Modelling Digital Quantum Simulation of the Rabi Model in Circuit QED:
Towards an Experimental Implementation of Deep-Strong Coupling Dynamics

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Modelling Digital Quantum Simulation of the Rabi Model in Circuit QED: Towards an Experimental Implementation of Deep-Strong Coupling Dynamics

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Abstract

The simplest form of dipole interaction between an atom and a single photon field mode, is described by the vacuum Rabi model. Strong atom-photon coupling, which is described by the simpler Jaynes-Cummings model, has been achieved in many platforms, such as cavity and circuit quantum electrodynamics (QED) and has brought a lot of success towards experimental quantum information processing over the past years. The full Rabi model dynamics can only be obtained in the so-called ultra-strong and deep-strong coupling regimes where the interaction coupling strength is comparable or higher than the natural system frequencies. However, due to our inability to achieve such high coupling strengths, these regimes remain largely unexplored in the lab. In this thesis, we investigate the possibility of reaching these regimes in a circuit QED setup, by means of a recently proposed analog-digital quantum simulation. Following a detailed numerical model of the proposed scheme, where we include the most important experimental limitations, we demonstrate the feasibility of the proposal using a transmon coupled to a 2D superconducting resonator, for a certain range of design parameters. Moreover, we show that the Wigner function representation of the resonator state in phase space is instrumental in order to probe the signature of deep-strong coupling in the system. Following these results, we design a device that will enable us to carry out the experiment with high fidelity measurements and perform direct Wigner tomography inside the resonator.
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Chapter 1

Introduction

1.1 Quantum light-matter interactions

The simplest model for quantum light-matter interaction was introduced by Rabi in 1936 [1], and describes the dipolar coupling of a two-level atom with a single mode photon field.

The ability to experimentally achieve couplings that are much bigger than the system decay rates has allowed for an extensive study of this interaction in many platforms, including atoms and ions in strongly confined cavity field [2, 3], as well as microfabricated artificial atoms coupled to 2D and 3D gigahertz resonators [4]. Typically, coupling strengths are much lower than the natural frequencies in the system and the dynamics reduce to those of the Jaynes-Cummings model [5], following a rotating wave approximation (RWA) where only the excitations-conserving terms are considered. The exact solvability of this toy model and the developed dressed atom formalism involving coherent population Rabi oscillations, have enabled precise control of these systems which has led to many milestones in quantum state engineering [6].

The full Rabi model dynamics, however, can only be explored in regimes where the couplings ($g$) are comparable or even greater than the system frequencies ($\omega$). In these, so-called ultra-strong ($g/\omega \lesssim 1$) and deep-strong coupling ($g/\omega \gtrsim 1$) regimes, the rotating wave approximation breaks down and counter-intuitive dynamics appear [7]. Despite the experimental approaches towards achieving ultra-strong coupling in many platforms, including circuit QED [8, 9], ratios of $g/\omega \sim 1$ still remain unachievable in the lab. Deep-strong coupling dynamics are therefore far from being achieved and their experimental investigation still remains a challenge.
Introduction

1.2 Simulating nature with quantum mechanics

Modelling and performing simulations of physical systems is a fundamental part of the huge scientific and technological progress nowadays. With the power of today’s classical computers, researchers in almost all scientific areas are able to extract information about systems of interest with speed and precision beyond human capabilities. However, the complexity of quantum systems, makes it difficult to calculate their properties due to the Hilbert space dimensions growing exponentially with the number of particles, setting a limit in the number of systems that we can simulate exactly.

In 1982, R. Feynman proposed the idea of using a quantum simulator, instead of a classical one, to simulate such systems [10]. His vision was that, if we have a quantum system which we are able to control with high precision, then by tuning some of the parameters of that system we might be able to reveal information about another quantum system of interest that shares the same dynamics. Provided that we are able to engineer and control such a quantum device, the amount of time required for a calculation will scale polynomially (not exponentially) with the number of particles. Moreover, as S. Lloyd has demonstrated in 1996, a universal quantum simulator can be built by using digital methods [11]. In particular, he observed that the dynamics of any quantum system can be approximated by a sequence of operations in very small time steps, using the Trotter decomposition [12]. Therefore, the reproduction of the dynamics of any system should be possible by applying a series of well-controlled quantum gates at very short time steps.

Quantum simulations offer not only the possibility to perform exact calculations of complex quantum systems, but also to experimentally access phenomena that have never been observed or even not exist in nature. It is a highly expanding field with many potential applications in a number of areas in physics, chemistry or even biology [13]. In fact, many proof-of-principle experiments have been realised so far in a variety of platforms, such as cold atoms [14, 15], trapped ions [16, 17], NMR [18, 19] and superconducting circuits [20–22].

1.3 Research focus and thesis overview

In this thesis, we investigate the possibility of experimentally reaching the largely unexplored deep-strong coupling regime of the Rabi model in a circuit QED setup, by means of an analog-digital quantum simulation that
1.3 Research focus and thesis overview

has recently been proposed by A. Mezzacapo et al. [23]. We want to achieve this in a circuit QED setup, using a superconducting transmon qubit coupled to a transmission line resonator. Our goal is to investigate whether this proposal is feasible with current state of the art architecture, and if so, in what parameter regimes. For this reason, we build up a numerical model description of the quantum simulations scheme including real-world experimental considerations of our system. Moreover, we want to understand the key features of the DSC dynamics and find possible ways to identify them in the experiment.

In chapter 2, we introduce the theoretical background as well as the motivations for this project. A brief description of the quantum Rabi model is followed by an introduction to circuit QED using superconducting transmon qubits. Finally, we present the ideas of universal quantum simulations and conclude with a description of the proposed analog-digital quantum simulation of the Rabi model in circuit QED.

In chapter 3, we present a step by step description of our numerical model. We discuss the master equation description of open quantum systems and implement all the necessary elements for an accurate modelling of the proposed quantum simulations scheme.

In chapter 4, we present our numerical results concerning the feasibility of the proposal in realistic circuit QED scenarios. Moreover, a detailed study of deep-strong coupling Rabi model dynamics is carried out in order to identify the key signatures of this regime.

In chapter 5, we describe the design of a chip based on these results. We implement all the necessary elements and design the key parameters for a high fidelity quantum simulation experiment.

Finally, in chapter 6, we summarise the conclusions of this work.
Chapter 2

Theory

2.1 The vacuum Rabi model

The quantum Rabi model [1, 2] describes the simplest interaction between quantum light and matter, i.e. the dipolar interaction of a two-level atom (qubit) coupled to a single quantized electromagnetic field mode. The dynamics of the coupled system are described by:

\[ H_R = \hbar \left[ \omega_R a^\dagger a + \frac{\omega_R}{2} \sigma^2 + g_R (\sigma^+ + \sigma^-)(a^\dagger + a) \right], \]

where the last term describes the interaction between the atomic dipole \((\sigma^+ + \sigma^-)\) and the photon electric field \((a^\dagger + a)\), governed by the coupling strength \(g_R\). Here, \(a^\dagger\) and \(a\) are the creation and annihilation operators of the bosonic field and \(\sigma^+ = |e\rangle \langle g|\), \(\sigma^- = |g\rangle \langle e|\), \(\sigma^2 = |g\rangle \langle g| - |e\rangle \langle e|\) are the Pauli operators of the qubit, where \(|g\rangle\) and \(|e\rangle\) denote the ground and excited states, respectively. When the atom and photon frequencies are nearly on resonance \((|\omega_R^g - \omega_R^e| \ll g_R\)), the interaction term is dominating and the atomic and bosonic fields become strongly correlated with the exchange of photon excitations.

On the theoretical side, despite its simplicity, an exact analytical solution of the quantum Rabi model in all parameter regimes has only been achieved recently by D. Braak [24]. More surprisingly, the generalised Dicke model [25] where \(N\) atoms are coupled to a photon field remains unsolvable for \(N > 3\).

Strong coupling is achieved when the coupling strength is much bigger than the decay rates \(\gamma, \kappa\) of the atom and photon field, respectively. Typically, the coupling strengths are much lower than the system natural
frequencies \( (g^R \ll \omega_q^R, \omega_r^R) \) and following a rotating wave approximation (RWA) where the fast oscillating counter-rotating terms \( \sigma^+ a^+ \), \( \sigma^- a^- \) are neglected \([26]\), the dynamics are accurately described by the Jaynes-Cummings model \([5]\):

\[
H = \hbar \omega_r a^+ a + \frac{\hbar \omega_q}{2} \sigma^z + g(\sigma^+ a^+ + \sigma^- a^-),
\]

which is exactly solvable.

Jaynes-Cummings physics have been studied extensively in many platforms, including cavity and circuit quantum electrodynamics (QED) \([2, 4]\), where atoms/qubits are strongly confined in cavities/resonators. An excitation in these systems is coherently reabsorbed and re-emitted several times, leading to entanglement between the qubit and the resonator, for very long timescales. The ability to control these dynamics provides one of the most important physical resources towards quantum information processing.

As one increases the atomic and photonic frequencies with respect to the coupling strength, however, the RWA breaks down and the Jaynes-Cummings model is not sufficient to describe the dynamics. One enters the so-called \textit{ultra-strong} \( (g/\omega_q,\omega_r \gtrsim 0.1) \) and \textit{deep-strong} \( (g \gtrsim \omega_q,\omega_r) \) coupling regimes of the Rabi model, that require the full Rabi Hamiltonian to be described.
Although ultra-strong coupling has been confirmed in many setups including circuit QED [8, 9], the deep-strong coupling regime which contains counter-intuitive dynamics, still remains largely unexplored experimentally.

## 2.2 Superconducting qubits

Superconducting qubits are effective nonlinear oscillators behaving as artificial atoms that are constructed by simple superconducting circuits based on LC oscillators [28]. A Josephson tunnel junction then introduces nonlinearity to the system, such that one obtains anharmonic oscillator behaviour that renders it an artificial two-level atom.

![Schematic of the anharmonic potential of a superconducting qubit.](image)

**Figure 2.2:** Schematic of the anharmonic potential of a superconducting qubit.

The main advantage over other architectures is that the Hamiltonian parameters are effectively designed using advanced fabrication techniques. One can in principle distinguish three types of superconducting qubits, namely the charge (also known as “Cooper-pair box”), the flux and the phase qubit.

### 2.2.1 From the Cooper-pair box to the transmon

The Cooper-pair box (CPB) [29] is a type of charge qubit, that consists of a small superconducting island connected to a superconducting reservoir via a Josephson junction. The energy needed for a Cooper-pair to cross the junction is set by the Josephson energy $E_J$, while the energy needed for adding extra pairs in the system is set by the charging energy $E_C = \frac{e^2}{2C}$. 
Here, $C_\Sigma = C_q + C_g$ is the total capacitance from the island to the environment, where $C_q$ is the capacitance between islands and $C_g$ the capacitance between island and gate.

Its dynamics are described by the following Hamiltonian:

$$H_{\text{CPB}} = 4E_C (\hat{n} - n_g)^2 - E_J \cos \hat{\phi}, \quad (2.3)$$

where $n_g = \frac{C_g V_g}{2e} + \frac{Q_r}{2e}$ is the just a normalised effective offset charge potentially caused by biasing the gate voltage $C_g V_g$ or coming from the environment ($Q_r$). The operators $\hat{n}$, $\hat{\phi}$ describe the Cooper-pair number transferred between the islands and the superconducting phase difference between them, respectively.

The main issue for this type of qubit is charge noise, which makes it loose its coherence faster. An approach for improving the CPB has been proposed in [30], which renders the qubit insensitive to charge noise by operating in the so-called transmon regime, where $E_J / E_C \gg 1$. The transmon qubit consists of two superconducting islands connected through two Josephson junctions. The transmon regime is achieved by adding a large shunting capacitance $C_B$ that connects the superconductors, while increasing $C_g$.

\[\text{Figure 2.3: a) Circuit diagram of the Cooper-pair box. b) Circuit diagram of the transmon qubit. Figure obtained from [30]}\]

The transmon Hamiltonian is the same as $H_{\text{CPB}}$ and the qubit ground-excited state transition energy is approximately given by

$$E_{01} \simeq \sqrt{8E_J E_C} - E_C. \quad (2.4)$$

An important quantity is the anharmonicity of the transmon defined as
the difference between the $0-1$ and $1-2$ levels transition frequencies

\[ \alpha = \frac{1}{\hbar} (E_{12} - E_{01}) .\]

The higher the anharmonicity, the more the transmon behaves like a qubit. Approximately the anharmonicity is given by $\alpha \simeq -E_C/\hbar$ [30].

