**Author**: Festi, Dino
**Title**: Topics in the arithmetic of del Pezzo and K3 surfaces
**Issue Date**: 2016-07-05

# Chapter 4

# A determinantal quartic K3 surface with prescribed Picard lattice

In this chapter we present a determinantal quartic K3 surface whose Picard lattice is isomorphic to a particular lattice of rank 2. This construction is made interesting by Oguiso in [Ogu15], where he showed that K3 surfaces with such a Picard lattice admit a fixed point free automorphism of positive entropy and can be embedded into $\mathbb{P}^3$ as quartic surfaces. In [FGvGvL13], it is shown that in fact such surfaces can be embedded as *determinantal* quartic surfaces, and an explicit example of such a surface is provided, giving also an explicit description of the automorphism predicted by Oguiso. Here the contribution of the author of this thesis to that paper is presented, except for Proposition 4.2.2 and Remark 4.2.3, due to Bert van Geemen and Alice Garbagnati. All the material presented here is part of a joint work with Alice Garbagnati, Bert van Geemen, and Ronald van Luijk, and it can be found in [FGvGvL13].

## 4.1 The main result

Let $k$ be any field, and let $x_0, x_1, x_2, x_3$ denote the coordinates of $\mathbb{P}^3_k$. Let $X \subset \mathbb{P}^3$ be a surface. We say that $X$ is *determinantal* if it is defined by an equation of the form

$$X\colon\ \det M = 0,$$

where $M$ is a square matrix whose entries are linear homogeneous polynomials in $x_0, x_1, x_2, x_3$.

Let $L = L_{(4,2,-4)}$ be the rank 2 lattice with Gram matrix

$$\begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}. \tag{4.1}$$

The following is the main result of this chapter.

**Theorem 4.1.1.** *Let $R = \mathbb{Z}[x_0, x_1, x_2, x_3]$ and let $M \in M_4(R)$ be any $4\times 4$ matrix whose entries are homogeneous polynomials of degree 1 and such that $M$ is congruent modulo 2 to the matrix*

$$M_0 = \begin{pmatrix} x_0 & x_2 & x_1 + x_2 & x_2 + x_3 \\ x_1 & x_2 + x_3 & x_0 + x_1 + x_2 + x_3 & x_0 + x_3 \\ x_0 + x_2 & x_0 + x_1 + x_2 + x_3 & x_0 + x_1 & x_2 \\ x_0 + x_1 + x_3 & x_0 + x_2 & x_3 & x_2 \end{pmatrix}. \tag{4.2}$$

*Denote by $X$ the complex surface in $\mathbb{P}^3$ given by $\det M = 0$. Then $X$ is a K3 surface and its Picard lattice is isometric to $L$.*

*Remark* 4.1.2. Let $\varphi \in \mathbb{R}$ be the real number given by

$$\varphi := \frac{1+\sqrt{5}}{2},$$

and let $K := \mathbb{Q}(\varphi)$ be the number field obtained by adjoining $\varphi$ to $\mathbb{Q}$; Notice that $K = \mathbb{Q}(\sqrt{5})$. Let $\mathcal{O}_K$ be the ring of integers of $K$. Then $\mathcal{O}_K = \mathbb{Z}[\varphi]$. The ring $\mathcal{O}_K$ has the structure of a $\mathbb{Z}$-module of rank 2, and $(1, \varphi)$ is a basis. If $x = a + b\varphi$ is an element of $\mathcal{O}_K$, denote by $\overline{x}$ the Galois conjugate of $x$. So, if $x = r + s\sqrt{5}$, then $\overline{x} = r - s\sqrt{5}$; it follows that $\overline{\varphi} = 1 - \varphi$, and hence

$$\overline{a + b\varphi} = a + b - b\varphi.$$

We define the bilinear form $b\colon \mathcal{O}_K \times \mathcal{O}_K \to \mathbb{Z}$ by

$$(x, y) \mapsto 2(x\overline{y} + \overline{x}y).$$

It is easy to see that $b$ is a symmetric, non-degenerate bilinear form of $\mathcal{O}_K$. Then $(\mathcal{O}_K, b)$ is an integral lattice of rank 2. If we consider the basis $(1, \varphi)$, we immediately see that $(\mathcal{O}_K, b)$ is isometric to the lattice $L$ defined in 4.1.

