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Chapter 3

The geometric Picard lattice of the K3 surfaces in a family

The geometric Picard lattice of a K3 surface can give information about the geometry as well as the arithmetic of the surface. A large literature is devoted to the computation of the Picard lattice of a K3 surface. In [PTvL15], Bjorn Poonen, Damiano Testa, and Ronald van Luijk give an algorithm to compute the geometric Néron-Severi group of any smooth, projective, geometrically integral variety X. The algorithm works under the assumption that it is possible to explicitly compute the Galois modules of X with finite coefficients, and it terminates if and only if the Tate conjecture holds for X. In [HKT13], Hassett, Kresch, and Tschinkel give an effective algorithm to compute the Picard lattice of a K3 surface of degree two. The algorithm is "effective" in the sense that given the equations defining the surface, it returns the Galois module structure of the geometric Picard lattice of the surface. Even though these algorithms show that in principle it is possible to compute the geometric Picard lattice of a K3 surface, in practice the computations involved are very hard to perform, making the algorithms highly impractical. The main problem in the task of computing the geometric Picard lattice is to find enough divisors to generate the whole geometric Picard lattice. This remains the main issue even if we are only interested in the geometric Picard number, that is, the rank of the geometric Picard lattice. Work of Charles, Elsenhans, Jahnel, Kloosterman, Kuwata, van Luijk, and others, show that there are different ways to provide lower and upper bounds for the geometric Picard number. See [vL05] and [vL07] for a method to give an upper bound of the geometric Picard number by looking at the reduction of the surface over different finite fields; this method is later applied by Stephan Elsenhans and Jörg Jahnel in [EJ08b] and [EJ08a]. Kuwata and Kloosterman, in [Kuw00] and [Klo07], provide explicit examples of elliptic K3 surfaces with geometric Picard number $\rho \geq r$, for $r = 0, 1, ..., 18$. In [Cha14], François Charles provides a non-deterministic algorithm to compute the geometric Picard number of a K3 surface. We suggest to consult [PTvL15] for a more accurate summary on this topic. All these methods, as well as the algorithm given by Charles, rely on the ability to explicitly find enough divisors on the surface. We are not aware of the existence of any practical algorithm that, given a surface X as input, returns a set of divisors on X generating the geometric Picard lattice of X.

In this chapter we consider a 1-dimensional family of K3 surfaces, and we give an explicit description of the geometric Picard lattice of the generic member of the family, providing also an explicit set of divisors generating the Picard lattice. This information can then be used to describe the geometric Picard lattice of every member of the family.

This chapter is part of joint work with Florian Bouyer, Edgar Costa, Christopher Nicholls, and Mckenzie West. The joint work has its roots in a question proposed by Anthony Várilly-Alvarado during the Arizona Winter School 2015 (see [VA15, Project 1]). We are also indebted with Alice Garbagnati for the comments that led to Proposition 3.7.6.

3.1 The main result

Let k be any field; recalling the notation introduced in Subsection 1.2.2, we will use $\mathbb{P}_k$ to denote the weighted projective space $\mathbb{P}_k(1, 1, 1, 3)$; also, let $\mathbb{A}^1_k$ denote the affine line over k. Sometimes we might drop the index k in $\mathbb{P}_k$ and $\mathbb{A}^1_k$, if no confusion arises.

Let $\mathbb{Q}$ be the field of rational numbers and fix an algebraic closure $\overline{\mathbb{Q}}$. Let t and x, y, z, w be the coordinates of $\mathbb{A}^1_{\mathbb{Q}}$ and $\mathbb{P}_{\mathbb{Q}}$, respectively.

Let $\mathfrak{X} \subset \mathbb{A}^1 \times \mathbb{P}$ be the threefold over $\mathbb{Q}$ defined by

$$\mathfrak{X}\colon w^2 = x^6 + y^6 + z^6 + tx^2y^2z^2. \tag{3.1}$$

Let $p\colon \mathfrak{X} \to \mathbb{A}^1$ be the projection of $\mathfrak{X}$ to $\mathbb{A}^1$, that is, the map defined by sending the point $(t_0, (x_0 : y_0 : z_0 : w_0)) \in \mathfrak{X}$ to the point $t_0 \in \mathbb{A}^1$.

Let t_0 be a point in $\mathbb{A}^1$. The fiber $p^{-1}(t_0) \subset \mathbb{A}^1 \times \mathbb{P}$ of $\mathfrak{X}$ over t_0 is given by the following equations

$$p^{-1}(t_0)\colon \begin{cases} w^2 &= x^6 + y^6 + z^6 + t_0x^2y^2z^2 \\ t &= t_0 \end{cases}$$

The fiber $p^{-1}(t_0)$ naturally embeds into $\mathbb{P}$, and we denote its image inside $\mathbb{P}$ by X_{t_0}; we also denote by B_{t_0} the plane sextic curve defined by

$$B_{t_0}\colon x^6 + y^6 + z^6 + t_0x^2y^2z^2 = 0. \tag{3.2}$$

Proposition 3.1.1. *Let t_0 be a point of $\mathbb{A}^1_{\mathbb{Q}} \setminus \{-3, -3\zeta_3, -3\zeta_3^2\}$, where ζ_3 is a primitive third root of unity. Then X_{t_0} is a K3 surface.*

Proof. The surface X_{t_0} is defined by the equation

$$X_{t_0}\colon w^2 = x^6 + y^6 + z^6 + t_0x^2y^2z^2,$$

and it is a double cover of $\mathbb{P}^2$ ramified above the sextic curve $B_{t_0} \subset \mathbb{P}^2$.

The curve B_{t_0} admits singular points if following system of equations admits solutions.

$$\begin{cases} 3x^5 + t_0xy^2z^2 &= 0 \\ 3y^5 + t_0x^2yz^2 &= 0 \\ 3z^5 + t_0x^2y^2z &= 0 \end{cases}$$

One can see that this happens if and only if $t_0 = -3, -3\zeta_3, -3\zeta_3^2$. So, for $t_0 \neq -3, -3\zeta_3, -3\zeta_3^2$, the curve B_{t_0} is smooth and, therefore, X_{t_0} is a K3 surface. □

Remark 3.1.2. Define $t_i := -3\zeta_3^i$, for $i \in \{0, 1, 2\}$. We claim that the surfaces X_{t_i}, for $i = 0, 1, 2$, are non-smooth and, therefore, are not K3 surfaces.

One can easily see that $X_{t_0} = X_{-3}$ has four ordinary double points: $(1 : \pm 1 : \pm 1 : 0)$.

For $i = 1, 2$, the map $(x : y : z : w) \mapsto (\zeta_3^i x : y : z : w)$ gives an isomorphism $X_{-3} \to X_{t_i}$. So also X_{t_i} has four ordinary double points, namely the points $(\zeta_3^i : \pm 1 : \pm 1 : 0)$, for $i = 1, 2$.

Nevertheless, for $i = 0, 1, 2$, blowing up X_{t_i} at its singular points, we do obtain a K3 surface.