### 2.2.2 Circuit quantum electrodynamics

In circuit quantum electrodynamics (circuit QED), superconducting qubits serve as the (artificial) atoms that are coupled to superconducting resonators which provide electromagnetic field modes [4]. In a 2D architecture, a resonator is made of a transmission line, that behaves as a chain of LC oscillators [31], effectively following quantum harmonic oscillator dynamics [32].

![Figure 2.4: Schematic of a transmon coupled to a transmission line resonator.](image)

The interaction of a transmon coupled to such a coplanar waveguide (CPW) resonator takes place via a dipole coupling term $\sim \hat{n}(a^\dagger + a)$ and the system is described by the generalised Rabi Hamiltonian [30]

\[ H = \hbar \sum_j \omega_j |jangle \langle j| + \hbar \omega_r a^\dagger a + \hbar \sum_{i,j} g_{i,j} |i\rangle \langle j| (a^\dagger + a) , \]  

(2.5)

where $g_{i,j} \propto \langle i|\hat{n}|j\rangle$ denotes the coupling strength of the interaction, and $\omega_r$, $\omega_q$ are the resonator and transmon eigenfrequencies.
Following a rotating-wave approximation (RWA), where the counter-rotating terms that simultaneously excite or de-excite the transmon and the resonator are neglected, the dynamics are reduced to those of the generalised Jaynes-Cummings Hamiltonian:

\[
H = \hbar \sum_j \omega_j |j\rangle \langle j| + \hbar \omega_r a^+ a + \hbar \left( \sum_i g_{i,i+1} |i\rangle \langle i+1| a^+ + \text{h.c.} \right).
\] (2.6)

2.3 Quantum simulations

In spite of the huge amount of progress in circuit QED over the past years, it has not been possible to move towards the DSC regime of the Rabi and Dicke models since it is impossible to design coupling strengths that are comparable to the natural frequencies of the system, with current state of the art architecture. In this section, we present a recent proposal for reaching these regimes in circuit QED by means of an analog-digital quantum simulation.

2.3.1 Analog and digital implementations

The notion of quantum simulations was firstly proposed by R. Feynman, in 1982, and refers to the intentional reproduction of the dynamics of a physical quantum system using another quantum system which can be precisely controllable [10]. A successful quantum simulator should consist of quantum systems with sufficient degrees of freedom and a set of appropriate interactions between the elements of the system. In addition, one must be able to prepare the system in arbitrary states as well as perform individual and collective measurements on the system.

There are two types of quantum simulations, namely the analog and the digital quantum simulation. In the first case, the simulators share the same dynamics as the simulated systems, such that simply by adjusting the system parameters, e.g. coupling strengths or transition frequencies, it is possible to simulate the desired Hamiltonian. Thus, it becomes apparent that analog simulators can simulate a limited number of systems and as a result one needs to find additional methods in order to be able to achieve universal quantum simulations.
2.3 Quantum simulations

Breaking the evolution into Trotter steps

In 1996, S. Lloyd proposed a method for simulating any local quantum system [11]. His argument begins with the observation that any Hamiltonian system with local interactions can be written as the sum of \( l \) local Hamiltonians

\[
H = \sum_{i=1}^{l} H_i,
\]  

(2.7)

where each one of them acts on a local Hilbert space of \( m_i \) dimensions.

The unitary evolution can thus be approximated by dividing time into \( n \) infinitesimal slices of duration \( t/n \) each, and applying sequentially the evolution operator of each local term for each time interval (Trotter steps). The sequence should then be repeated \( n \) times, i.e. implementing the Trotter formula [12]:

\[
e^{iHt} = \lim_{n \to \infty} (e^{iH_1t/n} \ldots e^{iH_l t/n})^n.
\]  

(2.8)

However, since the Hamiltonians do not generally commute, this is approximately true for a large number of steps \( n \). The error in the above approximation is determined by the Lie-Suzuki-Trotter formula [33]

\[
\sum_{i>j} [H_i, H_j] \frac{t^2}{2n} + \sum_{k=3}^{\infty} E(k),
\]  

(2.9)

where the higher order terms are bounded by the condition

\[
||E(k)||_{sup} \leq \frac{n||Ht/n||^k_{sup}}{k!}
\]  

(2.10)

and the total error doesn’t exceed \( ||n(e^{iHt/n} - 1 - iHt/n)||_{sup} \).

As a result, the efficiency in simulating a quantum system with \( N \) variables depends highly on the number of Trotter steps \( n \) on a given amount of time, i.e. the more steps one does the better the approximation is. Moreover, the number of local Hamiltonians \( l \) must be a polynomial function of \( N \) [11].

2.3.2 An analog-digital quantum simulation of the Rabi model in circuit QED

Recently, A. Mezzacapo et al. proposed an analog-digital quantum simulation of the quantum Rabi model that could possibly be implemented
experimentally in a circuit QED architecture [23]. The idea lies in the observation that the Rabi Hamiltonian in (2.1) can be decomposed in two parts, $H_R = H_{JC}^{(1)} + H_{AJC}^{(2)}$ where

$$H_{JC}^{(1)} = \hbar \left[ \frac{\omega_R}{2} a^+ a + \frac{\omega_q^{(1)}}{2} \sigma^z + g_R (\sigma^+ a + \sigma^- a^+) \right]$$

$$H_{AJC}^{(2)} = \hbar \left[ \frac{\omega_R}{2} a^+ a - \frac{\omega_q^{(2)}}{2} \sigma^z + g_R (\sigma^+ a^+ + \sigma^- a) \right]$$

with $\omega_q^{(1)} - \omega_q^{(2)} = \omega_R^q$.

The first part is simply the Jaynes-Cummings Hamiltonian whereas the second part (called the anti-Jaynes-Cummings) can be simulated by applying a local qubit $\pi$ rotation along the $\hat{x}$ axis before and after $H_{JC}$ with a different detuning for the qubit frequency:

$$H_{AJC}^{(2)} = e^{-i\pi \sigma_x/2} H_{JC}^{(2)} e^{i\pi \sigma_x/2}.$$  (2.13)

Moving to the interaction picture of a frame rotating at frequency $\tilde{\omega}$, the ladder operators transform as

$$a(t) = ae^{-i\tilde{\omega} t}, \sigma^-(t) = \sigma^- e^{-i\tilde{\omega} t}$$  (2.14)

and we have

$$H_{JC}^{(1)} = \hbar \left[ \Delta_r a^+ a + \Delta_q^{(1)} \sigma^z + g (\sigma^+ a + \sigma^- a^+) \right]$$

$$H_{AJC}^{(2)} = \hbar \left[ \Delta_r a^+ a - \Delta_q^{(2)} \sigma^z + g (\sigma^+ a^+ + \sigma^- a) \right]$$

with $\Delta_r = \omega_r - \tilde{\omega}$, $\Delta_q^{(1),(2)} = \omega_q^{(1),(2)} - \tilde{\omega}$.

The analog part of the simulation consists of simulating the Jaynes-Cummings and anti-Jaynes-Cummings terms, which can be realised straightforwardly in a circuit QED setup. A digital quantum simulation of the Rabi Hamiltonian $H_R$ can then be performed in Trotter steps where each step is constructed as

$$e^{iH_{JC}^{(1)} t/\hbar} e^{iH_{AJC}^{(2)} t/\hbar}.$$
This combination of an analog and a digital quantum simulation is universal and can simulate Rabi model dynamics in all parameter regimes, provided one chooses properly the system parameters such that

\[
\omega_r^R = 2\Delta_r, \quad \omega_q^R = \Delta_q^{(1)} - \Delta_q^{(1)}, \quad g^R = g.
\]
Chapter 3

Numerical model description

In this chapter, we implement a full numerical model that describes as close as possible the dynamics of a transmon coupled to a CPW resonator, based on the master equation description. Realistic system dissipation is included and qubit gates are implemented as in an experiment. We build all the necessary tools for modelling the proposed digital quantum simulation of the Rabi model (section 2.3.2) with the ultimate goal to decide whether an experimental realisation is feasible with our circuit QED architecture.

3.1 The need for a numerical simulation

One of the main limitations in an experimental implementation of a digital quantum simulation is the finite duration of the gates. As we have discussed in section 2.3, the error associated with each Trotter step decreases for smaller steps. Therefore, one of the things that we want to investigate is how small we should make this step in order to achieve a high fidelity quantum simulation for typical experimental parameters.

Furthermore, in our experimental setup the role of qubits is played by transmons which are in fact weakly anharmonic qutrits that can be restricted to the qubit subspace by tuning the anharmonicity, as we discussed in section 2.2. We therefore need to investigate the effect of the third level when simulating the two-level qubit of the Rabi model. For the $\pi/2$ qubit rotations, we need to address the transition between the ground and first excited levels while avoiding leakage to the third level. We achieve this by implementing optimised finite duration pulses, identical to the experimental ones, through a driving that is resonant with the
Numerical model description

In addition, control of the qubit frequency is required in order to be able to vary the detuning between qubit and resonator frequencies. For instance, in an experimental situation, the bit flip rotations are applied while the qubit is sufficiently off-resonance with the resonator. We model a realistic finite bandwidth control of the qubit frequency, as expected from the electronics setup.

Finally, we need to include the key dissipation mechanisms in the system and examine their effect on the simulation fidelity.

3.2 Master equation

The time evolution of the density operator of any quantum system is described by the von Neumann equation [34],

\[ \dot{\rho} = -\frac{i}{\hbar} [H, \rho(t)]. \]  

(3.1)

However, it is impossible to perfectly isolate a quantum system from the environment and in order to take dissipation mechanisms into account we need to consider the dynamics of open quantum systems where the system is coupled to a reservoir (environment). The non-unitary time evolution of an open quantum system can be described by a master equation of the Lindblad form [35] that is trace-preserving and completely positive:

\[ \dot{\rho} = -\frac{i}{\hbar} [H, \rho(t)] + \sum_i \gamma_i \mathcal{L}[C_i] \rho, \]  

(3.2)

where \( \mathcal{L}[C_i] \rho = (2C_i \rho C_i^\dagger - C_i^\dagger C_i \rho - \rho C_i^\dagger C_i) \) are the Lindblad superoperators for each source of dissipation and \( \gamma_i \) denotes the associated decay rate.

Thus, the time dependence of the density matrix is completely determined by solving the above master equation.

3.2.1 Master equation for the transmon-resonator system

Unitary evolution

The effective generalised Jaynes-Cummings Hamiltonian describing the dynamics of a transmon coupled to a superconducting resonator is given
3.3 Rotating frame transformation

by [30]:

\[ H_{JC} = \hbar \sum_{j=0}^{2} \omega_{j} |j\rangle \langle j| + \hbar \omega_{r} a^\dagger a + \hbar \left( \sum_{i=0}^{1} g_{i,i+1} |i\rangle \langle i+1| + a^\dagger \right). \tag{3.3} \]

Here, \( \omega_{r} \) denotes the resonator frequency, and \( \omega_{1} = \omega, \omega_{2} = (2\omega - \alpha) \) are the transition frequencies between levels 0–1 and 1–2 of the transmon, where \( \alpha \) is the transmon anharmonicity.

The resonator mode couples differently to the two transitions, with coupling strengths

\( g_{0,1} = g, g_{1,2} \approx \sqrt{2} g. \)

Defining the generalised ladder operators for the transmon as

\[ c = \sum_{i=0}^{1} \sqrt{i+1} |i\rangle \langle i+1|, \]

we can write

\[ H_{JC} = \hbar \sum_{j=0}^{2} \omega_{j} |j\rangle \langle j| + \hbar \omega_{r} a^\dagger a + \hbar g \left( a^\dagger c + ac^\dagger \right). \tag{3.4} \]

System dissipation

Qubits suffer from two main sources of dissipation, namely relaxation and dephasing. Relaxation of the excited state to the stable ground state occurs at a rate \( \gamma_{-} \), while dephasing (\( \gamma_{\phi} \)) refers to the loss of coherence in a superposition state that drives it into a statistical mixture. The resonator is dissipating at a decay rate \( \kappa \) which is a measure of the rate at which photon losses are happening.

