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Abstract

We prove two new density results about 16-ranks of class groups of quadratic number fields. They can be stated informally as follows.

Theorem A. *The class group of* $\mathbb{Q}(\sqrt{-p})$ *has an element of order* 16 *for one-fourth of prime numbers* p *of the form* $a^2 + 16c^4$.

Theorem B. *The class group of* $\mathbb{Q}(\sqrt{-2p})$ *has an element of order* 16 *for one-eighth of prime numbers* $p \equiv -1 \bmod 4$.

These density results are interesting for several reasons. First, they are the first non-trivial density results about the 16-rank of class groups in a family of quadratic number fields. Second, they prove an instance of the Cohen-Lenstra conjectures. Third, both of their proofs involve new applications of powerful sieving techniques developed by Friedlander and Iwaniec. Fourth, we give an explicit description of the 8-Hilbert class field of $\mathbb{Q}(\sqrt{-p})$ whenever p is a prime number of the form $a^2 + 16c^4$; the lack of such an explicit description for the 8-Hilbert class field of $\mathbb{Q}(\sqrt{d})$ is the main obstacle to improving the estimates for the density of positive discriminants d for which the negative Pell equation $x^2 - dy^2 = -1$ is solvable.

In case of Theorem B, we give an explicit description of an element of order 4 in the class group of $\mathbb{Q}(\sqrt{-2p})$ and we compute its Artin symbol in the 4-Hilbert class field of $\mathbb{Q}(\sqrt{-2p})$, thereby generalizing a result of Leonard and Williams. Finally, we prove a power-saving error term for a prime-counting function related to the 16-rank of the class group of $\mathbb{Q}(\sqrt{-2p})$, thereby giving strong evidence against a conjecture of Cohn and Lagarias that the 16-rank is governed by a Čebotarev-type criterion.