Let η be the generic point of $\mathbb{A}^1$ and let $K = \kappa(\eta)$ denote the residue field of η, that is, the function field $\mathbb{Q}(t)$. Fix an algebraic closure $\overline{K}$ of K such that $\overline{\mathbb{Q}} \subset \overline{K}$. Consider the fiber $p^{-1}(\eta)/K$ of $\mathfrak{X} \subset \mathbb{A}^1 \times \mathbb{P}$ above η. The fiber $p^{-1}(\eta)$ naturally embeds into $\mathbb{P}_K$. We denote by X_η the image of $p^{-1}(\eta)$ inside $\mathbb{P}_K$. Then X_η is the surface over K given by the equation

$$X_\eta \colon w^2 = x^6 + y^6 + z^6 + tx^2y^2z^2. \tag{3.3}$$

By Proposition 3.1.1, the surface $X_\eta \subset \mathbb{P}_K$ is a K3 surface.

The main goal of this chapter is to give a description of the geometric Picard lattice of X_η; using this we can get information about the geometric Picard lattice of any fiber of $\mathfrak{X}$.

The first step in order to achieve the description of $\operatorname{Pic} \overline{X_\eta}$, is to compute the geometric Picard number of X_η.

Proposition 3.1.3. *The geometric Picard lattice of X_η has rank* 19*, that is,* $\rho(\overline{X_\eta}) = 19$.

Proof. See Subsection 3.3.3. □

Using some explicit divisors of $\overline{X_\eta}$ it is then possible to give a complete description of $\operatorname{Pic} \overline{X_\eta}$, as shown by the main theorem below.

Theorem 3.1.4. *Let η be the generic point of $\mathbb{A}^1$. Then the generic fiber $X_\eta = p^{-1}(\eta)$ of $\mathfrak{X}$ is a K3 surface with geometric Picard lattice isometric to the lattice*

$$U \oplus E_8(-1) \oplus A_5(-1) \oplus A_2(-1) \oplus A_2(-4). \tag{3.4}$$

The proof of the theorem is given in two steps: first finding some divisors on the surface and computing the lattice Λ they generate, then proving that Λ is the full geometric Picard lattice.

3.2 An automorphism subgroup of the Picard lattice

Isometries of the Picard lattice of a K3 surface can be very useful in order to find divisors (cf. Section 3.3). We have seen that an automorphism of a K3 surface induces an (effective) isometry of the Picard lattice. In this section, using the symmetries of the equation defining X_η, we provide some automorphisms of X_η, and hence some (effective Hodge) isometries of $\operatorname{Pic} X_\eta$.

Let K and $\overline{K}$ be defined as before. Let $\zeta_{12} \in \overline{\mathbb{Q}} \subset \overline{K}$ be a primitive 12-th root of unity and define $\zeta_6 := \zeta_{12}^2$, $\zeta_4 := \zeta_{12}^3$, and $\zeta_3 := \zeta_{12}^4$.

Remark 3.2.1. Note that the element ζ_i is a primitive i-th root of unity, for $i \in \{3, 4, 6\}$.

Let $\mathbb{Q}(\zeta_3)$ be the number field obtained by adjoining ζ_3 to $\mathbb{Q}$, i.e., the 3rd cyclotomic field. Since $\zeta_6 = \zeta_3 + 1$, we have that $\zeta_6 \in \mathbb{Q}(\zeta_3)$.

Throughout this section, and also in the following ones, we will use the notation μ_n, C_n, D_n, and S_n to denote respectively the group of n-th roots of unity inside $\overline{\mathbb{Q}} \subset \overline{K}$, the cyclic group of order n, the group of symmetries of the regular n-polygon (that is, the dihedral group of order $2n$), and the permutation group of a set with n elements, for any positive integer n.

Consider the following automorphisms of $\mathbb{P}_{\mathbb{Q}(\zeta_3)}$:

- For any permutation σ of the set $\{x, y, z\}$ of coordinate functions of $\mathbb{P}_{\mathbb{Q}(\zeta_3)}$, consider the induced automorphism $\bar{\sigma}\colon \mathbb{P}_{\mathbb{Q}(\zeta_3)} \to \mathbb{P}_{\mathbb{Q}(\zeta_3)}$ defined by

$$\bar{\sigma}\colon P \mapsto (\sigma(x)(P) : \sigma(y)(P) : \sigma(z)(P) : w(P)).$$

- For any triple $(i, j, k) \in (\mathbb{Z}/6\mathbb{Z})^3$ such that $2(i+j+k) \equiv 0 \mod 6$ consider the automorphism $\psi_{i,j,k}\colon \mathbb{P}_{\mathbb{Q}(\zeta_3)} \to \mathbb{P}_{\mathbb{Q}(\zeta_3)}$ defined by

$$\psi_{i,j,k}\colon (x : y : z : w) \mapsto (\zeta_6^i x : \zeta_6^j y : \zeta_6^k z : w).$$

Remark 3.2.2. Since $\mathbb{P}_{\mathbb{Q}(\zeta_3)}$ is a weighted projective space with weights $(1, 1, 1, 3)$, we have that

$$(x : y : z : w) = (\zeta_3 x : \zeta_3 y : \zeta_3 z : w) = (\zeta_3^2 x : \zeta_3^2 y : \zeta_3^2 z : w)$$

and therefore $\psi_{4,4,4} = \psi_{2,2,2} = \mathrm{id}$. One can easily check that no other automorphism $\psi_{i,j,k}$ equals the identity.

The above automorphisms of $\mathbb{P}_{\mathbb{Q}(\zeta_3)}$ can be extended to automorphisms of $\mathbb{P}_{\overline{K}}$. Let $G_1, G_2 \subset \mathrm{Aut}(\mathbb{P}_{\overline{K}})$ be the subgroups of $\mathrm{Aut}(\mathbb{P}_{\overline{K}})$ generated by the extension to $\mathbb{P}_{\overline{K}}$ of the automorphisms $\bar{\sigma}$ and $\psi_{i,j,k}$, respectively. We define $G = \langle G_1, G_2 \rangle$ the subgroup of $\mathrm{Aut}(\mathbb{P}_{\overline{K}})$ generated by the elements of G_1 and G_2.

Let ψ and ς be two elements of G_1 and G_2, respectively. One can easily see that the automorphism given by $\varsigma^{-1}\psi\varsigma$ is an element of G_2. We can then define an action of G_1 on G_2, by sending $(\varsigma, \psi) \in G_1 \times G_2$ to $\varsigma^{-1}\psi\varsigma \in G_2$. Let $G_1 \ltimes G_2$ denote the semidirect product of G_1 and G_2, with G_1 acting on G_2 as described above. It is easy to see that $G = G_1 \ltimes G_2$.

With the following results we give a description of G_1, G_2, and G as abstract groups. Let $\Sigma \subset \mu_6^3$ be the subgroup of $\mu_6^3 = \mu_6 \times \mu_6 \times \mu_6$ defined by

$$\Sigma := \{(\zeta, \xi, \theta) \in \mu_6^3 \ : \ \zeta\xi\theta = \pm 1\}.$$

Remark 3.2.3. The group Σ is isomorphic to $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \{0, 3\}$. To see this, let (ζ, ξ, θ) be an element of Σ. Since $\zeta, \xi, \theta \in \mu_6$, there are $i, j, k \in \{0, 1, ..., 5\}$ such that $\zeta = \zeta_6^i, \xi = \zeta_6^j, \theta = \zeta_6^k$; since $\zeta\xi\theta = \pm 1$, we have that $i + j + k \in \{0, 3\}$. Then the map $\Sigma \to \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \{0, 3\}$ given by

$$(\zeta, \xi, \theta) \to (i, j, i + j + k)$$

is well defined and in fact it is an isomorphism of groups.