Therefore, the evolution of a transmon coupled to a transmission line resonator is described by the following master equation:

\[ \dot{\rho} = -\frac{i}{\hbar} [H_{JC}, \rho(t)] + \kappa \mathcal{L}[a] \rho + \gamma_{-} \mathcal{L}[c] \rho + \frac{\gamma_{\phi}}{2} \mathcal{L}[c^\dagger c] \rho. \tag{3.5} \]

3.3 Rotating frame transformation

According to the proposed quantum simulation scheme discussed in section 2.3.2, a necessary step is to write the Hamiltonian of the system in
Numerical model description

a rotating frame. Here, we modify the required rotating frame transformation for the case of a three level atom (qutrit) such as the transmon. We move to a frame rotating at frequency $\omega_{rf}$ by doing a rotating frame transformation given by the operator [36]

$$U(t) = \exp \left[ i\omega_{rf} t \left( a^\dagger a + \sum_{j=0}^{2} |j\rangle \langle j| \right) \right].$$

(3.6)

The Hamiltonian is then transformed as

$$\tilde{H}_{JC} = U H_{JC} U^\dagger - i U \dot{U}^\dagger,$$

(3.7)

and becomes

$$\tilde{H}_{JC} = \hbar \sum_{j=0}^{2} \Delta_j |j\rangle \langle j| + \hbar \Delta_r a^\dagger a + \hbar g \left( a^\dagger c + ac^\dagger \right),$$

(3.8)

where $\Delta_r = \omega_r - \omega_{rf}, \Delta_j = \omega_j - j\omega_{rf}$.

It is important to note here that the quantum simulation scheme is still efficient without the above transformation, since the dynamics do no change, however by moving to a certain rotating frame we increase the speed of the numerical simulations by a factor of $\sim 20$. For typical simulation times of $\sim 30$ min. this makes a huge difference.

3.4 Implementing the drive

Bit flip rotations of superconducting qubits are realised by applying driving microwave pulses resonant with the transition frequency that one wants to address. We model this using a driving term in our Hamiltonian description [36]:

$$H_d = \hbar \left[ \Omega(t) e^{-i\omega_d t} c^\dagger + \Omega^*(t) e^{i\omega_d t} c \right],$$

(3.9)

where $\omega_d$ is the frequency of the drive and the amplitude of the driving pulse is given by

$$\Omega(t) = \frac{1}{2} \left[ \Omega_x(t) + i\Omega_y(t) \right],$$

(3.10)

where $\Omega_x(t), \Omega_y(t)$ represent the two quadratures of the driving field. The choice of the pulse amplitude defines the nature of the applied pulse.
3.4 Implementing the drive

In a frame rotating at frequency $\omega_{rf}$, this term becomes:

$$\tilde{H}_d = \hbar \left[ \Omega(t) c^+ e^{-i(\omega_d - \omega_{rf})t} + \Omega^*(t) c e^{i(\omega_d - \omega_{rf})t} \right]. \quad (3.11)$$

In the case of an ideal two-level qubit, a driving pulse of the appropriate amplitude will have the same effect regardless of its shape being Gaussian or square, for example. However, in the case of weakly anharmonic qutrits the choice of the pulse shape is crucial. The reason for this is that the two transition frequencies in the transmon typically differ by a small fraction of $\sim 5\%$.

Therefore, applying a pulse resonant with the first transition to excite the qubit from $|g\rangle$ to $|e\rangle$ does not exclude the possibility of some leakage to the third level $|f\rangle$. Square pulses, for example, always result in some excitations out of the qubit subspace.

To reduce this effect, Gaussian waveform pulses can be used:

$$\Omega(t) = \Omega_{Amp} \exp \left[ -\frac{(\mu - t)^2}{2\sigma^2} \right], \quad (3.12)$$

where $\mu$, $\sigma$ denote the mean value and standard deviation of the Gaussian function, respectively.

Better control of the gates is achieved with increasing the pulse duration.

*Figure 3.1: Bloch sphere representation of a single qubit $\pi$ rotation around $\hat{x}$ in the qubit frame (left) and in a frame rotating at a frequency 0.5 GHz larger (right).*
3.4.1 DRAG pulses

Since decoherence of the qubits is a main limitation, we want to minimise the gate times as much as possible in order to achieve a high fidelity quantum simulation. There has been proposed a technique called Derivative Removal by Adiabatic Gate (DRAG) [37, 38], which allows for high fidelity pulses while reducing gate times down to 10 ns. This technique relies on controlling two quadratures of the driving field, $\Omega_x$ and $\Omega_y$, for effective phase modulation. One quadrature is proportional to the time-derivative of the other such that

$$\Omega(t) = \frac{1}{2} \left[ \Omega_x(t) + i \beta \dot{\Omega}_x(t) \right], \quad (3.13)$$

where $\beta$ is called the Motzoi parameter.

![Figure 3.2: Transmon level populations during a bit flip operation using DRAG. Amplitude: 111.899 MHz; $\beta = 0.00026$; gate time: 10 ns. At the end of the operation the population in $|f\rangle$ is $\sim 10^{-6}$.](image)

The procedure for optimising the bit flip operations is the following: We use a Gaussian waveform pulse ($\Omega_x$) and calculate the gate fidelity for several amplitudes. Then, using the optimal amplitude value, we start sweeping on the Motzoi parameter and repeat this procedure until we achieve the highest possible gate fidelity.
3.5 Flux control of qubit frequencies

In order to apply single qubit rotations to a transmon that is strongly coupled to a resonator, we need to effectively turn off the interaction. In an experiment this is achieved by detuning the transmon frequency $\omega_q$ far away from the resonator frequency $\omega_r$ such that $\frac{g}{\omega_q-\omega_r} \ll 1$. As we have seen in section 2.2, the qubit frequency depends on the Josephson energy. The design of the superconducting quantum interference device (SQUID) with two junctions connected in parallel, allows for tuning of the Josephson energy by applying an external magnetic flux in the SQUID loop (figure 3.3).

![Figure 3.3: Picture of a transmon coupled to a CPW resonator with an individual flux bias line for qubit frequency control. The central conductor of the CPW is coloured in green and the superconducting islands are depicted in blue and red. The flux bias line serves for introducing a current to the SQUID loop that results in changing the Josephson energy and tuning the qubit frequency.](image)

Usually, the two junctions are the same (symmetric junctions), and the dependence of the Josephson energy to an applied magnetic flux is given by the simple relation [39]:

$$E_J = E_J^{\text{max}} \left| \cos \left( \frac{\pi \Phi_{\text{ext}}}{\Phi_0} \right) \right|, \quad (3.14)$$

where $E_J^{\text{max}}$ is the sum of the Josephson energies of the two junctions, and $\Phi_0 = \frac{h}{2e}$ is the flux quantum.

When $\Phi_{\text{ext}}$ is an integer multiple of $\Phi_0$, the transmon is operating at a flux sweet spot, where the sensitivity to qubit dephasing via flux noise is diminished [30].
Asymmetric junctions

The general case where the two junctions are not the same, i.e. they have different phases $\phi_1 \neq \phi_2$ and Josephson energies $E_J^{(1)} \neq E_J^{(2)}$, is particularly interesting. In this case, the maximum bias current that passes through the SQUID when an external flux is applied, is the switching current $[40, 41]$:

$$I_{SW} = 2I_C \sqrt{\alpha^2 + (1 - \alpha^2) \cos^2 \left( \frac{\Phi_{ext}}{\Phi_0} \right)}, \quad (3.15)$$

where $I_C$ is the average critical current of the two junctions and the asymmetry factor $\alpha$ is given by

$$\alpha = \left| \frac{E_J^{(1)} - E_J^{(2)}}{E_J^{(1)} + E_J^{(2)}} \right|. \quad (3.16)$$

![Figure 3.4: Dependence of the switching current in the SQUID loop on an applied flux in the case of symmetric junctions ($\alpha = 0$). The points where the applied flux is an integer multiple of the flux quantum are called sweet spots because they are less sensitive to flux noise. Figure obtained from [41].](image)

The Josephson energy is given by [41]:

$$E_J = \frac{\Phi_0}{2\pi} I_{SW}. \quad (3.17)$$

Due to the junction asymmetry, the circulating supercurrent induced by the external magnetic flux never reaches the critical current $I_C$ and so
the minimum of the modulated $I_{SW}$ is not zero as in the symmetric case. Therefore, the transmon has two sweet spots, where it is less sensitive to dephasing, at $E_{j}^{\text{max}} = E_{j}^{(1)} + E_{j}^{(2)}$ and at $E_{j}^{\text{min}} = E_{j}^{(1)} - E_{j}^{(2)}$. As shown in figure 3.5, we can move from one sweet spot to the other by applying a magnetic flux $\Phi_{\text{ext}} = \Phi_{0}/2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3_5.png}
\caption{Envisaged design of qubit and resonator frequencies. The qubit frequency (——) has two sweet spots: one close to the resonator frequency (—) for the Jaynes-Cummings step, and the other resonant to the frequency of the drive (-- --) for the bit flip operations.}
\end{figure}

In order to eliminate qubit dephasing as much as possible in our experiment, we want to design a double sweet spot transmon. Therefore, for the Jaynes-Cummings evolution part there will be no applied flux such that the qubit frequency stays in the upper sweet spot, where it interacts strongly with the cavity. For the bit flip operations we will apply a $\Phi_{0}/2$ flux pulse which detunes the qubit frequency to the bottom sweet spot, where it is resonant with the frequency of the drive.

In order to implement this numerically, we first write the transmon Hamiltonian [Eq. (2.3)] in the basis of the Cooper-pair number operator $\hat{n}$ [42]

$$H_{\text{transmon}} = 4E_{C} \sum_{n} (\hat{n} - n_{g})^{2} |n\rangle\langle n| - \frac{E_{J}}{2} \sum_{n} |n\rangle\langle n+1| + |n+1\rangle\langle n|.$$ \hspace{1cm} (3.18)

By diagonalising this Hamiltonian, we obtain the transmon frequencies and we include the dependence on an applied flux using equations (3.15) and (3.17).
This implementation offers us exact control of the transmon level frequencies given the charging energy of the transmon and Josephson energies of the junctions. More importantly, it offers the possibility of modelling any effects that might be manifested to the flux pulses due to the electronic setup, as we shall see in the next chapter.

Figure 3.6: Sequence of two Trotter steps for the digital quantum simulation of the Rabi model. For the Jaynes-Cummings part of the simulation, the qubit interacts strongly with the resonator. The qubit is detuned with a $\Phi_0/2$ flux pulse and a resonant driving pulse is applied, which flips the state of the qubit. The Trotter step is completed with a second Jaynes-Cummings and a bit flip operation.

3.6 Trotter step description

In the previous sections we have introduced all the elements necessary for numerically simulating the digital quantum simulation of the Rabi model proposed in 2.3.2, in a circuit QED architecture. As shown in figure 3.6, a Trotter step consists of a Jaynes-Cummings evolution part followed by an anti-Jaynes-Cummings part.

The Jaynes-Cummings evolution is described by:

$$H_{JC}^{(1)} = \hbar \sum_j \Delta_j^{(1)} |j\rangle \langle j| + \hbar \Delta_r a^+ a + \hbar g \left( a^+ c + ac^+ \right),$$  \hspace{1cm} (3.19)

with $\Delta_r = \omega_r - \omega_{rf}$, $\Delta_q^{(1)} = \omega_q^{(1)} - \omega_{rf}$.

