Stellingen

behorende bij het proefschrift
On the 16-rank of class groups of quadratic number fields
door Djordjo Zeljko Milovic

If D is a fundamental discriminant, i.e., the discriminant of a quadratic number field, we let $\mathrm{Cl}(D)$ denote the narrow class group of the quadratic number field $\mathbb{Q}(\sqrt{D})$. Given a finite abelian group G, a prime number ℓ, and an integer $k \geq 1$, we define the ℓ^k-rank of G to be

$$\mathrm{rk}_{\ell^k} G = \dim_{\mathbb{F}_\ell} \left(\ell^{k-1}G/\ell^k G\right).$$

1. *Suppose p is a prime of the form $a^2 + c^4$, where a and c are integers.*

(i) If $a \equiv \pm 1 \bmod 16$ *and* $c \equiv 0 \bmod 4$*, then* $\mathrm{rk}_{16}\mathrm{Cl}(-4p) = 1$.

(ii) If $a \equiv \pm 3 \bmod 16$ *and* $c \equiv 2 \bmod 4$*, then* $\mathrm{rk}_{16}\mathrm{Cl}(-4p) = 1$.

(iii) If $a \equiv \pm 7 \bmod 16$ *and* $c \equiv 0 \bmod 4$*, then* $\mathrm{rk}_{16}\mathrm{Cl}(-4p) = 0$.

(iv) If $a \equiv \pm 5 \bmod 16$ *and* $c \equiv 2 \bmod 4$*, then* $\mathrm{rk}_{16}\mathrm{Cl}(-4p) = 0$.

2. *For all sufficiently large real numbers X, we have*

$$\#\{p \leq X : p \equiv 1 \bmod 4,\ \mathrm{rk}_{16}\mathrm{Cl}(-4p) = 1\} \geq \frac{X^{3/4}}{8 \log X}$$

and

$$\#\{p \leq X : p \equiv 1 \bmod 4,\ \mathrm{rk}_{16}\mathrm{Cl}(-4p) = \mathrm{rk}_8\mathrm{Cl}(-4p) - 1 = 0\} \geq \frac{X^{3/4}}{8 \log X}.$$

3. *For every $\epsilon > 0$, there is a constant $C_\epsilon > 0$ depending only on ϵ such that for every $X \geq 2$, we have*

$$\left| \sum_{\substack{p \leq X \\ p \equiv -1 \bmod 16}} \left(\frac{v}{u}\right) \right| \leq C_\epsilon X^{\frac{149}{150}+\epsilon},$$

where, for each p in the sum above, u and v are taken to be integers satisfying $p = u^2 - 2v^2$ and $u \equiv 1 \bmod 16$.

4. *The density of the set of prime numbers* $p \equiv -1 \bmod 4$ *for which* $\mathrm{rk}_{16}\mathrm{Cl}(-8p) = 1$ *is equal to*

$$\lim_{X \to \infty} \frac{\#\{p \leq X : p \text{ prime},\ p \equiv -1 \bmod 4,\ \mathrm{rk}_{16}\mathrm{Cl}(-8p) = 1\}}{\#\{p \leq X : p \text{ prime}\}} = \frac{1}{16}.$$

5. *The equality of Jacobi symbols*

$$\left(\frac{b}{a}\right) = \left(\frac{4a+9b}{9a+20b}\right)$$

holds true for all rational integers a and b such that a and $9a + 20b$ are odd and positive.

6. *Let $w = a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ be such that $\mathrm{Norm}(w)$ is a rational prime congruent to 1 modulo 8 and such that $|a + b| \equiv 3 \bmod 4$. Let $\overline{w} = a - b\sqrt{2}$, and let $\mathcal{I}(\overline{w})$ denote the group of all fractional ideals of $\mathbb{Z}[\sqrt{2}]$ coprime to $(\overline{w})$. Then the function*

$$\psi_w : \mathcal{I}(\overline{w}) \to S^1 = \{s \in \mathbb{C} : |s| = 1\}$$

defined by setting

$$\psi_w(\mathfrak{a}) = \left(\frac{z}{(\overline{w})}\right) \cdot \mathrm{sign}(\mathrm{Norm}(z)),$$

where z is any generator of the ideal $\mathfrak{a}$, is a Hecke character for the modulus $(\overline{w})\infty_1\infty_2$, where ∞_1 and ∞_2 are the two real embeddings of $\mathbb{Q}(\sqrt{2})$.

7. *Let $X > 1$ be a real number, let $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ be a function satisfying $\|f\|_\infty \leq 1$, and let $S(X; f)$ be defined as*

$$S(X; f) = \sum\sum_{\substack{pq \leq X \\ p<q}} f(p, q),$$

where p and q denote prime numbers. Let Y be a real number satisfying $1 < Y < X^{\frac{1}{4}}$. Suppose that there exist positive real numbers δ_1, δ_2, and δ_3 satisfying $\delta_3 < 2\delta_2$ such that

$$\text{(A)} \qquad A_p(X; f) = \sum_{q \leq X} f(p, q) \ll XY^{-\delta_1}$$

for all $p \leq Y$, where the implied constant is absolute, and such that

$$\text{(B)} \qquad \mathcal{B}(M, N; f, \Delta) = \sum\sum_{\substack{M<p\leq M+\Delta M \\ N<q\leq N+\Delta N}} f(p, q) \ll \Delta^{-\delta_2} \left(M^{-\delta_3} + N^{-\delta_3}\right) \Delta^2 MN,$$

for all $M, N > 1$ and $\Delta \in (0, 1)$ satisfying $\Delta M, \Delta N > 1$, where the implied constant is absolute. Then there exists a positive real number δ in $(0, 1)$ such that

$$S(X; f) \ll Y^{-\delta} X \log X,$$

where the implied constant is absolute. Moreover, we can take

$$\delta = \min\left(\frac{\delta_1}{2}, \frac{\delta_3}{2\delta_2}, \frac{\delta_3}{2}\right).$$

8. *Let p and q denote distinct prime numbers congruent to 1 modulo 4. Then we have*

$$\liminf_{X\to\infty} \frac{\#\{pq \leq X : \mathrm{rk}_4\mathrm{Cl}(8pq) = 2, \mathrm{rk}_8\mathrm{Cl}(8pq) \geq 1\}}{\#\{pq \leq X : \mathrm{rk}_4\mathrm{Cl}(8pq) = 2\}} \geq \frac{1}{8}.$$

9. *Boj ne bije svijetlo oružje, već boj bije srce u junaka.*