Let $\Delta \colon \mu_3 \hookrightarrow \mu_6^3$ be the embedding defined by

$$\Delta \colon \zeta \to (\zeta, \zeta, \zeta).$$

It is easy to see that the image of Δ is a normal subgroup of Σ. Let H denote the quotient group

$$H := \Sigma / \mathrm{im}(\Delta). \tag{3.5}$$

Remark 3.2.4. As an easy exercise in group theory, one can show that the group H is isomorphic to the group $C_2^2 \times C_6$.

Lemma 3.2.5. *The following statements hold:*

i) *G_1 is isomorphic to the symmetric group S_3;*

ii) *G_2 is isomorphic to the group H defined in* (3.5)*;*

iii) *G is isomorphic to $S_3 \ltimes H$, where the action of S_3 on H is given by permuting the coordinates of the elements of H.*

Proof. i) Trivial from the definition of G_1. In fact, recalling the definition of $\bar{\sigma}$, the map

$$\bar{\sigma} \mapsto \sigma$$

gives an isomorphism between G_1 and S_3.

ii) Let (ζ, ξ, θ) be an element of Σ and let i, j, k be defined as in Remark 3.2.3. Then $i+j+k \in \{0, 3\}$ or, equivalently, $2(i+j+k) \equiv 0 \mod 6$. We can then consider the map $\Sigma \to G_2$ given by

$$(\zeta, \xi, \theta) \mapsto \psi_{i,j,k}.$$

The map is clearly surjective; by Remark 3.2.2, it follows that the kernel is the subgroup $\{(0,0,0), (2,2,2), (4,4,4)\}$; so

$$G_2 \cong \Sigma/\{(0,0,0), (2,2,2), (4,4,4)\} = H,$$

concluding the proof.

iii) The statement trivially follows by recalling that $G = G_1 \ltimes G_2$ and then applying the isomorphisms used to prove points *i)* and *ii)*.

□

Corollary 3.2.6. *The group G has cardinality $2^4\,3^2$.*

Proof. By Lemma 3.2.5.(i), $G_1 \cong S_3$ and so $\#G_1 = 3! = 6$.

By Lemma 3.2.5.(ii), $G_2 \cong H$, with $H = \Sigma/\operatorname{im}(\Delta)$. The group Σ has cardinality $6^2\,2$ (cf. Remark 3.2.3); H is a quotient of Σ by a subgroup of order 3, hence $\#H = 6^2\,2/3 = 6 \cdot 2^2$. Alternatively, one can use Remark 3.2.4.

Since $G = G_1 \ltimes G_2$, it follows that $\#G = \#G_1 \cdot \#G_2 = 6 \cdot (6 \cdot 2^2)$, proving the statement. □

Lemma 3.2.7. *All the elements of G fix the surface X_η.*

Proof. To prove the statement it is enough to check that the automorphisms of G fix the equation defining X_η. □

Using Lemma 3.2.7, we can define the map $\mathrm{res}_\eta \colon G \to \mathrm{Aut}(X_\eta)$, sending an element of G to the automorphism of X_η it induces.

Lemma 3.2.8. *The map* $\mathrm{res}_\eta \colon G \to \mathrm{Aut}(X_\eta)$ *is injective.*

Proof. The statement is equivalent to saying that every element of G induces a non-trivial automorphism of X_η. The fixed subspace of a non-trivial element of G is a subspace defined by n linear equations, with $n \in \{1, 2, 3\}$, and therefore it cannot contain the surface X_η. □

With abuse of notation, we will use the symbols $\bar{\sigma}, \psi_{i,j,k}$ both for the automorphisms of $\mathbb{P}_{\overline{K}}$ and X_η; we will also use G to indicate both the subgroup of $\mathrm{Aut}(\mathbb{P}_{\overline{K}})$ and the image of res_η.

Remark 3.2.9. Since $\pi \colon X_\eta \to \mathbb{P}^2$ is a double cover of $\mathbb{P}^2$, one can consider the involution ι of X_η given by switching the elements inside the fibers of π. Keeping in mind the equation of X_η, we have that ι is given by

$$\iota \colon (x : y : z : w) \mapsto (x : y : z : -w).$$

Then it follows that $\iota = \psi_{3,3,3} \in G$.

Corollary 3.2.10. *Let P be a (not necessarily closed) point of $\mathbb{A}^1_{\overline{\mathbb{Q}}}$, and let X_P be the K3 surface corresponding to the fiber of $\mathfrak{X}$ over P. Assume that its geometric Picard group has odd rank and discriminant not a power of* 2*. Then the group G acts faithfully on* $\mathrm{Pic}\,\overline{X_P}$.

Proof. From Proposition 1.2.47, it follows that G embeds into $\mathcal{O}(\mathrm{Pic}\,\overline{X_P})$. □

Let $G_s \subset G$ denote the subgroup of G given by the symplectic automorphisms of X_η in G.

Lemma 3.2.11. *The subgroup G_s has cardinality* 72 *and it is generated by* $\psi_{3,3,3} \circ \overline{\sigma}_{(12)}$, $\overline{\sigma}_{(123)}$, $\psi_{2,4,0}$, $\psi_{0,3,3}$, *and* $\psi_{3,0,3}$.

Proof. First notice that the involution $\psi_{3,3,3}$ fixes infinitely many points of X (in fact, it fixes the ramification locus R), and so, by Proposition 1.2.49, it is not symplectic; it follows that G_s has index at least 2 inside G, and so $\#G_s \leq 72$.

Again using Proposition 1.2.49, one can check that all the automorphisms listed in the statement are symplectic. Then, by easy computations, one sees that they generate a subgroup of order 72. □

Remark 3.2.12. Using `MAGMA`, one can easily check that G_s is isomorphic to the group $\mathfrak{A}_{4,3}$; this group is called `SmallGroup(72,43)` in `MAGMA` and GAP. See [Fes16] for these computations. We refer to [Has12, Appendix: computations using GAP] and [GAP16] for more details about groups database in GAP.

Remark 3.2.13. The elements of G_s have either order 2 or 3. So, by Proposition 1.2.49, it follows that $\rho(\overline{X_\eta}) \geq 13$.

3.3 Some divisors on X_η

In this section we will explain how we found some divisors on X_η generating a rank 19 sublattice of $\operatorname{Pic} \overline{X_\eta}$.

We have three main tools to find divisors on X_η:

1. the structure on X_η of double cover of $\mathbb{P}^2$ (cf. Subsection 3.3.1);
2. the structure on X_η of double cover of a del Pezzo surface of degree 1 (cf. Subsection 3.3.2);
3. the automorphisms of X_η (cf. Section 3.2).

3.3.1 X_η as double cover of $\mathbb{P}^2$

Let $\pi\colon X_\eta \to \mathbb{P}^2_K$ be the map defined by

$$\pi\colon (x : y : z : w) \to (x : y : z).$$

Let $B_\eta \subset \mathbb{P}^2_K$ be the smooth sextic plane curve defined by

$$x^6 + y^6 + z^6 + tx^2y^2z^2 = 0. \tag{3.6}$$

Lemma 3.3.1. *The map π is a 2-to-1 map ramified above B_η.*

Proof. Let $P = (x_0 : y_0 : z_0)$ be a point of $\mathbb{P}^2_K$. Then it is easy to see that $\pi^{-1}(P) = \{(x_0 : y_0 : z_0 : w_0), (x_0 : y_0 : z_0 : -w_0)\}$, where $w_0 \in \overline{K}$ is a square root of the quantity $x_0^6 + y_0^6 + z_0^6 + tx_0^2y_0^2z_0^2$.