The anti-Jaynes-Cummings evolution part consists of a Jaynes-Cummings for $\Delta t = t_{JC}$ (possibly with different qubit frequency) sandwiched by two
3.6 Trotter step description

R_{\hat{x},\pi} rotations (bit flips):

\[ U_{\Lambda-JC} = R_{\hat{x},\pi} U_{JC} R_{\hat{x},\pi}. \]

The \( R_{\hat{x},\pi} \) rotations are implemented via a driving term, as discussed in section 3.4, and have a finite duration \( t_{\text{drive}} \sim 10 \) ns as in the experiment. The free energy evolution of the photon field in the resonator state needs to be taken into account during that time, therefore the Hamiltonian describing the system during the bit flip operation is:

\[ H_{\text{drive}} = \hbar \Delta_r a^\dagger a + \hbar \left( \Omega(t) c^\dagger e^{i(\omega_{rf} - i\omega_d)t} + \Omega^*(t) c e^{-i(\omega_{rf} - i\omega_d)t} \right). \tag{3.20} \]

Each Trotter step has, therefore, a finite time duration

\[ \tau = 2(t_{\text{drive}} + t_{JC}), \]

and the simulated Rabi model time after each step is \( t_{\text{Rabi}} = t_{JC} \). Notice that due to free evolution in the cavity state during the bit flips, we no longer have \( \omega_R^R = 2\Delta_r \) but \( \omega_R^R = \Delta_r (\tau/t_{JC}) \). Therefore, a quantum simulation in all parameter regimes can be achieved by choosing the appropriate parameters such that

\[ \omega_R^R = \Delta_r (\tau/t_{JC}), \quad \omega_q^R = \omega^{(1)} - \omega^{(2)}, \quad g^R = g. \]

In figure 3.7 we show the numerical evolution of the qubit and cavity states for three consecutive Trotter steps. We solve the master equation for the Hamiltonian describing the real time dynamics of the joint system:

\[
H = \hbar \sum_j \Delta_j |j\rangle \langle j| + \hbar \Delta_r a^\dagger a + \hbar g \left( a^\dagger c + ac^\dagger \right) \\
+ \frac{\hbar}{2} \left( [\Omega_x(t) + i\beta \dot{\Omega}_x(t)] e^{i(\omega_{rf} - \omega_q)t} c^\dagger + [\Omega_x(t) - i\beta \dot{\Omega}_x(t)] e^{-i(\omega_{rf} - \omega_q)t} c \right), \tag{3.21}
\]

where \( \Omega_x(t) = \Omega_{\text{Amp}} \exp \left[ -\frac{(\mu - t)^2}{2\sigma^2} \right] \) and \( \Omega_{\text{Amp}} = 0 \) during the Jaynes-Cummings evolution part. We consider realistic pulses with \( \sigma = 2 \) ns and 5\( \sigma \) width.

We plot the mean photon number inside the resonator, \( \langle a^\dagger a \rangle = \text{Tr} \left[ \rho_r a^\dagger a \right] \), as well as the transmon occupation probabilities for the ground \( \langle g | \rho_q | g \rangle \) and first excited \( \langle e | \rho_q | e \rangle \) states. The reduced density matrices of the qubit and the resonator

\[ \rho_q = \text{Tr}_r \rho_r, \quad \rho_r = \text{Tr}_q \rho_r. \]
Figure 3.7: Quantum simulation of Rabi model dynamics after three Trotter steps. The ideal Rabi model dynamics (---) of the qubit levels (top) and cavity photon number (bottom) are compared with the quantum simulation results at the end of each Trotter step (●). Real time evolution of the dynamics (—) shows collapses and revivals of the qubit population as expected.

are obtained from the partial trace of the joint density matrix over the qubit and resonator system, respectively.

The predictions of the Rabi model unitary evolution (described by $\rho^R$) are compared to the real time dynamics of the quantum simulation after each Trotter step. As a measure of how close the two systems we calculate the fidelity[43]

$$F(\rho, \rho^R) = \left( \text{Tr} \sqrt{\sqrt{\rho} \rho^R \sqrt{\rho}} \right)^2,$$

which ranges from 0, when there is no connection between them, to 1, when $\rho = \rho^R$. 


4 Numerical results

4.1 Simulations for ideal two-level qubits

4.1.1 Testing the limits

We first examine whether a quantum simulation of DSC Rabi model dynamics \((\omega_q^R = 0, \omega_r^R = g^R)\) can be achieved in a system described by the Jaynes-Cummings model, assuming a perfect two level qubit coupled to a resonator with a typical cQED coupling strength of \(g/2\pi = 80\) MHz, as proposed in [23].

![Image showing numerical results for mean photon number and oscillation period for different time steps.]

**Figure 4.1:** Numerically modelling the digital quantum simulation of an effective Rabi model with \((\omega_q^R = 0, \omega_r^R = g^R)\), for several time steps \(t_{JC}\), assuming a two-level qubit strongly coupled to a resonator with \(g/2\pi = 80\) MHz, as in typical circuit QED setups. (left) Mean photon number in the resonator (- - -) compared to the ideal dynamics (---). (right) Trotter error \([1 - F(\rho, \rho^R)]\) per step.
We try decreasing the time step ($t_{JC}$), in order to reduce the Trotter error, until a good agreement to the ideal dynamics is achieved, for at least one oscillation period ($t = 1/g^R$). As shown in figure 4.1, this would require implementing time steps $t_{JC} \lesssim 1$ ns. However, this would be impossible to achieve in the lab as it goes beyond the resolution limit set by current electronics used to control circuit QED experiments.

4.1.2 Finite time Trotter steps: Slowing down the dynamics

As we have seen in the previous section, a digital quantum simulation of the Rabi model in circuit QED would be impossible to implement for typical coupling strengths. Here, we present a different approach, i.e. to keep the Trotter step time fixed at a realistic value that can be achieved in the lab and vary the coupling strength $g$. The idea is that by lowering the coupling strength, we effectively "slow down" the dynamics of the Jaynes-Cummings evolution. The expectation is that for finite time steps this will result in decreasing the Trotter error.

![Figure 4.2: Numerically modelling the digital quantum simulation of the DSC Rabi model ($\omega_q^R = 0, \omega_r^R = g^R$) for different coupling strengths using a finite time Trotter step of $\tau = 40$ ns. The plots show simulated (●) vs ideal (—) mean photon number (top) and qubit level populations (bottom), after each Trotter step. Better agreement is observed for lower coupling strengths.](image)

We use realistic finite time gates for the Jaynes-Cummings and bit flip parts, $t_{JC} = t_{drive} = 10$ ns. A Trotter step, therefore, requires $\tau = 40$ ns.
of experimental time and corresponds to 10 ns of ideal Rabi evolution dynamics, which we rescale accordingly for the purposes of demonstration. Using the digital approximation

\[ e^{iH_R t} = \left( e^{iH^{(1)}_{JC} t/n} e^{iH^{(2)}_{A-JC} t/n} \right)^n + [H^{(1)}_{JC}, H^{(2)}_{A-JC}] \frac{t^2}{2n} + O(t^3), \]  

(4.1)

we want to simulate Rabi model dynamics in the deep-strong coupling (DSC) regime for the simple case where \( \omega^R_q = 0, \omega^R_r = g^R \).

In order to reduce the Trotter error, we set the rotating frame at the qubit frequency during the Jaynes-Cummings steps,

\[ \omega^1_q = \omega^2_q = \omega_r, \]

such that \( \Delta_q = 0 \), and choose the resonator frequency \( \omega_r \) such that

\[ \Delta_r = \omega_r - \omega_{rf} = g(t_{JC}/\tau) = g/4. \]

As shown in figure 4.2, good agreement can be achieved even for one period of Rabi model dynamics \( (g^R t) = 1 \), however for unusually low coupling strengths, below 10 MHz. The price that one has to pay when going to lower coupling strengths is that the experimental time should be extended and the fidelity of the simulation is going to be limited by decoherence mechanisms. We will examine this effect in section 4.2.2.

### 4.1.3 Eliminating first order Trotter errors

In the Trotter sequence of (4.1), the first order Trotter error is proportional to

\[ \sum_{i > j} [H_i, H_j] = [H^{(1)}_{JC}, H^{(2)}_{A-JC}]. \]  

(4.2)

We can reduce this error by setting the rotating frame to the qubit frequency, as in the simulations of the previous section, however, due to the finiteness of the bit flip operation we cannot eliminate it completely.

An alternative, is to try a symmetric implementation of the Trotter step:

\[ e^{iH_R t} \simeq \left( e^{iH^{(1)}_{JC} t/2n} e^{iH^{(2)}_{A-JC} t/n} e^{iH^{(1)}_{JC} t/2n} \right)^n, \]  

(4.3)

i.e. apply the Jaynes-Cummings part \( H^{(1)}_{JC} \) for half of the time \( t_{JC}/2 \), then the anti-Jaynes-Cummings part (as before) and finally another \( H^{(1)}_{JC} \) for \( t_{JC}/2 \).
Numerical results

In this case, the first order Trotter error

$$\sum_{i>j} [H_i, H_j] \frac{t^2}{2n} = \left( \left[ \frac{H_{JC}}{2}, H_{A-JC} \right] + \left[ H_{A-JC}, \frac{H_{JC}}{2} \right] \right) \frac{t^2}{2n}$$

(4.4)

should vanish, since the sum of the two commutators is zero.

We demonstrate this for two different models, one with parameters $$\omega_R^q = 0, \omega_R^r = g_R^R$$, such that $$\omega_q^{(1)} = \omega_q^{(2)}$$ (figure 4.3) and another with parameters $$\omega_q^R = \omega_r^R = g_R^R$$, where $$\omega_q^{(1)} \neq \omega_q^{(2)}$$ (figure 4.4).

Figure 4.3: Numerically modelling the digital quantum simulation of an effective Rabi model interaction with $$\omega_R^q = 0, \omega_R^r = g_R^R$$.

Model parameters: qubit frequencies $$\omega_q^{(1)}/2\pi = \omega_q^{(2)}/2\pi = 6 \text{ GHz}$$; resonator frequency $$\omega_r/2\pi = 6.0015 \text{ GHz}$$; coupling $$g/2\pi = 6 \text{ MHz}$$.

The plots show simulated (-•-) vs ideal (—) average cavity photon number (top) and qubit level populations (bottom). Simulations for non-symmetric Trotter step (left) are compared to the symmetric case (right).
4.1 Simulations for ideal two-level qubits

Figure 4.4: Numerically modelling the digital quantum simulation of an effective Rabi model interaction with $\omega_{q}^{R} = \omega_{r}^{R} = g^{R}$.

Model parameters: qubit frequencies $\omega_{q}^{(1)}/2\pi = 6\, \text{GHz}$, $\omega_{q}^{(2)}/2\pi = 5.994\, \text{GHz}$; resonator frequency $\omega_{r}/2\pi = 6.0015\, \text{GHz}$; coupling $g/2\pi = 6\, \text{MHz}$.

The plots show simulated (●) vs ideal (—) average cavity photon number (top) and qubit level populations (bottom). Simulations for non-symmetric Trotter step (left) are compared to the symmetric case (right).

At first, we notice that the effect of eliminating the first order Trotter error is more drastic in the second case, where $\Delta q \neq 0$, and it is manifested predominantly in the qubit level population plots. Looking at the plots more carefully, we realise that the envelopes of the real time dynamics are the same in both cases, with a relative shift. Therefore, from the experimentalist’s point of view, first order Trotter errors are eliminated simply by measuring the qubit and resonator states at different times.

In figure 4.5 we compare, for both models, the fidelity of the quantum simulation to the ideal Rabi model dynamics, with and without first order Trotter errors.
Numerical results

Figure 4.5: Symmetric vs non-symmetric implementation of the Trotter step. Fidelity of the quantum simulation to the ideal Rabi model dynamics with parameters $\omega_q^R = \omega_r^R = g^R$ (left) and $\omega_q^R = 0, \omega_r^R = g^R$ (right).

4.1.4 Simulations for various coupling strengths

Having eliminated first order Trotter errors, we model a number of quantum simulations for a range of coupling strengths from 1 to 10 MHz. As shown in figure 4.6, it is possible to reduce the Trotter error by several orders of magnitude by decreasing the coupling strength.

Figure 4.6: Trotter error evolution for a range of coupling strengths in a digital quantum simulation of an effective Rabi model with parameters $\omega_q^R = 0, \omega_r^R = g^R$. 
4.2 Simulations for a realistic circuit QED setup

4.2.1 From qubit to transmon - adding the third level

As a first step towards more realistic scenarios for a quantum simulation of the Rabi model using a circuit QED setup, we include the third level of the transmon. We run again the simulations of figure 4.3 for the same parameters \( \omega_q^R = 0, \omega_r^R = g^R \), including an anharmonic qutrit with typical transmon anharmonicities \(-500 \text{ MHz} \lesssim \frac{\alpha}{2\pi} \lesssim -200 \text{ MHz}\). In figure 4.7, we plot the simulation results for the same parameters as in figure 4.3, using a transmon with a typical anharmonicity of -300 MHz.

![Graph](image)

Figure 4.7: Numerically modelling the digital quantum simulation of an effective Rabi interaction with \( \omega_q^R = 0, \omega_r^R = g^R \), in a cQED setup using a three level transmon. Simulated (•) vs ideal (—) average cavity photon number (left) and transmon level populations (right) for the same parameters as in figure 4.3, including the third level \( \alpha/2\pi = -300 \text{ MHz} \).