## 4.2 Proof of the main result

In this section we give a proof of Theorem 4.1.1. Let $L$ be the lattice defined in 4.1, and let $R = \mathbb{Z}[x_0, x_1, x_2, x_3]$ and let $M \in M_4(R)$ be any $4 \times 4$ matrix whose entries are homogeneous polynomials of degree 1 and such that $M$ is congruent modulo 2 to the matrix $M_0$ given in (4.2). From now until the end of the section, let $X$ be the complex surface defined by $\det M = 0$.

We will first show that $X$ is a K3 surface with a Picard lattice admitting $L$ as sublattice. Then we will show that $X$ has Picard number at most 2, and finally we will prove that $L$ is the whole Picard lattice of $X$, hence proving Theorem 4.1.1.

**Lemma 4.2.1.** *Let $X$ be defined as before; then $X$ is smooth.*

*Proof.* Let $X_2$ be the surface over $\mathbb{F}_2$ defined by $\det M_0 = 0 \mod 2$. Using a computer, one can check that $X_2$ is smooth. Notice that $X$ equals the reduction of $X$ modulo 2. Then it follows that $X$ is smooth. □

**Proposition 4.2.2.** *Let $X$ be defined as before. Then $X$ is a complex K3 surface and $L$ can be embedded into* $\operatorname{Pic} X$.

*Proof.* It immediately follows from [FGvGvL13, Proposition 2.2]. □

*Remark* 4.2.3. As shown in the proof of [FGvGvL13, Proposition 2.2], it is easy to find two divisors of $X$ generating inside $\operatorname{Pic} X$ a sublattice isometric to $L$. By [Bea00, Proposition 6.2], the surface $X$ admits a projective normal curve of degree 6 and genus 3; let $C \in \operatorname{Pic} X$ be the class of that curve and let $H \in \operatorname{Pic} X$ be the hyperplane class. Then the sublattice $\langle H, C \rangle \subseteq \operatorname{Pic} X$ has Gram matrix

$$\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$$

and it is isometric to $L$.

The previous proposition implies that 2 is a lower bound for $\rho(X)$. To show that 2 is also an upper bound for $\rho(X)$, we follow [FGvGvL13, Section 5] and we use a method described in [vL07]. Recall the definition of the étale cohomology groups for schemes over finite fields, given in 1.2.37. The following results show how to give an upper bound for the geometric Picard number of $S$.

**Proposition 4.2.4.** *Let $K$ be a number field with ring of integers $\mathcal{O}$, let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ with residue field $k$, and let $\mathcal{O}_\mathfrak{p}$ be the localization of $\mathcal{O}$ at $\mathfrak{p}$. Let $\mathfrak{S}$ be a smooth projective surface over $\mathcal{O}_\mathfrak{p}$ and set $\overline{S} = \mathfrak{S} \times_{\mathcal{O}_\mathfrak{p}} \overline{K}$ and $S_{\overline{k}} = \mathfrak{S} \times_{\mathcal{O}_\mathfrak{p}} \overline{k}$. Let $\ell$ be a prime not dividing $q = \#k$. Let $F_q^*$ denote the automorphism of $H^2_{ét}(S_{\overline{k}}, \mathbb{Q}_\ell(1))$ induced by the $q$-th power Frobenius $F_q \in \operatorname{Gal}(\overline{k}/k)$.*

*The rank of $\operatorname{Pic} \overline{S}$ is at most the number of eigenvalues of $F_q^*$ that are roots of unity, counted with multiplicity.*

*Proof.* Combining Lemma 1.2.52 and Remark 1.2.37 we get a chain of primitive embeddings of lattices

$$\operatorname{Pic} \overline{S} \otimes_\mathbb{Z} \mathbb{Q}_\ell \hookrightarrow \operatorname{Pic} S_{\overline{k}} \otimes_\mathbb{Z} \mathbb{Q}_\ell \hookrightarrow H^2_{\text{ét}}(S_{\overline{k}}, \mathbb{Q}_\ell(1)) ,$$

and hence an upper bound for the rank of $\operatorname{Pic} S_{\overline{k}} \otimes_\mathbb{Z} \mathbb{Q}_\ell$ is an upper bound for the rank of $\operatorname{Pic} \overline{S}$ too.