The ramification points of π are then the points whose coordinates make that quantity vanish, that is, the points $(x : y : z) \in \mathbb{P}^2$ lying on the curve defined in (3.6). This concludes the proof. □

Recalling the notation introduced in Subsection 1.2.3, the curve B_η is the branch locus of π, and its pre-image R_η on X_η is the ramification locus. If no confusion arises, later we might drop the index η to denote B_η and R_η, writing just B and R.

Proposition 3.3.2. *Let $C \subset \mathbb{P}^2$ be an irreducible plane curve of degree $d \neq 6$, and let $D = \pi^*(C)$ be its pull-back via π. Assume that D splits into two irreducible components, say $D = D_1 + D_2$. Then neither D_1 nor D_2 is equal to a multiple of the hyperplane section in* $\operatorname{Pic} \overline{X_\eta}$.

Proof. Since π is a 2-to-1 map, the components D_1 and D_2 are both isomorphic to C, and they are switched by the involution $\psi_{3,3,3}$ (cf. Remark 3.2.9). This means that $D_1^2 = D_2^2$ and $D_1 \cdot H = D_2 \cdot H$, with H being the hyperplane section class. Since C has degree d and D is a double cover of C, we have that $D \cdot H = 2d$ and, by $D_1 \cdot H = D_2 \cdot H$, it follows that $D_1 \cdot H = D_2 \cdot H = d$.

The intersection $D_1 \cdot D_2$ is given by the points lying above the points of $C \cap B$. Recall that B is the branch locus, it has degree 6, and C intersects B with even multiplicity everywhere. Then $D_1 \cdot D_2 = 3d$. Combining this with

$$\begin{aligned} 2d^2 &= 2C^2 \\ &= \pi_*\pi^*(C)^2 \\ &= \pi_*(D)^2 \\ &= D^2 \\ &= (D_1 + D_2)^2 \\ &= D_1^2 + 2D_1 \cdot D_2 + D_2^2 \\ &= 2D_1^2 + 6d, \end{aligned}$$

and, therefore, $D_2^2 = D_1^2 = d^2 - 3d$.

Finally, recall that the hyperplane class H is the pull-back of the class of the line, and therefore $H^2 = 2$.

Then we can see that H and D_i, for any $i = 1, 2$, generate a lattice whose intersection matrix is

$$\begin{pmatrix} 2 & d \\ d & d^2 - 3d \end{pmatrix}. \tag{3.7}$$

The discriminant of the intersection matrix is $d(d-6)$. The integer d is the degree of a curve, so $d > 0$, and by hypotheses $d \neq 6$; therefore the discriminant is different from 0 and this proves that D_i, for $i = 1, 2$, is not linearly equivalent to any multiple of H. □

Remark 3.3.3. With the computations used to prove Proposition 3.3.2 one can also show that D_1 and D_2 are linearly independent: in fact, they generate a sublattice of $\operatorname{Pic} \overline{X_\eta}$ with Gram matrix

$$\begin{pmatrix} d^2 - 3d & 3d \\ 3d & d^2 - 3d \end{pmatrix}. \tag{3.8}$$

The determinant of (3.8) is $d^3(d-6)$, and so, if $d \neq 0, 6$, it is non-zero and it shows that D_1 and D_2 are linearly independent.

A sublattice of rank 2 is the most we can get from D_1, D_2 and H, even though these three divisor are pairwise linearly independent: recall that $D_1 + D_2 = D = dH$.

Remark 3.3.4. Combining Corollary 1.2.27 and Proposition 3.3.2 we have a useful criterion to find irreducible plane curves C such that the irreducible components C_1, C_2 of its pull-back on X_η are not linearly equivalent to the hyperplane section, and that therefore generate a sublattice of the geometric Picard lattice of rank 2. In order to find such a curve, we look for genus 0 plane curves intersecting B_η with even multiplicity everywhere.

The first try was given by looking for tri-tangent lines. We found that such lines do not exist. Then we started looking for plane conics. Looking for *all* the plane conics intersecting B_η with even intersection everywhere is complicated so, using the fact that B_η is given by a symmetric equation, we first looked for conics with symmetric equations too; in particular, we looked for diagonal conics and we found that all the diagonal conics with third roots of unity as coefficients intersect B_η with even multiplicity everywhere (cf. Proposition 3.3.13).

Remark 3.3.5. Even though it turned out that there exist no plane lines that are tri-tangent to B_η, it might happen that such lines exist for some special value of t. Indeed, we found that the branch locus B_{t_0} admits a tri-tangent line if and only if

$$t_0 = 0, -\frac{33}{2}\zeta, -5\zeta, \tag{3.9}$$

with $\zeta \in \mu_3$. For the computations see [Fes16].

3.3.2 X_η as double cover of a del Pezzo surface of degree 1

Let $\mathbb{P}_K(1,1,2,3)$ be the weighted projective space over $K = \mathbb{Q}(t)$ with coordinates x', y', z' and w'.

Let $\pi_z \colon \mathbb{P}_K \to \mathbb{P}_K(1,1,2,3)$ be the map defined by

$$\pi_z \colon (x : y : z : w) \mapsto (x : y : z^2 : w).$$

It is easy to see that the map π_z is a 2-to-1 map ramified along the plane $\{z = 0\} \subseteq \mathbb{P}_K$.

Let $X'_\eta \subset \mathbb{P}_K(1,1,2,3)$ be the surface defined by

$$X'_\eta \colon w'^2 = x'^6 + y'^6 + z'^3 + tx'^2y'^2z'.$$

Lemma 3.3.6. *The surface $X'_\eta \subset \mathbb{P}_K(1,1,2,3)$ is a del Pezzo surface of degree* 1*.*

Proof. From Proposition 1.2.59. □

Proposition 3.3.7. *The map $\pi_z \colon \mathbb{P}_K \to \mathbb{P}_K(1,1,2,3)$ induces a* 2*-to-*1 *morphism $X_\eta \to X'_\eta$, that is, $\pi_{z|X_\eta} \colon X_\eta \to X'_\eta$ is a double cover of X'_η.*

Proof. First notice that the map $\pi_z \colon (x : y : z : w) \to (x : y : z^2 : w)$ sends points of X_η to points of X'_η. Then notice that π_z is defined everywhere on $\mathbb{P}$, hence it is defined everywhere on X_η. Let $(x' : y' : z' : w')$ be a point of X'_η, and denote it by Q. It is easy to see that its preimage $\pi_z^{-1}(Q)$ in X_η is the set $\{(x' : y' : \pm\zeta : w')\}$, where ζ is an element in $\overline{K}$ such that $\zeta^2 = z'$. □

Remark 3.3.8. In fact, X'_η is not the only del Pezzo doubly covered by X_η. Exploiting the symmetry of X_η it easy to see, using the same argument as for X'_η, that the morphisms

$$\pi_x\colon (x:y:z:w) \mapsto (x^2:y:z:w),$$
$$\pi_y\colon (x:y:z:w) \mapsto (x:y^2:z:w),$$

from $\mathbb{P}_K$ to $\mathbb{P}(2,1,1,3)_K$ and $\mathbb{P}(1,2,1,3)_K$ respectively, induce on X_η a double cover structure of the del Pezzo surfaces of degree 1

$$X''_\eta\colon w''^2 = x''^3 + y''^6 + z''^6 + tx''y''^2z''^2$$
and
$$X'''_\eta\colon w'''^2 = x'''^6 + y'''^3 + z'''^6 + tx'''^2y'''z'''^2.$$

Remark 3.3.9. The structure of double cover of a del Pezzo surface of degree 1 on X_η can be used to obtain more divisors that are linearly independent. In fact, the Picard lattice of a del Pezzo surface of degree 1 over an algebraically closed field has rank 9 (cf. Corollary 1.2.58). Let $E_1, ..., E_9$ be generators of $\operatorname{Pic} \overline{X'_\eta}$. Then their pullbacks $\pi^*(E_1), ..., \pi^*(E_9)$ are nine linearly independent divisors on $\overline{X_\eta}$. Pulling back also nine generators of $\operatorname{Pic} \overline{X''_\eta}$ and $\operatorname{Pic} \overline{X'''_\eta}$ (see the definition of these surfaces in Remark 3.3.8) or, equivalently, considering the orbits of $\pi^*(E_1), ..., \pi^*(E_9)$ under the action of G_1, one gets $9 \times 3 = 27$ divisors on $\overline{X_\eta}$, generating a sublattice of $\operatorname{Pic} \overline{X_\eta}$ of rank 13.