We observe that the addition of the third level has a considerable effect on the simulation fidelity, despite the fact that it is not being populated \( \lesssim 10^{-5} \) after each Trotter step, as a result of our optimised DRAG pulses. We examine this more carefully in figure 4.8, where we compare the simulation fidelity for a range of coupling strengths using a three level transmon and varying the anharmonicity.

As expected, the simulation fidelity gets better as \( |\alpha| \) is increased, which is practically achieved by increasing the charging energy, \( E_C/\hbar \sim |\alpha| \). However, as we have discussed in section 2.2, we need \( E_J/E_C \gtrsim 30 \) in order to be in the transmon regime, where the qubit is not sensitive to charge noise. For the design that we intend to use (figure 3.5), we estimate the Josephson energy around \( E_J/\hbar = 9 \text{ GHz} \) during the bit flip operation.
Figure 4.8: Fidelity plots for the quantum simulation of the DSC Rabi model \((\omega_R^q = 0, \omega_R^r = g^R)\) using a three level transmon coupled to a resonator for a range of coupling strengths and anharmonicities.

which suggests that we need \(E_C/\hbar \sim 300\,\text{MHz}\).

From figure 4.8, we conclude that a high fidelity (90\%) quantum simulation is achievable for coupling strengths \(g/2\pi \lesssim 5\,\text{MHz}\) and an anharmonicity \(|\alpha|/2\pi \gtrsim 300\,\text{MHz}\).

### 4.2.2 Dissipation

We implement dissipation mechanisms by adding the appropriate Lindblad superoperator associated with each decay rate in the master equation (see section 3.2). The key sources of decoherence are the transmon and resonator relaxation times \(T_1 = 1/\gamma_-, T_{\text{cav}} = 1/\kappa\) as well as the transmon dephasing time \(T_2 = 1/\gamma_\phi\).
Comparing dissipation mechanisms

We first want to identify the most important form of dissipation. We examine the case where \( g/2\pi = 4 \text{ MHz} \), which we have found to give high fidelities before, and study the effect of each decay mechanism on the simulation fidelity. The experimental time of the simulation is \( \sim 1 \mu s \) and we add one by one the relative dissipation times (of 10 \( \mu s \) each).

![Fidelities for different decay mechanisms (g=4 MHz)](image)

**Figure 4.9:** Fidelity plots comparing the impact of each decay mechanism on the quantum simulation for \( g/2\pi = 4 \text{ MHz} \).

As shown in figure 4.9 the resonator decay is the most limiting factor, therefore, we should aim for a high quality resonator when designing the experiment.

Realistic decay rates

As we have discussed earlier, the transmon that is simulating the Rabi qubit, has two sweet spots in order to eliminate flux noise both during the Jaynes-Cummings evolution and the bit flip operations, in the experiment. However, we expect an uncertainty of \( \sim 100 \text{ MHz} \) in targeting the qubit frequencies, due to the fabrication process of the Josephson junctions, while for the implementation of the quantum simulations scheme we need a detuning between the qubit and resonator around 1 MHz (since \( g/2\pi = 4 \text{ MHz} \)). For this reason, we have to design the top sweetspot at least 200 MHz higher than the resonator frequency. We would then possibly have to be slightly detuned from the top sweet spot during the Jaynes-Cummings operation, which would result in lower dephasing time.
$T_2 \sim 1 \mu s$. This is the price that we have to pay in order to strongly couple a qubit and a resonator with such a low coupling strength.

In figure 4.11 we plot the fidelity of the quantum simulation for the most likely and feasible decay rates that we expect in the experiment. We aim to have a high quality factor resonator with a relaxation time of $15 \mu s$ and a typical qubit with $T_1 = 10 \mu s$. The dephasing time is set to be $1 \mu s$ during the Jaynes-Cummings and $10 \mu s$ during the bit flip operation, as expected from the design in figure 4.10 (right).

**Figure 4.10:** Design of qubit and resonator frequencies. *(left)* Ideal frequency scheme as a function of flux. *(right)* Most realistic frequency scheme based on possible deviation in targeting the transmon frequencies.

**Figure 4.11:** Fidelity plots of the quantum simulation with realistic decay mechanisms for $g/2\pi = 4$ MHz.
4.2.3 Finite-bandwidth flux control: RC filter

In our numerical model, the transmon frequencies can be tuned by applying square flux pulses $\Phi_{\text{ext}}$ from 0 to $\Phi_0/2$ and then diagonalising the transmon Hamiltonian (see section 3.5). This implementation allows us to model bandwidth limited flux pulsing, naturally arising from our electronics setup.

In an experimental situation, the applied voltage that induces the flux pulses to the SQUID loop, is attenuated due to the microwave electronics. This effect can be simulated as a single pole low pass filter with a cut-off frequency $f_{\text{cut}} = \frac{1}{2\pi RC}$, where RC is the time constant of the filter. Therefore, any input voltage $V_{\text{in}}$ is modified as $V_{\text{out}} = \frac{V_{\text{in}}}{1+j\omega RC}$, in the frequency domain.

As a result, the response of the flux bias line is slowed down and the flux pulse will no longer be a step function. In order to model this effect numerically, we calculate the response in the time domain, by taking the Laplace transform of the transfer function $\frac{1}{1+j\omega RC}$. Therefore, an applied square flux pulse is modified as

$$\Phi_{\text{out}} = \Phi_{\text{in}} e^{-t/RC} \frac{e^{-t/RC}}{RC}. \quad (4.5)$$

![Simulation fidelity for filtered flux pulses](image)

**Figure 4.12:** Impact of a single pole low-pass filter of RC=1.25 ns in the simulation fidelity.

From figure 4.12, we conclude that a single pole low-pass RC filter, with an estimated time constant RC=1.25 ns, does not affect dramatically the simulation fidelity.
4.2.4 Final real-world quantum simulations

In figure 4.13, we show that an analog-digital quantum simulation of the Rabi model in a realistic circuit QED setup with a three-level transmon coupled to a resonator, is feasible with a good fidelity including experimental parameters such as finite time pulses, bit flips implemented via optimised DRAG pulses, bandwidth-limited flux pulsing for qubit frequency detuning between Trotter steps, as well as cavity and transmon relaxation and dephasing processes.

**Figure 4.13:** Numerically modelling the digital quantum simulation of an effective Rabi interaction with $\omega_q^R = 0$, $\omega_r^R = g^R$. Model parameters: qubit-cavity detuning = 1 MHz; coupling $\frac{g}{2\pi} = 4$ MHz; transmon anharmonicity $\frac{\alpha}{2\pi} \simeq -300$ MHz; photon decay time $T_c = 30 \mu$s; transmon relaxation $T_1 = 10 \mu$s and dephasing $T_2 = 1 - 10 \mu$s. The plots on the top show simulated (●) and ideal (—) transmon level populations (left) and average cavity photon number (right). Fidelity of the simulated system state to the ideal state after each Trotter step is plotted on the bottom.
4.3 Exploring the dynamics of the deep-strong coupling regime

In this section, we explore the dynamics of the Rabi model in regimes beyond ultra-strong coupling. We show that deep-strong coupling (DSC) provides extraordinary dynamics exhibiting special types of hybrid entanglement between the qubit and macroscopic Schrödinger cat states. We identify the Wigner function as a key tool in probing these dynamics, and we demonstrate that they can be reproduced with the digital quantum simulations scheme discussed so far.

4.3.1 Strong coupling and beyond

The Jaynes-Cummings regime

As we have previously discussed, the strong coupling regime of the Rabi model, when \( g^R \ll \omega^R_q, \omega^R_r \), is described by the Jaynes-Cummings model of equation (2.2). In the absence of non-conserving excitation terms, it is obvious that nothing happens when we start without any excitation in the system \(|g\rangle|0\rangle\). When an excitation is added to the system, e.g. the qubit initially in the excited state \(|e\rangle\), we observe the well-known Rabi oscillations \(|e\rangle|0\rangle \leftrightarrow |g\rangle|1\rangle\) that lead to a highly entangled superposition between the qubit and a photon Fock state:

\[
\frac{1}{\sqrt{2}} (|0\rangle|e\rangle + |1\rangle|g\rangle).
\] (4.6)

Parity chains

As the coupling strength becomes comparable to the qubit and cavity frequencies, the RWA breaks down and the counter-rotating terms \( \sigma^+ a^\dagger, \sigma^- a \) of the Rabi Hamiltonian [Eq. (2.1)] become dominant. The dynamics can be described in the Hilbert space by two unconnected parity chains depending on the initial state [7]:

\[
|g\rangle|0\rangle \leftrightarrow |e\rangle|1\rangle \leftrightarrow |g\rangle|2\rangle \leftrightarrow |e\rangle|3\rangle \leftrightarrow |g\rangle|4\rangle \ldots \quad (p = +1),
\]

\[
|e\rangle|0\rangle \leftrightarrow |g\rangle|1\rangle \leftrightarrow |e\rangle|2\rangle \leftrightarrow |g\rangle|3\rangle \leftrightarrow |e\rangle|4\rangle \ldots \quad (p = -1),
\]

where \( p \) is the eigenvalue of the parity operator: \( \Pi = \sigma^z (-1)^{a^\dagger a} \).

The ratio \( g^R/\omega^R \) determines how far a state can propagate along these chains. When \( g^R/\omega^R \ll 1 \) the chains connecting states through non-
Figure 4.14: Rabi model energy level structure (left) and evolution of dynamics in Hilbert space (right). The blue arrows indicate interactions via the energy-conserving terms $\sigma^+ a$, $\sigma^- a^\dagger$ while the red arrows stand for interactions via the counter rotating terms $\sigma^+ a^\dagger$, $\sigma^- a$. The latter are absent in the Jaynes-Cummings regime and the Hilbert space is restricted to excitation conserving Jaynes-Cummings doublets.

conserving excitations terms break into the Jaynes-Cummings doublets that we discussed before.

4.3.2 Superpositions of coherent states

We now examine more carefully the DSC regime of the Rabi model in the simple case that we mostly discussed in the previous chapter, with parameters: $g^R = \omega^R$, $\omega^R_q = 0$. As we have already observed (figure 4.13 for example), even when starting without any excitation in the system, the mean photon number oscillates in a coherent way between zero and $\langle a^\dagger a \rangle = 4$, i.e. the system evolves via the first parity chain ($p = +1$).

In order to better understand these dynamics we look at the evolution in phase space, which contains more information than just the evolution of the photon number. We, therefore, reconstruct the Wigner quasiprobability distribution of the cavity state which is defined as [44]

$$ W(\alpha) = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \langle n | D^{-1}(\alpha) \rho | D(\alpha) | n \rangle. \quad (4.9) $$

$D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a]$ is the displacement operator of the mode. It is also defined by $D(\alpha) | 0 \rangle = | \alpha \rangle$, i.e. when acting on the vacuum it creates a coherent state $| \alpha \rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} | n \rangle$. In general, $D(\alpha)$ displaces a pho-
4.3 Exploring the dynamics of the deep-strong coupling regime

A photon state in phase space by a magnitude and direction set by the complex number \( \alpha \).

Looking at the Wigner function of the cavity state, allows us to extract very interesting information about the photon field and its evolution in phase space. A remarkable feature about this function is that it becomes negative for non-classical states such as Fock or Schrödinger cat states, and has therefore been proposed as a measure of the non-classicality of states [45].

![Cavity Wigner function evolution](image)

**Figure 4.15:** Evolution of the Wigner function in the DSC regime of the Rabi model with parameters \( g^R = \omega_R \), \( \omega_q^R = 0 \). The cavity initially in the vacuum state (top left) turns into a superposition of two coherent states \( |\alpha\rangle \) and \( |-\alpha\rangle \) with \( \alpha = 2 \) (top right). This point corresponds to a photon number peak at \( \langle a^\dagger a \rangle = |\alpha|^2 = 4 \).

In figure 4.15, we plot the evolution of the Wigner function of the cavity in phase space for the DSC Rabi model with \( g^R = \omega_R \), \( \omega_q^R = 0 \), for half an oscillation period, i.e. until the photon number peak at \( \langle a^\dagger a \rangle = 4 \). The cavity evolves into a superposition state between two coherent states \( |\alpha\rangle \)
Numerical results

and $|−α⟩$ with $α = 2$.