Let $c$ be an element of $\operatorname{Pic} X_{\overline{k}}$; then $c$ is represented by a divisor of $X_{\overline{k}}$, say $c = [C]$, for some $C \in \operatorname{Div} X_{\overline{k}}$. Since $\operatorname{Pic} S_{\overline{k}} \cong \operatorname{NS} S_{\overline{k}}$ is finitely generated (cf. Theorem 1.2.7), it follows that some power of Frobenius acts as the identity on $\operatorname{Pic} \overline{k}$. This means that the rank of $\operatorname{Pic} S_{\overline{k}}$ is at most the number of eigenvalues of $F_q^*$ that are roots of unity, counted with multiplicity, and therefore the rank of $\operatorname{Pic} \overline{S}$ is so too.

See also [FGvGvL13, Proposition 5.2] and/or [vL07, Proposition 6.2 and Corollary 6.4]. □

**Proposition 4.2.5.** *Let $S$ be a K3 surface over a finite field $k \cong \mathbb{F}_q$. As in Proposition 4.2.4, let $F_q^*$ denote the automorphism of $H^2_{ét}(S_{\overline{k}}, \mathbb{Q}_\ell(1))$ induced by the $q$-th power Frobenius $F_q \in \operatorname{Gal}(\overline{k}/k)$, and for any $n$, let $\operatorname{Tr}((F_q^*)^n)$ denote the trace of $(F_q^*)^n$. Then we have*

$$\operatorname{Tr}\left((F_q^*)^n\right) = \frac{\#S(\mathbb{F}_{q^n}) - 1 - q^{2n}}{q^n}.$$

*Furthermore, the characteristic polynomial* $f(t) = \det(t - F_q^*) \in \mathbb{Q}[t]$ *of* $F_q^*$ *has degree* 22 *and satisfies the functional equation*

$$t^{22} f(t^{-1}) = \pm f(t).$$

*Proof.* Let $F_S$ be the $q$-th power absolute Frobenius of $S$, which acts as the identity on the $k$-rational points of $S$ and by raising to the $q$-th power on the coordinate rings of affine open subsets of $X$. The geometric Frobenius $\varphi = F_S \times 1$ on $S \times_k \overline{k} = \overline{S}$ is an endomorphism of $\overline{S}$ over $\overline{k}$ (cf. [Mil80, proof of V.2.6 and pages 290–291]). The set of fixed points of $\varphi^n$ is $S(\mathbb{F}_{q^n})$. The Weil conjectures (see [Mil80, Section VI.12 ], recall that these were proven by Deligne) state that the eigenvalues of $\varphi^*$ acting on $H^i_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell)$ have absolute value $q^{i/2}$. Since $S$ is a K3 surface, we have $\dim H^i_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell) = 1, 0, 22, 0, 1$ for $i = 0, 1, 2, 3, 4$, respectively (see 1.2.37), so the Lefschetz trace formula for $\varphi^n$ (see [Mil80, Theorems VI.12.3 and VI.12.4]) yields

$$\begin{aligned} \#S(\mathbb{F}_{q^n}) = \sum_{i=0}^{4} (-1)^i \operatorname{Tr}\left((\varphi^*)^n | H^i_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell)\right) &= \\ = 1 + \operatorname{Tr}\left((\varphi^*)^n | H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell)\right) + q^{2n}&. \end{aligned} \tag{4.3}$$