3.3.3 Explicit divisors

We have seen that X_η can be endowed with two structures: the structure of double cover of the plane, and the structure of double cover of a del Pezzo surface of degree 1. Using these two structures we have been able to explicitly compute some divisors on X_η. Some of these divisors are not defined over $K = \mathbb{Q}(t)$, but only over some algebraic extension of K. In order to define them, we need to introduce some elements of $\overline{K}$.

Let ζ_{12}, ζ_6 and ζ_3 be defined as before (cf. 3.2), and define $\zeta_4 := \zeta_{12}^3$.

Remark 3.3.10. The element $\zeta_4 \in \overline{K}$ is a primitive 4-th root of unity.

Consider the elements $t + 3\zeta_3^i \in K(\zeta_3)$ for $i \in \{0,1,2\}$, and let $\beta_i \in \overline{K}$ be a square root of $t + 3\zeta_3^i$, i.e. $\beta_i^2 = t + 3\zeta_3^i$ for $i \in \{0,1,2\}$.

We denote by K_1 the field $K(\zeta_{12}, \beta_0, \beta_1, \beta_2)$.

Let $h(v) \in K[v]$ be the polynomial $h := v^3 + tv^2 + 4$ and let c_0, c_1 and c_2 be its roots in $\overline{K}$. The polynomial h has discriminant $\Delta = -16(t^3 + 27) = (4\zeta_4\beta_0\beta_1\beta_2)^2$. Let δ denote the element $4\zeta_4\beta_0\beta_1\beta_2$ inside K_1; one can easily check that δ is a square root of Δ. Let K_2 be the field obtained by adjoining c_0 to K_1, that is, K_2 is the field $K(\zeta_{12}, \beta_0, \beta_1, \beta_2, c_0)$. We will see later (cf. Lemma 3.4.2) that also c_1 and c_2 are contained in K_2.

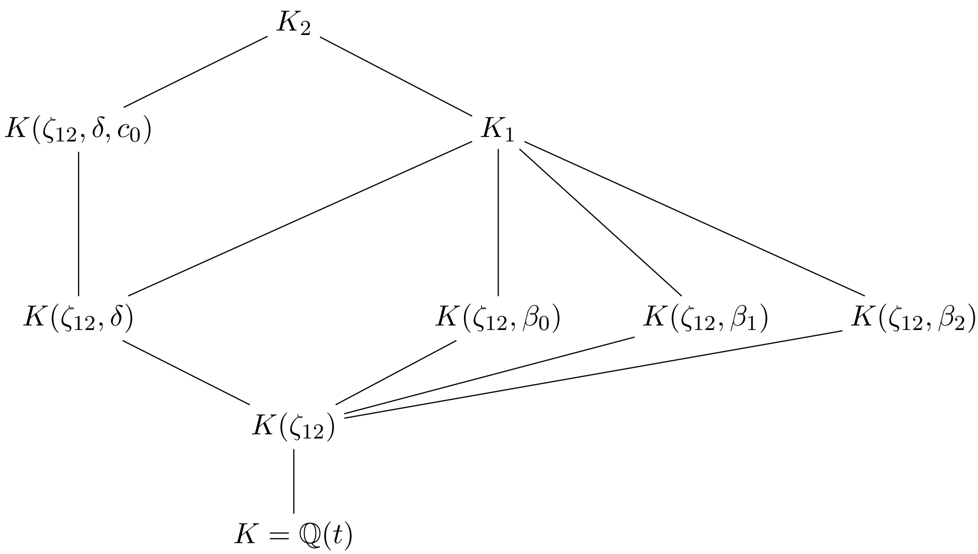

Figure 3.1: The field-diagram showing the construction of K_2 stated above.

Let $\mathcal{D}' = \{D'_1, ..., D'_4\}$ be the set of divisors on X_η given by

$$D'_1\colon \begin{cases} x^2 + y^2 + \zeta_3 z^2 & = 0 \\ w - \beta_1 xyz & = 0 \end{cases}$$

$$D'_2\colon \begin{cases} x^2 + \zeta_3 y^2 + \zeta_3^2 z^2 & = 0 \\ w - \beta_0 xyz & = 0 \end{cases}$$

$$D'_3\colon \begin{cases} c_0\delta x^2 - 2(9c_0^2 + 3tc_0 - 2t^2)xy + 2\delta y^2 - \delta z^2 & = 0 \\ (x^3 + a_3x^2y + b_3xy^2 + c_3y^3)(c_0^2c_1 + 2) - 2w & = 0 \end{cases}$$

$$D'_4\colon \begin{cases} x^2 + y^2 + \zeta_3^2 z^2 & = 0 \\ w - \beta_2 xyz & = 0 \end{cases}$$

where

$$\begin{aligned} a_3 &= \frac{9c_0 + 6t}{4(t^3 + 27)}\delta, \\ b_3 &= -c_0^2 - tc_0, \\ c_3 &= \frac{18 - 3t^2c_0 - 3tc_0^2}{8(t^3 + 27)}. \end{aligned}$$

Remark 3.3.11. One can easily check that the curves $D'_1, ..., D'_4 \subseteq \mathbb{P}$ lie on X_η.

Remark 3.3.12. Although all the divisors listed above look like the pull-back of a plane conic, divisor D'_3 was originally found as the pull-back of a generator of $\operatorname{Pic} \overline{X'_\eta}$.

For every $i = 1, ..., 4$, the divisor $D'_i \subset X_\eta$ is defined by two equations, namely $f_i = w - g_i = 0$, where f_i and g_i are two homogeneous polynomials in x, y, z of degree 2 and 3 respectively. Since the polynomial f_i has no w-term, we denote by C_i the conic of $\mathbb{P}^2_K$ it defines.

Proposition 3.3.13. *For every $i \in \{1, ..., 4\}$, the following statements hold:*

1. *the conic $C_i \subset \mathbb{P}^2_K$ intersects the branch locus B_η of π with even multiplicity everywhere;*
2. *the divisor D'_i of $\overline{X_\eta}$ is an irreducible component of the pull-back of C_i via π;*

3. the curve $D_i' \subset \overline{X_\eta}$ is isomorphic to the conic C_i.

Proof. 3. The restriction of π to D_i' induces an isomorphism to C_i. The inverse is given by the map $C_i \to D_i'$ sending $(x : y : z)$ to $(x : y : z : g_i(x, y, z))$.