In order to understand this evolution we look at the interaction term in the Rabi Hamiltonian,

$$H_I = \hbar g^R \sigma_x (a e^{-iω_r t} + a^+ e^{iω_r t}).$$

(4.10)

The cavity density matrix evolves as $U_I^\dagger \rho U_I$, where

$$U_I = \exp \left[ -i \int_0^t dτ g^R \sigma_x \left( a e^{-iω_r τ} + a^+ e^{iω_r τ} \right) \right] = \exp \left[ \sigma_x \left( \int_0^t dτ (-i g^R) e^{iω_r τ} a^+ \right) \left. \right|_0^t \right].$$

(4.11)

This interaction is, therefore, equivalent to a time-dependent displacement operation on the cavity $D(α) = \exp \left[ α a^+ - α^∗ a \right]$, that is determined by the complex number $α = \int_0^t dτ (-i g^R) e^{iω_r τ}$ and depends on the qubit state ($σ^x$ operator).

For example, if the qubit is initially in an eigenstate of $σ_x$ and the cavity in the vacuum state, this will result in the creation of a coherent state, as: $σ_x D(α)|±⟩|0⟩ \rightarrow |±⟩|∓α⟩$.

**Maximum displacement**

We can calculate the modulus of the cavity displacement

$$|α| = |i g^R \int_0^t dτ e^{iω_r τ}|$$

$$= g^R \sqrt{ω_r |e^{iω_r t} - i|^2}$$

$$= g^R \sqrt{\cos^2 ω_r t + (\sin ω_r t - 1)^2}$$

$$= g^R \sqrt{1 - 2 \cos ω_r t}.$$ 

(4.12)

At $t = \frac{π}{ω_r}$ the displacement reaches its maximum value:

$$|α|_{max} = \frac{2g^R}{ω_r}.$$ 

(4.13)

Therefore, the size of these coherent states, and the quantum superposition, increases as we increase the coupling strength relatively to the natural frequencies in the system. Notice that in the strong coupling regime,
4.3 Exploring the dynamics of the deep-strong coupling regime

\( g^R \ll \omega^R \), there is no cavity displacement because the cavity terms rotate much faster than the coupling strength.

4.3.3 Hybrid discrete - continuous variable entanglement

For a better understanding of the photon-qubit dynamics, we first calculate the negativity of the joint system, which is defined as the absolute value of the sum of negative eigenvalues of the partial transpose of the system density matrix \( \rho^T \) [46]

\[
N(\rho) = \sum_i |\lambda_i| - \lambda_i. \tag{4.14}
\]

This quantity vanishes for states which are not entangled [47].

From figure 4.16, we verify the existence of entanglement in the system, however in order to verify its nature we need to look at the state in the cavity after conditioning on different qubit bases.

The cavity density matrix after conditioning on the qubit being in a certain state \( |\psi_q\rangle \) is

\[
\rho^\text{cond}_r = \text{Tr}_q \left[ \rho \left( |\psi_q\rangle\langle\psi_q| \otimes I \right) \right], \tag{4.15}
\]

which experimentally amounts to measuring the qubit in \( |\psi_q\rangle \).

In figure 4.17, we plot the evolution of the Wigner function of the cavity state conditioned on measuring the qubit in the \( |+\rangle \) state. The plot shows...
Figure 4.17: Evolution of the Wigner function of the cavity state conditioned on measuring the qubit in $|+\rangle$, for the same parameters as in figure 4.15. The plots show the gradual creation of a coherent state $|\alpha\rangle$ from vacuum (top left) until $\alpha = -2$ (bottom right).

the gradual evolution of the coherent state $|-\alpha\rangle$, i.e. the left blob of figure 4.15. Similarly, the conditioned on $|-\rangle$ cavity state corresponds to the right blob $|\alpha\rangle$.

Moreover, in figure 4.18, we plot the cavity Wigner function after conditioning on the $\sigma_z$ basis, which shows the gradual creation of a very special state, known as Schrödinger cat. These states are quantum superpositions of two macroscopic coherent states with opposite phases $\{|\alpha\rangle \pm | -\alpha\rangle\}$ and can be recognised by the existence of negative/positive fringes of the Wigner distribution in phase space [2].

There is therefore no doubt that the qubit-cavity entanglement is of the form
\[
\frac{1}{\sqrt{2}} (| -\alpha\rangle|+\rangle + |\alpha\rangle|-\rangle), \tag{4.16}
\]
4.3 Exploring the dynamics of the deep-strong coupling regime

Figure 4.18: Evolution of the Wigner function of the cavity state conditioned on measuring the qubit in the ground state, for the same parameters as in figure 4.15. The plots show the gradual evolution of an even Schrödinger cat state \{ |α⟩ + |−α⟩ \} up to α = 2.

or equivalently:

\[ \frac{1}{2} \{ (|α⟩ + |−α⟩) |g⟩ - (|α⟩ - |−α⟩) |e⟩ \}. \quad (4.17) \]

This type of entanglement between the qubit and macroscopic coherent states in the cavity is called hybrid discrete-continuous variable entanglement and has many potential applications in quantum information theory [48] and quantum key distribution protocols for quantum cryptography [49].

Moreover, Schrödinger cat states are well known for their increased sensitivity to displacements, increasing with the fraction of negative/positive fringes in the Wigner function distribution, which can be used for high precision metrology experiments [50].

Being able to experimentally access the DSC dynamics of the Rabi model...
Numerical results

will, therefore, give us the opportunity to generate and control interesting forms of entanglement, as well as to deterministically create and manipulate Schrödinger cat states in the lab.

4.3.4 Creating Schrödinger cat states

Using the numerical model for a digital quantum simulation of the Rabi model in circuit QED, developed in chapter 3, we attempt to reproduce the interesting features of the DSC regime that we discussed in the previous section.

In figure 4.19 we plot the Wigner function distribution inside the resonator after conditioning on the transmon being in the ground and the excited state. Having included all the key experimental limitations, as discussed in section 4.2.4, we observe that using our digital quantum simulation scheme we are able to reproduce the Schrödinger cat states, as expected from the deep-strong coupling dynamics of the Rabi model, with high fidelity $\sim 90\%$.

This result is an indisputable proof that we can achieve DSC dynamics experimentally in a typical circuit QED system, by means of a digital quantum simulation.

![Figure 4.19: Creation of even and odd the Schrödinger cat states as predicted by the Rabi model (figure 4.18), using a digital quantum simulation with a three-level transmon coupled to a resonator in circuit QED. Model parameters: qubit-cavity detuning = 1 MHz; coupling $\frac{g}{2\pi} = 4$ MHz; transmon anharmonicity $\frac{\alpha}{2\pi} \simeq -300$ MHz; photon decay time $T_c = 30$ µs; transmon relaxation $T_1 = 10$ µs and dephasing $T_2 = 1 - 10$ µs. Fidelity to the ideal cat state: $\sim 90\%$.](image)
4.3 Exploring the dynamics of the deep-strong coupling regime

Probing the Rabi model regimes using the Wigner function negativity

So far we have focused on the reproduction of DSC dynamics, however, the quantum simulations protocol that we have implemented, provides access in all parameter regimes of the Rabi model. We can tune $\omega^R$ simply by tuning the detuning between the resonator and qubit frequency. This is something that can also be done in the experiment.

We run the simulation in all parameter regimes, for the model that we are using so far (starting with no initial excitations) and calculate the integrated Wigner function negativity, i.e. the volume of the negative parts, when conditioning on the $|g\rangle$ and $|e\rangle$ states, for a range of ratios $g^R/\omega^R$. As shown in figure 4.20, the different regimes are very well distinguished. When $g^R \ll \omega^R$ there is no interaction and the resonator is always in the vacuum state. Moving towards ultra-strong coupling, the Wigner function negativity starts increasing, however the coupling strength is not big enough to create coherent states. In this regime, the Wigner function, when conditioning on the excited state, resembles that of a single photon Fock state. The creation of even and odd Schrödinger cat states is only possible when $g^R$ approaches $\omega^R$, where the integrated negativity is the same for both conditionings.

![Figure 4.20: Integrated Wigner function negativity of the resonator state after measuring/conditioning on the qubit being in the ground (••) and excited state (••).](image)

Figure 4.20: Integrated Wigner function negativity of the resonator state after measuring/conditioning on the qubit being in the ground (••) and excited state (••).
Chapter 5

Towards an experimental implementation: designing the chip

Following the numerical results of the previous chapter, here, we discuss our steps towards realising a device that features a transmon coupled to a coplanar waveguide (CPW) resonator with the appropriate parameters, in order to realise a digital quantum simulation of the Rabi model in circuit QED. We first present our approaches towards measuring the qubit and resonator states, as well as performing a direct Wigner tomography of the resonator state. We then estimate the key parameters that are necessary for optimal high fidelity measurements and discuss the design process of the chip elements.

5.1 Overview of the experiment and readout process

5.1.1 Readout of qubit and cavity states

When the transmon and resonator frequencies are far detuned ($|\omega_q - \omega_r| \ll g$), the interaction term in the generalised Jaynes-Cummings Hamiltonian [Eq. (2.6)] can be treated as a perturbation. The transmon can be restricted to the qubit Hilbert space, provided we take into account the higher order tran-
Towards an experimental implementation: designing the chip

sitions, and the effective Hamiltonian in this dispersive regime is [30]

\[ H = \hbar \omega'_r a^\dagger a + \hbar \omega'_q \frac{1}{2} \sigma_z + \hbar \chi \sigma_z a^\dagger a. \]  

(5.1)

The frequencies acquire a Lamb shift

\[ \omega'_r = \omega_r - \chi_{12}/2, \quad \omega'_q = \omega_q + \chi_{01}, \]

where \( \chi_{ij} = \frac{g_{ij}^2}{\omega_{ij} - \omega_r} \) and the effective interaction is described by the dispersive shift given by

\[ \chi = \chi_{01} - \chi_{12}/2. \]

For the transmon \( \omega_{12} = \omega_{01} + \alpha, g_{12} = \sqrt{2}g_{01} \) and therefore the dispersive shift can be written as

\[ \chi = \frac{g^2 \alpha}{\Delta(\Delta + \alpha)}, \]  

(5.2)

where \( \Delta = \omega_{01} - \omega_r. \)

\[ \begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dispersive_shift.png}
\caption{Dispersive shift of the qubit and cavity frequencies. Figure obtained from [42].}
\end{figure} \]

Notice that the resonator frequency is shifted by \( \omega'_r \rightarrow \omega'_r + \chi \sigma_z, \) i.e. it is higher or lower depending on whether the qubit is in its ground or excited state. This effect is widely used in circuit QED setups in order to perform quantum non-demolition (QND) measurements of the qubit state, via a readout resonator coupled dispersive to it. Using power spectroscopy to readout the resonator frequency via a feedline, then by measuring the sign of the dispersive shift we determine the state of the qubit.
5.1 Overview of the experiment and readout process

The qubit frequency is also shifted by $\omega_q' \rightarrow \omega_q' + 2\chi a^\dagger a$, i.e. depending on the photon number in the resonator. As shown in figure 5.1, photon number peaks can be resolved in the qubit spectrum provided the induced dispersive shift $\chi$ is larger than the cavity linewidth $\kappa$. This requirement defines the strong-dispersive regime, in which a QND measurement of the photon number in the resonator can be realised by coupling a qubit dispersively to the resonator. One then needs a readout resonator in order to measure the state of that qubit.

5.1.2 Wigner tomography

We aim to readout the cavity and perform Wigner state reconstruction of the field in the resonator, using a technique similar to that of [51, 52], where a transmon, is coupled dispersively to the resonator described by the following Hamiltonian:

$$H = \hbar \omega_q |e\rangle\langle e| + \hbar \omega_r a^\dagger a - \hbar \chi a^\dagger a |e\rangle\langle e|.$$  \hfill (5.3)

As we discussed in 5.1.1, the above dispersive interaction results in a state-dependent shift on the qubit and cavity frequencies. This can be described as a cavity phase shift conditional on the qubit state

$$C_{\Phi} = e^{i\chi \tau a^\dagger a} |e\rangle\langle e| = I \otimes |g\rangle\langle g| + e^{ia^\dagger a \chi \tau} \otimes |e\rangle\langle e|,$$  \hfill (5.4)

where $\tau$ is the interaction time, which determines the phase acquired by the cavity, $\Phi = \chi \tau$.