For the remainder of this proof we restrict our attention to the middle cohomology, so $H^i_{\text{ét}}$ with $i = 2$. By the (proven) Weil conjectures, the characteristic polynomial $f_\varphi(t) = \det(t - \varphi^* | H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell))$ is a polynomial in $\mathbb{Z}[t]$ satisfying the functional equation $t^{22} f_\varphi(q^2/t) = \pm q^{22} f_\varphi(t)$ (note that the polynomial $P_2(X, t) = \det(1 - \varphi^* t | H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell))$ of [Mil80, Section VI.12], is the reverse of $f_\varphi$). Let $\varphi^*(1)$ denote the action on $H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell(1))$ (with a Tate twist) induced by $\varphi$. Note that the fact that $\varphi^*(1)$ acts on the middle cohomology is not reflected in the notation. The eigenvalues of $\varphi^*(1)$ differ from those of $\varphi^*$ on $H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell)$ by a factor $q$ (see [Tat65]), so we have

$$\operatorname{Tr}\left((\varphi^*)^n | H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell)\right) = q \cdot \operatorname{Tr}\left(\varphi^*(1)^n\right), \tag{4.4}$$

and the characteristic polynomial $f_\varphi^{(1)} \in \mathbb{Q}[t]$ of $\varphi^*(1)$ satisfies the functional equation $q^{22} f_\varphi^{(1)}(t) = f_\varphi(qt)$, and thus also the equation $t^{22} f_\varphi^{(1)}(1/t) = \pm f_\varphi^{(1)}(t)$. It follows that the eigenvalues, and hence the characteristic polynomials, of $\varphi^*(1)$ and $\varphi^*(1)^{-1}$ coincide. Finally, the product of the geometric Frobenius $\varphi = F_S \times 1$ and the Galois automorphism $1 \times F_q$ on $S \times_k \overline{k} = \overline{S}$ is the absolute Frobenius $F_{\overline{S}}$, which acts as the identity on the cohomology groups, so the maps $\varphi^*(1)$ and $F_q^*$ act as inverses of each other (see [Mil80, Lemma VI.13.2 and Remark VI.13.5,] and [Tat65, Chapter 3]). We conclude $f = f_\varphi^{(1)}$ and $\mathrm{Tr}\left((F_q^*)^n\right) = \mathrm{Tr}(\varphi^*(1)^{-n}) = \mathrm{Tr}(\varphi^*(1)^n)$, which, together with (4.3) and (4.4), implies the proposition. □

**Proposition 4.2.6.** *Let $X$ be defined as at the beginning of the section Then $\rho(X) \leq 2$.*

*Proof.* Let $\mathfrak{S}$ denote the surface over the localization $\mathbb{Z}_{(2)}$ of $\mathbb{Z}$ at the prime 2 given by $\det M = 0$, and write $S'$ and $\overline{S'}$ for the reductions $\mathfrak{S}_{\mathbb{F}_2}$ and $\mathfrak{S}_{\overline{\mathbb{F}}_2}$, respectively. One checks that $S'$ is smooth and $\mathfrak{S}$ is reduced, for instance with `MAGMA` [BCP97]. Since $\mathrm{Spec}\,\mathbb{Z}_{(2)}$ is integral and regular of dimension 1, the scheme $\mathfrak{S}$ is integral, and the map $\mathfrak{S} \to \mathrm{Spec}\,\mathbb{Z}_{(2)}$ is dominant, it follows from [Har77, Proposition III.9.7], that $\mathfrak{S}$ is flat over $\mathrm{Spec}\,\mathbb{Z}_{(2)}$. Since the fiber over the closed point is smooth, it follows from [Liu02, Definition 4.3.35], that $\mathfrak{S}$ is smooth over $\mathrm{Spec}\,\mathbb{Z}_{(2)}$. Therefore, $S = \mathfrak{S}_{\mathbb{C}}$ is smooth as well, so $S$ and $S'$ are K3 surfaces. Let $F_2^*$ denote the automorphism of $H^2_{\text{ét}}(\overline{S'}, \mathbb{Q}_\ell(1))$ induced by Frobenius $F_2 \in \mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2)$.