2. It follows from 3.

1. The curve D_i' maps 1-to-1 to C_i, so it is not the only component. The statement follows from Corollary 1.2.27.

□

Let $G\mathcal{D}'$ denote the set $\{ sD_i' \ : \ s \in G,\ i \in \{1, 2, 3, 4\} \}$, obtained by letting the automorphisms of G act on the elements of $\mathcal{D}'$.

Let Λ' be the sublattice of $\operatorname{Pic} \overline{X_\eta}$ generated by the elements of $G\mathcal{D}'$.

Proposition 3.3.14. *The lattice Λ' is an even lattice of rank 19, signature $(1, 19)$, discriminant $2^{21}3^3$ and discriminant group isomorphic to $C_2^{16} \times C_6 \times C_{12}^2$.*

Before presenting the proof, we introduce some notations that will be useful in the proof and later in this chapter too.

Let k be any field, let A be the polynomial ring $k[v]$ and let F be the field of fractions of A, that is, $F = \operatorname{Frac} A = k(v)$. Fix an algebraic closure $\overline{F}$ of F, and let v_0 be an element inside $\overline{F}$. We define the *specialization of the field F to v_0*, denoted by F_{v_0}, the field $\operatorname{Frac} k[v_0]$. Note that F_{v_0} is a finite algebraic extension of k.

Example 3.3.15. Let t_0 be an element of $\overline{\mathbb{Q}}$. Then the specialization of $K = \mathbb{Q}(t)$ at t_0 is the number field $\mathbb{Q}(t_0)$.

Let $t_0 \in \mathbb{Z}$ be an integer, fix an integral model Ξ_{t_0} for the surface X_{t_0}. Let $p \in \mathbb{Z}$ be a prime of good reduction for Ξ_{t_0}, and let $\mathbb{F}_p$ denote the field with p elements.

Let K_{2,t_0} be the number field obtained by specializing

$$K_2 = \mathbb{Q}(\zeta_{12}, \beta_0, \beta_1, \beta_2, c_0)(t)$$

to $t = t_0$, let $\mathcal{O}_{t_0}$ denote the ring of integers of K_{2,t_0}, and let $\mathfrak{p}$ be a prime of $\mathcal{O}_{t_0}$ lying above p. Let $\kappa(\mathfrak{p})$ be the residue field $\mathcal{O}_{t_0}/\mathfrak{p}$. The field $\kappa(\mathfrak{p})$ is isomorphic to $\mathbb{F}_{p^m}$, for some $m \in \mathbb{Z}_{>0}$.

Let $X_{t_0,p}/\mathbb{F}_p$ be denote the reduction of Ξ_{t_0} modulo p. Let $B_{t_0,p} \subseteq \mathbb{P}^2_{\mathbb{F}_p}$ denote the branch locus of $X_{t_0,p}$.

Let D be one of the divisors of X_η in $G\mathcal{D}'$, and let $\overline{D}$ denote its Zariski closure inside $\mathfrak{X}$. Then $\overline{D}$ is a divisor of $\mathfrak{X}$. We define D_{t_0} to be the *specialization of* $\overline{D}$ *at* t_0, that is, the divisor on X_{t_0} obtained by taking the fiber of $\overline{D}$ above t_0. Note that not all the divisors of $G\mathcal{D}'$ can be specialized to any $t_0 \in \overline{\mathbb{Q}}$: in fact, for example, D_3' cannot be specialized to $t = -3$. Assume that D can be specialized to t_0 and that $\mathfrak{p} \in \mathcal{O}_{t_0}$ is a prime of good reduction for Ξ_{t_0}. Then let $\overline{D_{t_0}}$ be the Zariski closure of D_{t_0} inside Ξ_{t_0}. We define $D_{t_0,\mathfrak{p}}$ to be the reduction modulo $\mathfrak{p}$ of $\overline{D_{t_0}}$. The curve $D_{t_0,\mathfrak{p}}$ is a divisor on $X_{t_0,p}/\mathbb{F}_p$ that can be defined over $\mathbb{F}_{p^m}$. Notice that the procedure of going from a divisor of X_η to a divisor of $X_{t_0,p}$ consists of the same step, repeated twice: taking the closure of a divisor of the generic fiber of a family and specialising it to a special fiber.

Proof of Proposition 3.3.14. The main step in order to prove the statement is to compute the intersection matrix $[D \cdot D']_{D,D' \in G\mathcal{D}'}$, that is, the intersection numbers $D \cdot D'$ for all the elements D, D' in the set $G\mathcal{D}'$. In doing so, it is helpful to recall that: the intersection form is symmetric, and so $D \cdot D' = D' \cdot D$; the surface X_η is a K3 surface and then, from the adjunction formula, it follows that if D is the divisor given by an irreducible curve with arithmetic genus g then $D^2 = 2g - 2$. Let D be any divisor in $G\mathcal{D}'$. From Proposition 3.3.13, D is isomorphic to a plane conic C and, therefore, it has genus $g = 0$. From this it follows that $D^2 = -2$, that is, all the divisors in $G\mathcal{D}'$ have self intersection -2.

Computing the intersection number of two divisors defined over a function field is an expensive computation for a computer, this is why we reduce our computations to computations over finite fields. Fix an integer $t_0 \in \mathbb{Z}$, and an integral model for X_{t_0}. Let p be a prime of good reduction for the fixed integral model of X_{t_0} and, recalling the notation introduced before starting the proof, let K_{2,t_0} be the specialization of K_2 to t_0, $\mathcal{O}_{t_0}$ be the ring of integers of K_{2,t_0} and $\mathfrak{p}$ be a prime of $\mathcal{O}_{t_0}$ lying above p. Using lemmas 1.2.51 and 1.2.52, if D, D' are two divisors on X_η, then $D \cdot D' = D_{t_0,\mathfrak{p}} \cdot D'_{t_0,\mathfrak{p}}$. Since all divisors $D \in G\mathcal{D}'$ are defined over K_2, all the divisors $D_{t_0,p}$ are defined over the finite field $\mathbb{F}_{p^m}$, for some $m \in \mathbb{Z}_{>0}$.

If $D_{t_0,p}$ and $D'_{t_0,p}$ have no components in common, then the intersection $D_{t_0,p} \cap D'_{t_0,p}$ is a zero-dimensional scheme over $\mathbb{F}_{p^m}$. Using `MAGMA` (cf. [BCP97]) it is possible to compute its degree. Since we are considering divisors on a smooth surface, the degree of the zero-dimensional scheme given by the intersection of the two divisors equals the sum of the intersection multiplicities of the points of intersection of the two divisors (see [HS00, A.2.3]), and so the degree of $D_{t_0,p} \cap D'_{t_0,p}$ is the intersection number $D_{t_0,p} \cdot D'_{t_0,p} = D \cdot D'$. In this way we get the intersection matrix of the lattice Λ' generated by $D \in G\mathcal{D}'$. Using the intersection matrix of the generators of a lattice, one is able to compute the rank, the signature, the determinant, and the discriminant group of the lattice. One can find the `MAGMA` code used to perform these computations, and that led to the results in the statement, in [Fes16]. $\square$

Remark 3.3.16. Let X be a surface over a field k. In Theorem 1.2.4, we state that there is a unique integral pairing of $\operatorname{Div} \overline{X}$ satisfying the intuitive conditions that an intersection pairing should satisfy. Such intersection pairing can be explicitly defined as the alternating sum of the length of the Tor groups of the two divisors. On smooth surfaces, the only non-zero term of this sum is the first term, that coincides with the degree of zero-dimensional scheme defined by the intersection of two divisors; this is what we used in proving Proposition 3.3.14, in order to compute the intersection numbers of the divisors.