For instance, if $\tau = \pi / \chi$, there will be a conditional $\pi$ shift per photon. In this case, we can use this type of interaction to realise a phase gate on the qubit, conditioned on there being an odd number of photons in the cavity [51].

The Wigner function [Eq.(4.9)] can also be written as

$$W(\alpha) = \frac{2}{\pi} \text{Tr} \left[ D^\dagger(\alpha) \rho_r D(\alpha) P \right]$$  \hfill (5.5)

where $D(\alpha)$ is the displacement operator and $P = e^{i\pi a^\dagger a}$ is the photon number parity operator.

Thus, as shown in figure 5.2, the Wigner tomography scheme for the resonator state $\rho_r$, consists of two parts: displacing the cavity by $D(\alpha)$ and measuring the mean photon parity $\langle P \rangle$ for each displacement. The displacements can be realised with coherent driving pulses directly into
the cavity. The number of displacements needed, depends on how much of the phase space \((\text{Re}(\alpha), \text{Im}(\alpha))\) we want to resolve.

The cavity displacement can be achieved by applying a coherent driving pulse

\[
\epsilon_d(t) \left( a^+ e^{i(\omega_d - \omega_r)t} + ae^{-i(\omega_d - \omega_r)t} \right)
\]

inside the cavity, where the magnitude and direction of the displacement are set by the pulse amplitude \(\epsilon_d\) and frequency of the drive \(\omega_d\), respectively.

For the photon parity measurement we need to perform a Ramsey experiment with an ancillary qubit coupled dispersively to the resonator. The sequence is depicted in figure 5.2, and goes as follows:

*Figure 5.2: Wigner tomography scheme. A displacement pulse \(D(\alpha)\) is applied to the cavity, which is coupled dispersively to a qubit. A single qubit \(\pi/2\) rotation around \(\hat{y}\) \(R_{\hat{y}, \pi/2}\) brings the qubit (initially in its ground state) to the superposition state \(|+\rangle = \frac{1}{\sqrt{2}} (|g\rangle + |e\rangle)\), where it interacts dispersively with the cavity for time \(\tau = \pi/\chi\). Depending on the photon parity, the qubit will end up in the \(|+\rangle\) (even) or in the \(|-\rangle\) (odd) state. The measurement is concluded with a second \(R_{\hat{y}, \pi/2}\) followed by readout of the qubit state.*

The qubit is initialised in the ground state. We then need to apply a single qubit \(\pi/2\) rotation around \(\hat{y}\), which will bring the qubit to the \(|+\rangle\) state:

\[
R_{\hat{y}, \pi/2} |g\rangle = \frac{1}{2} (|g\rangle + |e\rangle).
\]

According to (5.4), letting the qubit-cavity interact for time \(\tau = \pi/\chi\) is
5.1 Overview of the experiment and readout process

Equivalent to the operation:
\[ C_\pi |+\rangle = \frac{1}{2} (|g\rangle + e^{i\pi a^\dagger a}|e\rangle) = \frac{1}{2} (|g\rangle + (-1)^{a^\dagger a}|e\rangle), \]

i.e. for odd photon numbers \( n = \langle a^\dagger a \rangle \) the qubit will end up in state \( |-\rangle \), whereas for an even photon number in the resonator it will end up in \( |+\rangle \).

Finally, a second \( R_{y, \pi/2} \) pulse will bring the qubit states to \( |e\rangle \) or \( |g\rangle \) depending on whether they were in \( |-\rangle \) or \( |+\rangle \), respectively.

The photon parity measurement is concluded with a QND measurement of the qubit state via a readout resonator, as described in section 5.1.1.

5.1.3 Schematic of the device

Figure 5.3 shows a photograph of the designed chip.

![Photograph of the designed chip](image)

**Figure 5.3: Photograph of the designed chip.** The central conductors of the CPWs have been coloured in order to distinguish between different resonators.

The resonators are featured using coplanar waveguides (CPWs), which consist of a central conductor with separated grounded tracks on both sides. One end is open, such that the voltage reaches its peak value at this point, while the other is shorted so that there is zero voltage. This design creates a \( \lambda/4 \) resonator because precisely a quarter of the mode’s wavelength fits from one end to the other.

The \( \lambda/4 \) resonator featuring the Rabi cavity (red) is coupled to the transmon featuring the Rabi qubit (blue). During the Jaynes-Cummings
Towards an experimental implementation: designing the chip

part of the simulation, they will be strongly coupled with a detuning of \( \sim 1 \) MHz. Notice that the qubit is placed near the shorted end of the resonator in order to achieve a low coupling strength (\( g \sim 4 \) MHz).

The Rabi resonator is also capacitively coupled to a CPW (white), through which we can apply a coherent drive for the cavity displacements, \( D(\alpha) \). An ancillary transmon (green) is also coupled to the Rabi resonator, for the Wigner tomography scheme discussed in section 5.1.2.

Each transmon is coupled to a readout resonator (yellow) and each one of these resonators is capacitively coupled to a feedline, through which microwave pulses for readout spectroscopy can be applied. The rightmost feedline, is also used for applying the driving pulses on the Rabi qubit.

Both transmons are addressed by individual flux bias lines, for control of the qubit frequency. An applied voltage through these lines results in change of the current and flux in the SQUID loop. As we have earlier discussed, this changes the Josephson energy and effectively the frequency of the qubit. The bias line should be slightly off-centre with respect to the loop, otherwise it would cause the creation of cancelling currents from the two sides of the loop that would result in zero net flux.

The fabrication of this device has been done by N. K. Langford and A. Bruno. In the first stage of the fabrication, a NbTiN thin film is deposited on a Si substrate. The CPWs and notches for the qubits are defined using e-beam lithography and reactive ion etching. In the second stage, the Josephson junctions, made from Al/AlOx/Al, are added by double angle shadow evaporation. The ground plane is etched into thin grids in order to trap the propagation of unwanted magnetic vortices. Finally, air-bridges made of Al/Ti are used to connect ground planes.

5.2 Main considerations in designing the parameters

5.2.1 Eliminating the Purcell effect

One of the most significant forms of qubit relaxation is the Purcell effect [30]. It refers to the enhancement of the transmon relaxation rate due to the presence of the resonator and occurs at a rate

\[
\frac{1}{T_{\text{Purcell}}} = \kappa \left( \frac{g}{\Delta} \right)^2.
\] (5.6)
5.2 Main considerations in designing the parameters

This rate should be much smaller than the qubit relaxation rate $\gamma$, in order for the qubit not to be Purcell limited. In the case of the Rabi qubit and resonator, which are near resonance ($\Delta \ll$), this is achieved with a low-loss resonator ($\kappa \ll$).

In the case of readout resonators dispersively coupled to the qubits, where the decay rate $\kappa$ is designed to be large for faster readout, this can be controlled by a sufficiently large detuning $\Delta$.

5.2.2 Suppressing non-linear terms

The dispersive Hamiltonian in equation (5.1) is valid as long as the mean photon number inside the resonator $\bar{n}$ is well below the critical photon number [53]

$$n_{\text{crit}} = \frac{\Delta^2}{4g^2}. \quad (5.7)$$

Above that limit, higher order nonlinear terms become significant and the perturbative expansion breaks down. Therefore, we need to carefully choose the coupling strength and the qubit resonator detuning in order to allow room for sufficient photon population during the readout because, as we shall see in the next section, the measurement fidelity increases with the number of photons.

In the strong dispersive regime ($\chi > \kappa$), one needs to define another limit in the photon number [42]

$$n_{\chi} = \frac{\kappa \Delta}{\chi^2}. \quad (5.8)$$

The reason is that, while at $\bar{n} \ll n_{\text{crit}}$ the perturbation theory is accurate, when $\bar{n} \sim n_{\chi}$ higher order terms might become significant compared to the cavity linewidth due to the strong qubit/resonator coupling, i.e. when $g \gg \kappa$ the cavity might become significantly nonlinear.

5.2.3 High fidelity qubit measurements

Signal-to-noise ratio

The signal-to-noise ratio ($SNR$) for measuring the qubit state via a readout resonator is determined by how fast the measurement rate $\Gamma_m$ is, compared to the decay rate $\gamma$:

$$SNR = \eta \frac{\Gamma_m}{\gamma}. \quad (5.9)$$
where $\eta$ is the measurement efficiency.

The SNR is optimal when the radiation through the feedline is resonant with the resonator’s frequency and is given by [42]:

$$SNR = \eta \frac{\kappa}{\gamma} \frac{\bar{n}\chi^2}{\kappa^2/4 + \chi^2}, \quad (5.10)$$

where $\bar{n}$ is the maximum number of photons in the resonator and $\kappa, \gamma$ are the decay rates of the resonator and the qubit. For conventional amplifiers the efficiency is given by $\eta = \hbar \omega_r / k_B T_N$, where $T_N$ is the amplifier’s noise temperature and $k_B$ is Boltzmann’s constant. The SNR is maximised when $\kappa = 2\chi$.

### Readout fidelity

The expected readout fidelity over a measurement time $\tau_m$ is [54]:

$$F = e^{-\tau_m/2} \operatorname{erf} \left( \sqrt{\frac{SNR \tau_m}{2}} \right), \quad (5.11)$$

where we have introduced the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

From the above formula, one expects an enhancement in fidelity for short times (depending on the magnitude of SNR) until the exponential, which accounts for the qubit dissipation, starts to dominate.

### 5.2.4 Summary of the designed parameters

Taking the all the above factors into consideration, we present here our final choice of parameters for the chip design. In figure 5.4, we show the frequency scheme of resonators and transmons as a function of applied flux.

The Rabi qubit is a double sweet spot transmon, as we have already discussed. For the Jaynes-Cummings part of the simulation it will be strongly coupled to the Rabi resonator ($\omega_r/2\pi = 6$ GHz) with a detuning $\Delta/2\pi = 1$ MHz. During the bit flip part it will be detuned at 4.3 GHz (bottom sweet spot) by applying a magnetic flux of $\Phi_0/2$ (driving frequency: $\omega_d/2\pi = 4.3$ GHz). At the bottom sweet spot we also plan to measure its state via a readout resonator at $\omega_{\text{read}}/2\pi = 7$ GHz.

The Wigner qubit is also designed to be a double sweet spot transmon. At the bottom sweet spot (7 GHz) it will be coupled dispersively to the Rabi cavity ($g/2\pi = 50$ MHz, $\chi/2\pi = 1.07$ MHz) for the parity operation.
5.2 Main considerations in designing the parameters

We then plan to measure its state via a readout resonator at $\omega_{\text{read}}/2\pi = 8.5$ GHz. It is designed to be a double sweet spot transmon so that during the Trotter sequence it will be far detuned at its top sweet spot (10 GHz), to avoid any shifts of the Rabi resonator frequency.