The divisor classes in $H^2_{\text{ét}}(\overline{S'}, \mathbb{Q}_\ell(1))$ defined by the hyperplane class and the curve $C$ as in Remark 4.2.3 span a two-dimensional subspace $V$ on which $F_2^*$ acts as the identity. We denote the linear map induced by $F_2^*$ on the quotient $W := H^2_{\text{ét}}(\overline{S}_2, \mathbb{Q}_\ell(1))/V$ by $F_2^*|_W$, so that $\mathrm{Tr}(F_2^*)^n = \mathrm{Tr}(F_2^*|_V)^n + \mathrm{Tr}(F_2^*|_W)^n = 2 + \mathrm{Tr}(F_2^*|_W)^n$ for every integer $n$. From Proposition 4.2.4, we obtain

$$\mathrm{Tr}(F_2^*|_W)^n \;=\; -2 \;+\; \frac{\#S'(\mathbb{F}_{2^n}) - 1 - 2^{2n}}{2^n}\;.$$

We counted the number of points in $S'(\mathbb{F}_{2^n})$ for $n = 1, \ldots, 10$ with `MAGMA`. The results are in the table below.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\#S'(\mathbb{F}_{2^n})$ | 6 | 26 | 90 | 258 | 1146 | 4178 | 17002 | 64962 | 260442 | 1044786 |
| $\mathrm{Tr}(F_2^*\|_W)^n$ | $-\frac{3}{2}$ | $\frac{1}{4}$ | $\frac{9}{8}$ | $-\frac{31}{16}$ | $\frac{57}{32}$ | $-\frac{47}{64}$ | $\frac{361}{128}$ | $-\frac{1087}{256}$ | $-\frac{2727}{512}$ | $-\frac{5839}{1024}$ |

If $\lambda_1, \ldots, \lambda_{20}$ denote the eigenvalues of $F_2^*|_W$, then the trace of $(F_2^*|_W)^n$ equals

$$\mathrm{Tr}(F_2^*|_W)^n \;=\; \lambda_1^n + \ldots + \lambda_{20}^n \;,$$

i.e., it is the $n$-th power sum symmetric polynomial in the eigenvalues of $F_2^*|_W$. Let $e_n$ denote the elementary symmetric polynomial of degree $n$ in the eigenvalues of $F_2^*|_W$ for $n \geq 0$. Using Newton's identities

$$ne_n = \sum_{i=1}^{n} (-1)^{i-1} e_{n-i} \cdot \mathrm{Tr}(F_2^*|_W)^i$$

and $e_0 = 1$, we compute the values of $e_n$ for $n = 1, \ldots, 10$. They are listed in the following table.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $e_n$ | $-\frac{3}{2}$ | 1 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | 0 | $-1$ | 2 |

We denote the characteristic polynomial of a linear operator $T$ by $f_T$, so that

$$f_{F_2^*} = f_{F_2^*|_V} \cdot f_{F_2^*|_W} = (t-1)^2 f_{F_2^*|_W} \;.$$

Because $f_{F_2^*}$ satisfies the functional equation of Proposition 4.2.5, the polynomial $f_{F_2^*|_W}$ satisfies $t^{20} f_{F_2^*|_W}(t^{-1}) = \pm f_{F_2^*|_W}(t)$. Since the middle coefficient $e_{10} = 2$ of $t^{10}$ in $f_{F_2^*|_W}$ is nonzero, the sign in this functional equation is $+1$, so $f_{F_2^*|_W}$ is palindromic and we get

$$\begin{aligned} f_{F_2^*|_W} &= t^{20} - e_1 t^{19} + e_2 t^{18} - \cdots + e_{10} t^{10} - e_9 t^9 + \cdots - e_1 t + 1 \\ &= t^{20} + \tfrac{3}{2} t^{19} + t^{18} - \tfrac{1}{2} t^{13} + t^{11} + 2t^{10} + t^9 - \tfrac{1}{2} t^7 + t^2 + \tfrac{3}{2} t + 1. \end{aligned}$$

With `MAGMA`, one checks that this polynomial is irreducible over $\mathbb{Q}$, and as it is not integral, its roots are not algebraic integers, so none of its roots is a root of unity. Hence, the polynomial $f_{F_2^*} = (t-1)^2 f_{F_2^*|_W}$ has exactly two roots that are a root of unity. This implies that $F_2^*$ has only two eigenvalues (counted with multiplicity) that are roots of unity, and so, by Proposition 4.2.4, it follows that the rank of the Picard group $\operatorname{Pic} S \cong \operatorname{Pic} \mathfrak{S}_{\overline{\mathbb{Q}}}$) is bounded by two from above. □

We have now all the elements to prove Theorem 4.1.1.