The above definition of the intersection pairing can be generalised to schemes of higher dimension, and in general it is not true that the intersection number equals the degree of scheme defined by the intersection of the two divisors. Also notice that in higher dimension, the intersection of the two divisors does not need to be zero-dimensional.

For the explicit definition of the intersection pairing and more details about this topic, see [Har77, Appendix A].

Remark 3.3.17. The divisors $D'_1, ..., D'_4$ are only some of the divisors we found using the methods described in subsections 3.3.1 and 3.3.2. They have been presented here because they form a minimal set of independent divisors such that their orbits under the action of G generate a rank 19 sublattice of $\operatorname{Pic} \overline{X_\eta}$. In fact, for any $j \in \{1, ..., 4\}$ the set

$$G(\mathcal{D}' - \{D'_j\}) = \{sD_i : s \in G,\ i \in \{1, ..., 4\} - \{j\}\}$$

generates a sublattice of $\operatorname{Pic}\overline{X_\eta}$ of rank at most 17 (see [Fes16] for the computations).

Having a finite-index sublattice has been very important in order to saturate Λ' and obtain the full geometric Picard lattice: in fact, it tells us which field all the classes of $\operatorname{Pic}\overline{X_\eta}$ are defined over (cf. Proposition 3.4.8), helping us in finding more divisors with computational methods (see Remark 3.5.1).

Proposition 3.3.14 tells us that the geometric Picard number of the generic fiber of $\mathfrak{X}$ is at least 19. A priori the rank of $\operatorname{Pic}\overline{X_\eta}$ could be also 20. In order to prove that this is not the case, that is, in order to prove Proposition 3.1.3, we need to show that the family $\mathfrak{X}$ is non-isotrivial. So we will show that there are two smooth fibers with different geometric Picard number (cf. Lemma 3.3.21). In fact, on the one hand it is possible to show that 19 is an upper bound for $\rho(\overline{X_{t_0}})$, for several values of $t_0 \in \mathbb{Q}$ (cf. Remark 3.3.18); on the other hand, it is possible to exhibit a concrete example of a fiber X_{t_0} with geometric Picard number equal to 20 (cf. Example 3.3.20).

Remark 3.3.18. As we have seen in the introduction of this section, there are several methods to give an upper-bound for the Picard number of a $K3$ surface. During the Arizona Winter school 2015, using methods described in [vL07], [EJ08b], [EJ08a], and [Har15], Stephan Elsenhans computed an upper bound for $\rho(\overline{X_{t_0}})$, for every $t_0 \in \mathbb{Q}$ with naïve height at most 10^4. For such a $t_0 \in \mathbb{Q}$, let $\ell(t_0)$ denote the upper bound computed by Elsenhans. Then $\ell(t_0) = 19$ for all the values considered, except for

$$t_0 = -255/4, -33/2, -5, 0, 8, 15/4, 24, 240, 1320. \tag{3.10}$$

For these values of t_0 the upper-bound trivially turns out to be 20.

These computations, together with Proposition 3.3.14, show that if $t_0 \in \mathbb{Q}$ is a number for which $\ell(t_0)$ has been computed and equals 19, then $\rho(\overline{X_{t_0}}) = 19$.

Remark 3.3.19. Notice that the rational values of t_0 for which B_{t_0} admits a tri-tangent line, i.e., the real values listed in (3.9), Remark 3.3.5, are contained in the values listed in (3.10).

Example 3.3.20. Let $t_0 = 0$ and consider the K3 surface $X_0 = X_{t_0}$. The ramification locus

$$B_0 \colon x^6 + y^6 + z^6 = 0$$

of X_0 admits tri-tangent lines, for example the line $L\colon y = \zeta_{12}z$. The pull-back π^*L of L on X_0 via π splits into two irreducible components:

$$L_i = \begin{cases} y - \zeta_{12}z &= 0 \\ w + (-1)^i x^3 &= 0 \end{cases}$$

for $i = 1, 2$.

Let $\mathcal{D}_0 \subset \operatorname{Pic} \overline{X_0}$ be the set of classes of divisors of $\overline{X_0}$ given by the specialisation of the classes inside $\mathcal{D}'$ to $t_0 = 0$, and the class $[L_1]$. Let $G\mathcal{D}_0$ be the union of the G-orbits of the elements of $\mathcal{D}_0$.

Using the same technique used to prove Proposition 3.3.14, one can prove that the elements of $G\mathcal{D}_0$ generate a sublattice of $\operatorname{Pic} \overline{X_0}$ of rank 20. See [Fes16] for the explicit computations. To our knowledge, Masahiro Nakahara, a graduate student of Anthony Várilly-Alvarado, has been the first to point out that X_0 is a singular K3 surface.

Lemma 3.3.21. *The family $\mathfrak{X}$ is not isotrivial, that is, not all the smooth fibers are isomorphic.*

Proof. Remark 3.3.18 shows that there are many smooth fibers with geometric Picard number 19; Example 3.3.20 shows that the fiber $p^{-1}(0)$ is smooth and has geometric Picard number equal to 20. Then $\mathfrak{X}$ has at least two smooth fibers that are not isomorphic. □

We are now able to prove Proposition 3.1.3, that says that the geometric Picard number of X_η is at most 19.

Proof of Proposition 3.1.3. The family $\mathfrak{X}$ is parametrised by the affine line $\mathbb{A}^1$, which has dimension 1. Lemma 3.3.21 shows that $\mathfrak{X}$ is not isotrivial and so, by Theorem 1.2.50, the geometric Picard number of X_η can be at most 19.

On the other hand, the Picard lattice $\operatorname{Pic} \overline{X_\eta}$ contains the lattice Λ', that has rank 19, therefore $\operatorname{Pic} \overline{X_\eta}$ has rank at least 19.

The statement follows. □

Remark 3.3.22. It is possible to prove that $\rho(\overline{X_\eta}) \geq 19$ also without using the explicit divisors listed above. In fact we will show that there is a dominant rational map from X_η to a surface having geometric

Picard number at least 19. From this, using Proposition 1.2.36.(4), it follows that also the geometric Picard number of X_η is at least 19.

Consider the dominant rational morphism $\phi\colon \mathbb{P} \dashrightarrow \mathbb{P}$ defined by

$$(x : y : z : w) \mapsto ((yz)^2 : (xz)^2 : (xy)^2 : (xyz)^3 w).$$

It is easy to see that ϕ maps the surface X to

$$X'\colon w^2 = (yz)^3 + (xz)^3 + (xy)^3 + t(xyz)^2.$$

The surface $X' \subset \mathbb{P}$ is not smooth: it has three D_4-singularities at $(0 : 0 : 1 : 0), (0 : 1 : 0 : 0), (1 : 0 : 0 : 0)$. Blowing up X' at $Q = (0 : 0 : 1 : 0)$ we obtain the surface $X'' = \mathrm{Bl}_Q X' \subset \mathbb{P} \times \mathbb{P}^1(s_1, s_2)$ defined by

$$\begin{cases} s_1 y &= s_2 x \\ w^2 &= (yz)^3 + (xz)^3 + (xy)^3 + t(xyz)^2. \end{cases}$$

Consider the affine patch $X'' \cap \{x \neq 0, s_1 \neq 0\}$, defined by the equation

$$W^2 = s^3 Z^3 + Z^3 + s^3 + ts^2 Z^2,$$

where $W = w/x^3, Z = z/x$ and $s = s_2/s_1$. Eventually, after the change of variables

$$\begin{cases} W' &= (s^3 + 1)W \\ Z' &= (s^3 + 1)Z \end{cases}$$

one can see that $X'' \cap \{x \neq 0, s_1 \neq 0\}$ is birational to the surface given by

$$W'^2 = Z'^3 + ts^2 Z'^2 + s^3(s^3 + 1)^2,$$

that is a cubic base-change of the surface

$$E\colon W^2 = Z^3 + ts^2 Z^2 + s^5(s + 1)^2,$$

obtained by sending s to s^3.

