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Introduction

The present thesis is a collection of results about problems that, during the last four years, have challenged the author. The line connecting the works presented here is the study of the arithmetic of surfaces that are double covers of the projective plane, ramified along a curve of low degree: in particular del Pezzo and K3 surfaces.

In Chapter 1, we recall some preliminary results about lattice theory and algebraic geometry. After giving the definition of a lattice and basic properties of integral lattices, the focus shifts towards algebraic geometry. Namely, the definitions of weighted projective spaces, double covers of surfaces, Picard groups, K3 surfaces, and del Pezzo surfaces are given, together with some properties of these objects that will be of use at a later stage.

The topic of Chapter 2 is the arithmetic of del Pezzo surfaces of degree 2 over finite fields. Del Pezzo surfaces can be classified using their *degree*, that is always an integer between 1 and 9. Morally, the higher the degree the *easier* the surface. For example, the projective plane $\mathbb{P}^2$ is a del Pezzo surface of degree 9; the blow-up of $\mathbb{P}^2$ at one point, and $\mathbb{P}^1 \times \mathbb{P}^1$ are del Pezzo surfaces of degree 8; smooth cubics in $\mathbb{P}^3$ are del Pezzo surfaces of degree 3; double covers of $\mathbb{P}^2$ ramified along a smooth quartic curve give examples of del Pezzo surfaces of degree 2.

It is a fact that every del Pezzo surface over an algebraically closed field is birationally equivalent to $\mathbb{P}^2$ (see [Man86, Theorem IV.24.4]). Over arbitrary fields, the situation is more complicated, and so it is easier to look at weaker notions. Let k be any field and let X be a variety of dimension n over k. The variety X is said to be *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$, defined over k.

Work of B. Segre, Yu. Manin, J. Kollár, and M. Pieropan prove that every del Pezzo surface of degree $d \geq 3$ defined over k is unirational, provided that the set $X(k)$ of rational points is non-empty. C. Salgado, D. Testa, and A. Várilly-Alvarado prove that all del Pezzo surfaces of degree 2 over a finite field are unirational as well, except possibly for three isomorphism classes of surfaces (see [STVA14, Theorem 1]). In Chapter 2 it is shown that these remaining three cases are also unirational, thus proving the following theorem.

Theorem A. *Every del Pezzo surface of degree 2 over a finite field is unirational.*

A more general criterion for unirationality of del Pezzo surfaces of degree 2 is also given.

Theorem B. *Suppose k is a field of characteristic not equal to 2, and let $\overline{k}$ be an algebraic closure of k. Let X be a del Pezzo surface of degree 2 over k. Let $B \subset \mathbb{P}^2$ be the branch locus of the anti-canonical morphism $\pi\colon X \to \mathbb{P}^2$. Let $C \subset \mathbb{P}^2$ be a projective curve that is birationally equivalent with $\mathbb{P}^1$ over k. Assume that all singular points of C that are contained in B are ordinary singular points. Then the following statements hold.*

1. *Suppose that there is a point $P \in X(k)$ such that $\pi(P) \in C - B$. Suppose that B contains no singular points of C and that all intersection points of B and C have even intersection multiplicity. Then the surface X is unirational.*

2. *Suppose that one of the following two conditions hold.*

   (a) There is a point $Q \in C(k) \cap B(k)$ that is a double or a triple point of C. The curve B contains no other singular points of C, and all intersection points of B and C have even intersection multiplicity.

   (b) There exist two distinct points $Q_1, Q_2 \in C(\overline{k}) \cap B(\overline{k})$ such that B and C intersect with odd multiplicity at Q_1 and Q_2

and with even intersection multiplicity at all other intersection points. Furthermore, the points Q_1 and Q_2 are smooth points or double points on the curve C, and B contains no other singular points of C.

Then there exists a field extension ℓ of k of degree at most 2 for which the preimage $\pi^{-1}(C_\ell)$ is birationally equivalent with $\mathbb{P}^1_\ell$; for each such field ℓ, the surface X_ℓ is unirational.

All these results are part of joint work with Ronald van Luijk; Theorem A has been published in [FvL16]; everything contained in Chapter 2 can also be found in [FvL15].

While Chapter 2 is devoted to the study of the arithmetic of del Pezzo surfaces, Chapter 3 deals with the arithmetic of K3 surfaces. K3 surfaces are a possible 2-dimensional generalisation of elliptic curves, and in the last sixty years they have attracted a growing attention since they are on the boundary between those surface whose geometry and arithmetic we understand pretty well, and those whose geometry and arithmetic is still obscure to us. Smooth quartic surfaces in $\mathbb{P}^3$ are examples of K3 surfaces, as well as double covers of $\mathbb{P}^2$ ramified along a smooth sextic curve.

