2} = 0 \), so \( E_5^{-1,0} = \text{Dft}(J_l) \). Thus, by the equation (3.6), for \( l < 2d - 2 \), there are isomorphisms

\[
H^0(Z, K) = \ker \left( (S/J)^\vee \xrightarrow{d_4} (I/J)_l \right) \oplus \text{Dft}(J_l).
\]

By Lemma 3.3.3, the map \( d_4 \) is surjective, so we can compute the dimension of \( H^0(Z, K) \) as

\[
h^0(K) = \dim S_{p-l} - \dim J_{p-l} - \dim I_l + \dim J_l + \dim \text{dft}(J_l).
\]

Now we restrict to the cases where \( 2d - 5 \leq l \leq 2d - 3 \). Severi [Sev46, §14] showed that the set of nodes on a nodal surface is independent in degrees \( \geq 2d - 5 \), so \( \dim S_l - \dim I_l = k \).

Recall that \( \text{dft}(J_l) + \dim J_l = 4\dim S_{l-d+1} \). Thus, for \( 2d - 5 \leq l \leq 2d - 3 \), we have

\[
h^0(K) = \dim S_{p-l} - 4 \dim J_{p-l-d+1} + \dim \text{dft}(J_{p-l}) - \dim S_l + k + 4\dim J_{l-d+1}.
\]
Since \( h^0(K) \) is independent of \( l \), we can compare the dimensions for \( l = 2d - 5 \) and \( l = 2d - 3 \) to obtain \( \text{dft}(J_{2d-3}) = \text{dft}(J_{2d-5}) = 0 \) by Lemma 3.3.7.

**Remark 3.3.10.** Proposition 3.3.9 gives the largest possible degree for algebraic independence. One can verify on Explicit examples 4.1.15 and 4.2.7 that the partial derivatives \( F_i \) are no longer algebraically independent in degree \( 2d - 2 \).

**Lemma 3.3.11.** Let \( l < 2d - 4 \) and \( G \in I_l \). Suppose \( Gz_j \in J_{l+1} \) for all \( j = 0, \ldots, 3 \), then \( G \in J_l \).

**Proof.** Let \( F_k = \frac{\partial F}{\partial z_k} \) and \( Gz_j = \sum_k H_{jk} F_k \) for each \( j \) with \( H_{jk} \) of degree \( l - d + 2 \leq l \). Then, for any \( i \neq j \),

\[
0 = G(z_j z_i - z_i z_j) = \sum_{k=0}^3 (z_i H_{jk} - z_j H_{ik}) F_k.
\]

By Proposition 3.3.9, the \( F_i \)'s are algebraically independent in degree \( l + 2 \leq 2d - 3 \), so the coefficients \( z_i H_{jk} - z_j H_{ik} = 0 \) for all \( i, j, k \). In particular, \( z_j \) divides \( H_{jk} \) for each \( j \). Let \( H_{jk} = H'_{jk} z_j \), then \( G = \sum_k H'_{jk} F_k \) lies in \( J_l \). \( \square \)

We can now conclude:

**Proposition 3.3.12.** Let \( X \subset \mathbb{P}^3 \) be a nodal surface of degree \( d \geq 4 \). Then, the map

\[
dP^2 : H^1(X, \hat{T}_X) \to \text{Hom}(H^0(X, \hat{\omega}_X), H^1(X, \hat{\Omega}_X))
\]

is injective.

**Proof.** The map \( dP^2 \) factors through

\[
\Pi : H^1(X, \hat{T}_X) = (I/J)_d \rightarrow \text{Hom}(H^0(X, \hat{\omega}_X), H^1(X, \hat{\Omega}^1_X)_{\text{prim}}) = \text{Hom}(S_{d-4}, (I/J)_{2d-4})
\]

which is given by multiplication of polynomials. Suppose \([P] \in (I/J)_d \) is in the kernel of \( \Pi \), that is, \( P \cdot Q \in J_{2d-4} \) for all \( Q \in S_{d-4} \). We shall prove that \( P \in J_d \), so \([P] = 0 \in (I/J)_d \).

Suppose, to the contrary, that \( P \notin J_d \), then there exists a homogeneous polynomial \( R \in S_{<d-4} \) of maximal degree such that \( P \cdot R \notin J \). By hypothesis, \( P \cdot R \cdot z_i \in J \) for all \( i = 0, \ldots, 3 \). Lemma 3.3.11 then implies that \( P \cdot R \in J \), giving the contradiction. \( \square \)
Remark 3.3.13. In [EM15, Corollaire 5.2.2], Eyssidieux and Mégy proved a similar result, that

\[ \Pi : (I/J)_d \to \text{Hom}((S/J)_{d-n-2}, (I/J)_{2d-n-2}) \]

is injective. However, they needed to impose the condition that \((I/J)_{d-n-2} \geq (I/J)_{2d-n-2}\) is generated by \((I/J)_{d-n-2}\). Our result, though much more restricted, avoids this condition. In fact, for all our examples of nodal sextic surfaces in Chapter 4, \((I/J)_{d-n-2} = (I/J)_{2d-n-2}\) are zero.