*Proof of Theorem 4.1.1.* By Proposition 4.2.2 we have $\rho(X) \geq 2$ and $L$ can be embedded into $\operatorname{Pic} X$.

By Proposition 4.2.6 we have that $\rho(X) \leq 2$. It follows that $\rho(X) = 2$ and $L$ is a finite index sublattice of $\operatorname{Pic} X$.

Since $\det L = -20$, from Lemma 1.1.5 it follows that the index $[\operatorname{Pic} X : L]$ can only be 1 or 2. Assume $[\operatorname{Pic} X : L] = 2$, and let $D$ be an element of $\operatorname{Pic} X$ that is not in $L$. Let $H, C$ be defined as in Remark 4.2.3, (namely the hyerplane section class and the class of a curve of degree 6 and degree 3), then $(H, C)$ is a basis of $L$ and $D = \frac{aH+bC}{2}$ It follows that $D^2 = a^2 + 3ab + b^2$. Since $L$ is an even lattice, $D^2$ is even and so $a$ and $b$ are both even. Then $D = \frac{a}{2}H + \frac{b}{2}C$ is inside $L$, getting a contradiction. The contradiction comes from the assumption that $[\operatorname{Pic} X : L] = 2$. So $[\operatorname{Pic} X : L] = 1$ and this concludes the proof. □

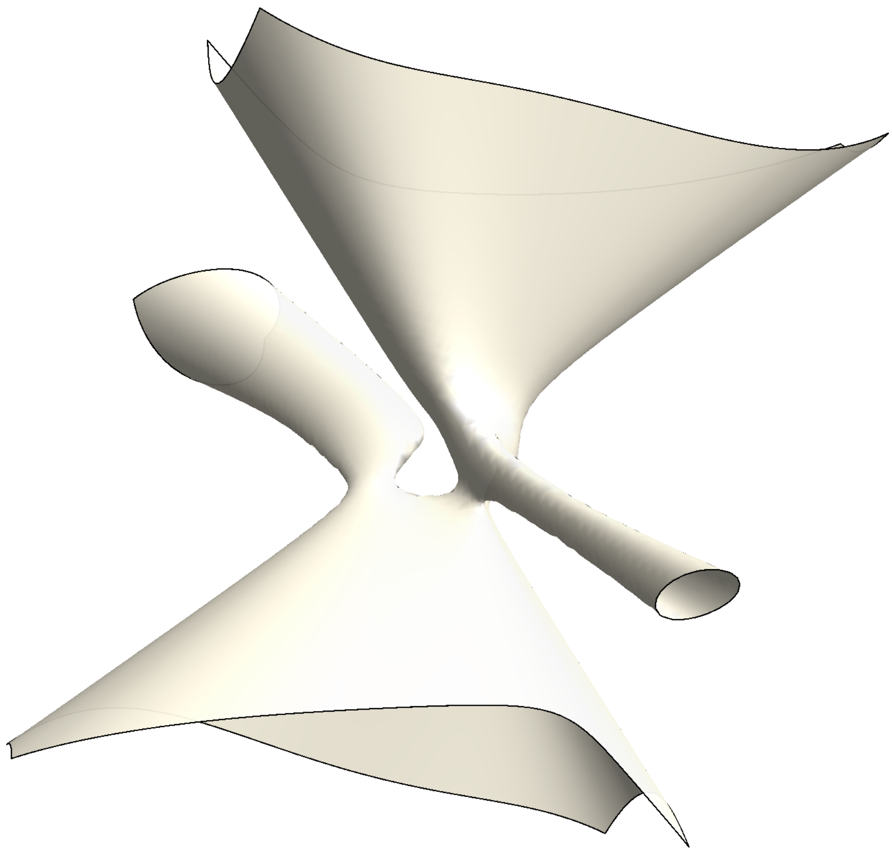

Figure 4.1: A visual rendition of the real points of an affine patch of the complex K3 surface given by $\det M_0 = 0$.