Let X be a K3 surface. The study of the Picard lattice $\operatorname{Pic} X$ can give information about the arithmetic and the geometry of X. Even though during the last years a range of techniques and theoretical algorithms to compute the Picard lattice have been developed (see Chapter 3 and [PTvL15] for references), we do not know yet of any practical algorithm to compute the Picard lattice of a K3 surface.

In the chapter, the following family of K3 surfaces over $\mathbb{Q}$ is considered:

$$\mathfrak{X}\colon w^2 = x^6 + y^6 + z^6 + tx^2y^2z^2.$$

Let t_0 be an element of $\overline{\mathbb{Q}}$. Then X_{t_0} denotes denotes the member of $\mathfrak{X}$ for $t = t_0$, that is, X_{t_0} is the surface over $\overline{\mathbb{Q}}$ given by the equation $w^2 = x^6 + y^6 + z^6 + t_0x^2y^2z^2$. The main result of the chapter is a description of the Picard lattice of the elements of $\mathfrak{X}$, given by the following theorem.

Theorem C. *Let* $t_0 \in \overline{\mathbb{Q}}$ *be an algebraic number. Then the surface* X_{t_0} *has Picard number* $\rho(X_{t_0}) \in \{19, 20\}$*.*

If $\rho(X_{t_0}) = 19$*, then the Picard lattice* $\operatorname{Pic} X_{t_0}$ *is an even lattice of rank* 19*, determinant* $2^5\, 3^3$*, signature* $(1, 18)$*, and discriminant group isomorphic to* $C_6 \times C_{12}^2$*.*

A more explicit description is given in Theorem 3.1.4. This theorem can be used to rule out information about the geometry and the arithmetic of the elements of the family $\mathfrak{X}$. In the last section of the chapter we give some corollaries in this spirit.

The whole Chapter 3 is part of joint work with Florian Bouyer, Edgar Costa, Christopher Nicholls, and Mckenzie West, and it comes from a problem proposed by Anthony Várilly-Alvarado during the Arizona Winter School 2015 (see [VA15, Project 1]).

In Chapter 4 we continue our study of K3 surfaces. Let k be any field, and let x_0, x_1, x_2, x_3 denote the coordinates of $\mathbb{P}^3_k$. Let $X \subset \mathbb{P}^3$ be a surface. We say that X is *determinantal* if it is defined by an equation of the form

$$X\colon\ \det M = 0,$$

where M is a square matrix whose entries are linear homogeneous polynomials in x_0, x_1, x_2, x_3.

Let $L_{(4,2,-4)}$ be the rank 2 lattice with Gram matrix

$$\begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}.$$

In [Ogu15], Oguiso shows that a K3 surface S with Picard lattice isometric to $L_{(4,2,-4)}$ admits a fixed point free automorphism g of positive entropy and can be embedded into $\mathbb{P}^3$ as a quartic surface. In the same paper, Oguiso states that "it seems extremely hard but highly interesting to write down explicitly the equation of S and the action of g in terms of the global homogeneous coordinates of $\mathbb{P}^3$, for at least one of such pairs" (cf. [Ogu15, Remark 4.2]). In [FGvGvL13], it is shown that in fact such surfaces can be embedded as determinantal quartic surfaces. In Chapter 4, as well as in the paper, we provide an explicit example of a determinantal quartic surface over $\mathbb{Q}$ with Picard lattice isometric to $L_{(4,2,-4)}$.

Theorem D. *Let $R = \mathbb{Z}[x_0, x_1, x_2, x_3]$ and let $M \in M_4(R)$ be any 4×4 matrix whose entries are homogeneous polynomials of degree 1 and such that M is congruent modulo 2 to the matrix*

$$M_0 = \begin{pmatrix} x_0 & x_2 & x_1+x_2 & x_2+x_3 \\ x_1 & x_2+x_3 & x_0+x_1+x_2+x_3 & x_0+x_3 \\ x_0+x_2 & x_0+x_1+x_2+x_3 & x_0+x_1 & x_2 \\ x_0+x_1+x_3 & x_0+x_2 & x_3 & x_2 \end{pmatrix}.$$

Denote by X the complex surface in $\mathbb{P}^3$ given by $\det M = 0$. Then X is a K3 surface and its Picard lattice is isometric to $L_{(4,2,-4)}$.

This result is part of joint work with Alice Garbagnati, Bert van Geemen, and Ronald van Luijk; all the results contained in Chapter 4 are also exposed in [FGvGvL13]. In the same paper, an explicit description of the action of the fixed point free automorphism with positive entropy of X is also provided, giving a full answer to Oguiso's remark.