Remark 3.3.14. The proof of Proposition 3.3.9 suggests that there is some form of duality between the algebraic independence of \(F_i\) in degree \(l\) and the independence of the nodes in degree \(\rho - l\), for example, an equality \(dft(J_{\rho - l}) = k'_l\) where \(k'_l \leq k\) is the rank of the matrix \((e_i(p_j))_{i,j}\) determining the dependence of the nodes.

However, such an equality cannot hold for \(l \geq 2d - 2\) because the term \(\text{im} \left( \bigoplus_{0 \leq i < j \leq 3} S_{l-2d+2} \to \bigoplus_{i=0}^3 S_{l-d+1} \right) \) in \(E^{-1,0}_5\) is non-zero. If we can understand this term better, it may be possible to use Mustață and Popa results [MP16, Corollary H] on independence of nodes to prove the injectivity of the infinitesimal period map for hypersurfaces in higher dimensions with isolated singularities.

To prove the infinitesimal Torelli theorem, we further assume that \(X = M/G\) is a quotient surface and \(G\) is small in the neighbourhoods of all points on the singular locus of \(X\). This assumption is not essential, but we have only defined the infinitesimal period map for quotient surfaces, and not for general V-manifolds. We need to show that the Hodge numbers are constant in an open neighbourhood of \(X\) on a versal family of \(G\)-equivariant deformations of \(M\).

Proposition 3.3.15. Let \(X \subset \mathbb{P}^{n+1}\) be a projective hypersurface of degree \(d\) such that \(X\) is a V-manifold. Suppose \(n \geq 2\) and \((n, d) \neq (2, 4)\). Then all small deformations of \(X\) parametrized by \(H^1(X, \hat{T}_X)\) are projective hypersurfaces of degree \(d\), preserving the singularities of \(X\).

Proof. By Proposition 2.3.7 and Remark 2.3.8, there is an isomorphism

\[ H^1(X, \hat{T}_X) = H^1(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(-\log X)) \]

under the hypothesis \(n \geq 2\) and \((n, d) \neq (2, 4)\). By Example 3.1.3, the latter parametrizes all infinitesimal deformations of \(X\) in \(\mathbb{P}^{n+1}\) that preserve the singularities. Since there is a unique linear system \(\mathcal{O}_{\mathbb{P}^{n+1}}(d) = \mathcal{O}_{\mathbb{P}^{n+1}}(X)\) in \(\mathbb{P}^{n+1}\) of degree \(d\), all deformations of \(X\) are projective hypersurfaces of degree \(d\). \(\square\)
**Theorem 3.3.16.** Let $X \subset \mathbb{P}^3$ be a nodal surface of degree $d \geq 5$. Then $X$ satisfies the infinitesimal Torelli property.

**Proof.** The injectivity of the infinitesimal period map was already proven in Proposition 3.3.12. It remains to show that the period domain is well-defined in a small neighbourhood $U$ of $X$ in the moduli family parametrized by $H^1(X, T_X)$.

Suppose $X$ has $k$ nodes. By Proposition 3.3.15, any $X' \in U$ is also a nodal surface of degree $d$ with $k$ nodes. The Hodge numbers of a nodal surface of degree $d$ is only dependent on the number of nodes, hence the Hodge numbers are constant on $U$ and the period domain is well-defined. □

**Remark 3.3.17.** Deformations of nodal surfaces have been studied in greater depth by Burns and Wahl [BW74]. They showed that the formal moduli space of surfaces of degree $d$ with at most nodal singularities extends that of smooth surfaces of degree $d$, and is a reduced complete intersection of dimension $\binom{d+3}{3} - 16$ for $d \geq 5$ (Corollary 2.11).

They further showed that $G$-equivariant deformations of a nodal surface $X$ of degree $d$ is unobstructed if and only if the set $Z$ of nodes is independent in degree $d$ [BW74, Corollary 4.3]. Thus, if $d \leq 5$, all $G$-equivariant deformations are unobstructed by Severi’s result on independence of nodes. For the two examples we give in Chapter 4 with $d = 6$, we show directly in both cases that the infinitesimal deformations are unobstructed (Propositions 4.1.14 and 4.2.12), and hence the nodes are independent in degree 6. This gives an improvement over Severi’s result.