In the tables below, we summarise the choice of parameters during the Jaynes-Cummings part, as well as during the parity operation and qubit readout:

**Parameters during Jaynes-Cummings interaction**

<table>
<thead>
<tr>
<th>Qubit</th>
<th>Resonator</th>
<th>$\Delta/2\pi$</th>
<th>$g/2\pi$</th>
<th>$\chi/2\pi$</th>
<th>$T_{\text{Purcell}}$</th>
<th>$n_{\text{crit}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rabi (6 GHz)</td>
<td>Rabi (6.001 GHz)</td>
<td>1 MHz</td>
<td>4 MHz</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rabi (6 GHz)</td>
<td>Readout (7 GHz)</td>
<td>1 GHz</td>
<td>50 MHz</td>
<td>-0.58 MHz</td>
<td>2.2 ms</td>
<td>100</td>
</tr>
<tr>
<td>Wigner (10 GHz)</td>
<td>Rabi (6 GHz)</td>
<td>4 GHz</td>
<td>50 MHz</td>
<td>-50.7 kHz</td>
<td>96 ms</td>
<td>1600</td>
</tr>
</tbody>
</table>

**Parameters during the parity operation**

<table>
<thead>
<tr>
<th>Qubit</th>
<th>Resonator</th>
<th>$\Delta/2\pi$</th>
<th>$g/2\pi$</th>
<th>$\chi/2\pi$</th>
<th>$T_{\text{Purcell}}$</th>
<th>$n_{\text{crit}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wigner (7 GHz)</td>
<td>Rabi (6 GHz)</td>
<td>1 GHz</td>
<td>50 MHz</td>
<td>1.07 MHz</td>
<td>6 ms</td>
<td>100</td>
</tr>
</tbody>
</table>

---

**Figure 5.4:** Frequency scheme of resonators and transmons as a function of applied flux.
Towards an experimental implementation: designing the chip

Parameters during qubit readout

<table>
<thead>
<tr>
<th>Qubit</th>
<th>Resonator</th>
<th>$\Delta/2\pi$</th>
<th>$g/2\pi$</th>
<th>$\chi/2\pi$</th>
<th>$T_{\text{Purcell}}$</th>
<th>$n_{\text{crit}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rabi (4.3 GHz)</td>
<td>Readout (7 GHz)</td>
<td>2.7 GHz</td>
<td>50 MHz</td>
<td>-0.093 MHz</td>
<td>15 ms</td>
<td>729</td>
</tr>
<tr>
<td>Rabi (4.3 GHz)</td>
<td>Rabi (6 GHz)</td>
<td>1.7 GHz</td>
<td>4 MHz</td>
<td>-1.4 kHz</td>
<td>2.7 s</td>
<td>28900</td>
</tr>
<tr>
<td>Wigner (7 GHz)</td>
<td>Readout (8.5 GHz)</td>
<td>1.5 GHz</td>
<td>50 MHz</td>
<td>-0.28 MHz</td>
<td>4.9 ms</td>
<td>225</td>
</tr>
</tbody>
</table>

Moreover, following equations (5.10) and (5.11), we calculate the expected fidelity as a function of measurement time (figure 5.5), assuming qubit lifetimes of $\sim 10 \mu$s. We estimate an efficiency $\eta \sim 0.3$ for the measurement of the Rabi qubit for which a Josephson parametric amplifier (JPA) [55] can be used. The frequency of the readout resonator for the Wigner qubit (8.5 GHz) is out of the JPA's frequency range, therefore we will use a conventional amplifier. Assuming a noise temperature $T_N = 10 \text{ K}$ we estimate the measurement efficiency $\eta = \frac{\hbar \omega_{\text{read}}}{k_B T_N} \approx 0.04$, for this qubit.

![Figure 5.5: Estimated readout fidelity versus measurement time for each qubit.](image)

The estimated measurement times, for which the fidelity is maximised, are summarised below:
### 5.3 Designing the chip elements

#### 5.3.1 Resonator quality factors

For a $\lambda/4$ resonator of length $l$, the resonator frequency is completely determined by the CPW’s dimensions [56]

$$\omega_r = \frac{\pi}{2l\sqrt{LC}}$$

(5.12)

where $C$ and $L$, are the capacitance and inductance per unit length.

In figure 5.6 we show the network representation of a $\lambda/4$ readout resonator coupled to a feedline, from where the transmission and reflection from the ports can be derived [57]:

$$S_{21} = S_{12} = \frac{2}{2 + Z_0/Z_L'}$$

$$S_{11} = S_{22} = -\frac{Z_0/Z_L}{2 + Z_0/Z_L'}$$

(5.13)

(5.14)

where $Z_L$ is the loaded impedance, i.e. the sum of the resonator and capacitor impedances.

The impedance is defined as the ratio of applied voltage to current flowing through the line. The characteristic impedance of a transmission line is given by [57]

$$Z_0 = \sqrt{\frac{R + i\omega L}{G + i\omega C}}$$

(5.15)

where $C$, $L$ are the capacitance and series inductance per unit length. The losses in the transmission line are determined by the shunt conductance and series resistance per unit length, $G$ and $R$ respectively.

The loaded quality factor of a resonator describes the competence between energy stored and energy leaking out of the system. It is given by the ratio of resonator frequency to the decay rate:

$$Q_L = \frac{\omega_r}{\kappa}$$

(5.16)
Towards an experimental implementation: designing the chip

Figure 5.6: Network representation of a $\lambda/4$ readout resonator coupled to a feedline. Figure obtained from [56].

It is related to the coupling quality factor $Q_C$, which describes losses due to coupling to other elements, and the internal quality factor $Q_i$, which describes intrinsic losses of the transmission line, via the following relation:

$$\frac{1}{Q_l} = \frac{1}{Q_C} + \frac{1}{Q_i}.$$  \hspace{1cm} (5.17)

Using the simulation software Sonnet, which uses a finite element method for solving Maxwell’s equations for different boundary conditions, we can determine the required capacitive coupling between the resonator and the feedline in order to obtain a desired coupling quality factor. Keeping the gap fixed and varying the coupler length, we obtain a series of transmission amplitudes for various frequencies. For the readout resonators that are coupled to the feedline as in figure 5.6, the coupling quality factor is determined by the transmission amplitude $S_{31}$ by [56],

$$Q_C = \frac{\pi}{2|S_{31}|^2}.$$  \hspace{1cm} (5.18)

Therefore, the choice of coupler length between resonator and feedline determines $S_{31}$ and therefore the quality factor and decay rate $\kappa$.

The Rabi resonator needs to have a very low decay rate in order to achieve a high fidelity quantum simulation. Ideally, it would not be coupled to the outside world, however we use an input coupler to a CPW through which the cavity displacements, necessary for the Wigner tomography, will be applied. The transmission amplitude in this case (only two ports), is related to the quality factor by:

$$Q_C = \frac{\pi}{|S_{21}|^2}.$$  \hspace{1cm} (5.19)
5.3.2 Josephson junctions

The transmon frequencies are determined by $\omega_{01} \simeq (\sqrt{8E_JE_C} - E_C)/\hbar$, where $E_J$ is flux tuneable with $E_J^{\text{max, min}} = E_J^{(1)} \pm E_J^{(2)}$. Since we plan to use double sweet spot transmons, the Josephson energies of the two junctions have to be different. Assuming a charging energy of $E_C/\hbar = 260 \text{ MHz}$ we calculate, for both transmon qubits, the Josephson energies required for each junction.

The Rabi qubit, has two sweet spots at $6.2 \text{ GHz} (E_J^{\text{max}}/\hbar = 20 \text{ GHz})$ and $4.3 \text{ GHz} (E_J^{\text{min}}/\hbar = 10 \text{ GHz})$ which can be achieved with junctions at $E_J^{(1)}/\hbar = 15.03 \text{ GHz}$, $E_J^{(2)}/\hbar = 5.033 \text{ GHz}$. Notice that $E_J/E_C$ is always above 38 and thus we are in the transmon regime.

The Wigner qubit has two sweet spots at $10 \text{ GHz} (E_J^{\text{max}}/\hbar = 50 \text{ GHz})$ and $7 \text{ GHz} (E_J^{\text{min}}/\hbar = 25.3 \text{ GHz})$ which requires $E_J^{(1)}/\hbar = 37.975 \text{ GHz}$, $E_J^{(2)}/\hbar = 12.635 \text{ GHz}$.

The Josephson energies are determined by the shape and width of the junction, since this changes the resistance from one conductor to the other. The relationship between Josephson energy and resistance $R$ at room temperature, is given by [39, 56]

$$E_J = \frac{\hbar\Delta_{\text{gap}}}{8e^2R}, \quad (5.20)$$

where $\Delta_{\text{gap}}$ is the bandgap of the superconductor (Al).

Therefore, we estimate the Josephson energy by measuring the normal-state resistance for several junction shapes and widths at room temperature.

5.3.3 Charging energy

The charging energy is determined by the total capacitance ($C_\Sigma$) between the transmon islands,

$$E_C = \frac{e^2}{2C_\Sigma}. \quad (5.21)$$

The total capacitance between the islands of the transmon is obtained using the software Ansoft Maxwell, which simulates the capacitance network of the two transmons coupled to the CPWs (figure 5.7) and determines the capacitance matrix $C$. This is done by calculating the total energy of the network

$$E = \frac{1}{2} \mathbf{Q} \cdot \mathbf{V}, \quad (5.22)$$
Figure 5.7: Design (top) and capacitance network (bottom) of a transmon coupled to two CPWs. The central conductors of the CPWs are indicated with light and dark green. The capacitance matrix is obtained for different separations between the superconducting islands (blue and red). Bottom figure obtained from [56].

where $Q = (Q_1, Q_2, ...)$, $V = (V_1, V_2, ...)$ are the charge and voltage vectors, respectively, and $Q = CV$.

We can then calculate the energy associated with the tunnelling of a Cooper-pair from one island to the other, by setting $Q_1 = e$, $Q_2 = -e$. Moreover, the charge at the centre of the CPW’s is set to zero ($Q_3 = Q_4 = 0$) and $V_5 = 0$ for the ground plane. This energy is now equal to $E_C$ and we are able to extract $C_C$ for various separations between the two islands and for several combinations of finger number and width. Therefore, we obtain the desired value for $E_C$ by varying these parameters.
5.3 Designing the chip elements

5.3.4 Coupling strength

The coupling strength for a transmon capacitively coupled to a CPW is given by [30]

\[ g_{j,j+1} = \frac{eV_{\text{rms}}^0}{\hbar} \frac{C_g}{C_\Sigma} \sqrt{2(j+1)} \left( \frac{E_J}{8E_C} \right)^{1/4}, \]  

(5.23)

where \( C_g \) is the gate capacitance and \( V_{\text{rms}}^0 \) the vacuum voltage fluctuations.

The total electric energy stored in the resonator (with capacitance \( C_r \)) is given by [56]

\[ \frac{1}{2} C_r (V_{\text{rms}}^0)^2 = \frac{1}{2} \left( \frac{1}{2} \hbar \omega_r \right). \]  

(5.24)

For \( \lambda/4 \) resonators the total capacitance of the resonator is determined by the characteristic impedance \( Z_0 \) and resonator frequency [56]:

\[ C_r = \frac{\pi}{4\omega_r Z_0}. \]  

(5.25)

Using the simulation software *Ansoft Maxwell*, for the design shown in figure 5.7, we obtain different coupling strengths by varying the gap between the central conductor of the CPW and the transmon island, as well as the coupler lengths.

For the coupling to the readout resonators, the transmons are placed near the open end of the resonators, and we aim for coupling strengths of 50 MHz which is achieved with a gap of 2 \( \mu \)m and a coupler length of 250 \( \mu \)m and 175 \( \mu \)m for the Rabi and Wigner qubits, respectively.

Concerning the coupling to the Rabi resonator, the Wigner qubit is placed near the open end with a gap of 2 \( \mu \)m and a coupler length of 280 \( \mu \)m for an expected \( g/2\pi = 50 \) MHz.

For the Rabi qubit, we want to achieve a much lower coupling strength of 4 MHz. For this reason, we calculate the coupler length required to achieve a \( g/2\pi = 40 \) MHz near the open end, which is 230 \( \mu \)m, and then we work out how close to the shorted end we need to place the transmon in order to reduce the coupling strength to 10% of this value. We do this as follows:

Since this is a \( \lambda/4 \) resonator of length \( l \), we have \( \lambda = 4l \). The dependence of the voltage from the antinode (where it is maximum) is given by:

\[ \cos \left( \frac{2\pi x}{\lambda} \right) = \cos \left( \frac{\pi x}{2l} \right), \]  

where \( x = 0 \) at the antinode and \( x = l \) at the open end. Therefore, we need to place the qubit at the point where \( \cos \left( \frac{\pi x}{2l} \right) = 0.1 \).
Conclusions and future work

In summary, using a full master equation numerical description, we have shown that it is possible to reach the deep-strong coupling regime of the Rabi model in a realistic circuit QED setup, by means of an analog-digital quantum simulation. Including the most important experimental considerations, we have identified a region of feasible operating parameters to achieve this, using a transmon qubit coupled to a 2D superconducting resonator. We have found that, for an accurate quantum simulation, we need to engineer unusually low coupling strengths (~ 5 MHz). Furthermore, we have demonstrated that first order errors in the Trotter approximation can be eliminated completely via a symmetric implementation of Trotter steps, offering substantial enhancement in the simulation fidelity. Following an extensive study of the deep-strong coupling regime, we have identified the Wigner function as a key tool to probe these dynamics, which are very accurately reproduced using our quantum simulation scheme. Finally, we have designed a device to enable the realisation of the experiment and carry out a direct Wigner tomography of the resonator state with high fidelity qubit measurements.

Once an experimental verification of these dynamics is achieved in the context of the Rabi model, it would be very interesting to extend these ideas in quantum systems with more qubits coupled to a resonator. Then, it would be possible to study the DSC regime in the context of the Dicke model [25], which predicts the existence of superradiant phase transitions.
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References


References