One can easily notice that the natural projection of $E \subseteq \mathbb{A}^2 \times \mathbb{A}^1$ onto $\mathbb{A}^1$, defined by

$$\pi_s\colon ((Z, W), s) \mapsto s,$$

induces an elliptic fibration $E \to \mathbb{A}_1$. The generic fiber E_ϵ of E has j-invariant

$$j(E_\epsilon) = -2^8 \frac{t^6 s^2}{(s+1)^2(27s^2 + (4t^3+54)s + 27)}$$

and discriminant

$$\Delta(E_\epsilon) = -2^4 s^{10}(s+1)^2(27s^2 + (4t^3+54)s + 27).$$

It follows that the fibers above the points $s = 0, -1, \gamma_1, \gamma_2$, with γ_i such that $\gamma_i^2 + (4t^3/27 + 2)\gamma_i + 1 = 0$, for $i = 1, 2$, are singular. Doing analogous computations on the affine patch $X'' \cap \{x \neq 0, s_2 \neq 0\}$ one can see that also the fiber above the point $\infty = (0:1)$ is singular. So E has five singular fibers, namely the fibers above the points

$$(0:1), (1:0), (1:-1), (1:\gamma_1), (1:\gamma_2).$$

Using the characterisation of singular fibers (for example, see [Sil94, Table IV.9.4.1]), the fibers above these points are of type II^*, II^*, I_2, I_1, and I_1, respectively (here we use Kodaira's notation for singular fibers, see [Kod64]) and hence they have $9, 9, 2, 1,$ and 1 irreducible components, respectively. From Tate-Shioda formula (see [Shi90, Theorem 1.3 and Corollary 5.3]), it follows that $\rho(\overline{X''}) \geq 19$.

$$\begin{array}{ccccc} X & \overset{\phi}{\dashrightarrow} & X' & \overset{\mathrm{bl}_Q}{\dashrightarrow} & X'' \\ & & & & \Big\downarrow{\scriptstyle \pi_s} \\ & & & & \mathbb{P}^1 \end{array}$$

Since the composition $\mathrm{bl}_Q \circ \phi$ is a dominant rational map, $\rho(\overline{X}) = \rho(\overline{X''})$ (cf. Proposition 1.2.36.(4)), and so $\rho(\overline{X}) \geq 19$.

Corollary 3.3.23. *The lattice Λ' is a proper finite-index sublattice of* $\operatorname{Pic} \overline{X_\eta}$.

Proof. By construction, the lattice Λ' is contained in $\operatorname{Pic} \overline{X_\eta}$, and they both have rank 19. Hence Λ' is a finite-index sublattice of $\operatorname{Pic} \overline{X_\eta}$.

Then all we need to show is that Λ is not equal to $\operatorname{Pic} \overline{X_\eta}$. Assume, by contradiction, that $\Lambda' = \operatorname{Pic} \overline{X_\eta}$. Then the transcendental lattice

$T(\overline{X_\eta}) = (\operatorname{Pic}\overline{X_\eta})^\perp \subset H^2(X_\eta, \mathbb{Z})$ has rank $3 = 22 - 19$ and the discriminant lattice A_T is isomorphic to the discriminant lattice A_P of $\operatorname{Pic}\overline{X_\eta}$ (see Proposition 1.1.12). By Proposition 3.3.14 the discriminant group A_P is isomorphic to $C_2^{16} \times C_6 \times C_{12}^2$, implying that it cannot be generated by fewer than 19 elements, i.e., $\ell(A_P) = 19$. Since A_P and A_T are isomorphic, $\ell(A_T) = 19$. From Lemma 1.1.11 it then follows that $19 = \ell(A_T) \leq \operatorname{rk}(T(\overline{X_\eta})) = 3$, getting a contradiction. □

Corollary 3.3.24. *The automorphism group* $\operatorname{Aut}(X_\eta)$ *embeds into the group of isometries of* $\operatorname{Pic}\overline{X_\eta}$*.*

Proof. By Proposition 1.2.47, the statement is true if $\operatorname{Pic}\overline{X_\eta}$ is a lattice of odd rank with discriminant not a power of 2. From Corollary 3.3.23 we know that $\operatorname{Pic}\overline{X_\eta}$ has rank 19 and that Λ' is a finite-index sublattice of $\operatorname{Pic}\overline{X_\eta}$. The lattice Λ' has discriminant $2^{21}\, 3^3$ (cf. Proposition 3.3.14) and therefore, using Lemma 1.1.5, the discriminant $\operatorname{Pic}\overline{X_\eta}$ is congruent to 6 up to square factors. Hence, it cannot be a power of 2. □

3.4 The field of definition of $\operatorname{Pic}\overline{X_\eta}$

Even though we know that Λ' cannot be the full geometric Picard lattice of X_η (see Corollary 3.3.23), the fact that Λ' has finite index inside $\operatorname{Pic}\overline{X_\eta}$ allows us to say something about the field of definition of $\operatorname{Pic}\overline{X_\eta}$ (cf. Proposition 3.4.8).

Let us recall the notation introduced in Subsection 3.3.3. K is the field $\mathbb{Q}(t)$, we fixed an algebraic closure $\overline{K}$ of K such that $\overline{\mathbb{Q}} \subset \overline{K}$. The element $\zeta_n \in \overline{\mathbb{Q}}$ is a n-th root of unity, for $n \in \{3, 4, 6, 12\}$, such that $\zeta_n = \zeta_{12}^{12/n}$. For $i = 0, 1, 2$, the element $\beta_i \in \overline{K}$ is such that $\beta_i^2 = t + 3\zeta_3^i$. The field K_1 is the field $K = K(\zeta_{12}, \beta_0, \beta_1, \beta_2)$.

Remark 3.4.1. The elements $t+3, t+3\zeta_3$, and $t+3\zeta_3^2$ generate a subgroup B of order 8 inside $K(\zeta_{12})^\times/(K(\zeta_{12})^\times)^2$. By Kummer theory (cf. Theorem [Mil15, Theorem 5.28]), the subgroup B corresponds to the extension of $K(\zeta_{12})$ obtained by adjoining the square roots of all the elements of B. The field we obtain adjoining these square roots is $K_1 = K(\zeta_{12}, \beta_0, \beta_1, \beta_2)$. Then, by Theorem [Mil15, Theorem 5.28], the extension $K_1/K(\zeta_{12})$ has degree 8 and exponent 2.

We defined $c_i \in \overline{K}$, for $i = 0, 1, 2$, to be the roots of the polynomial $h(v) := v^3 + tv^2 + 4 \in K[v]$. The polynomial h has discriminant $\Delta = -16(t^3 + 27) = (4\zeta_4\beta_0\beta_1\beta_2)^2$, and δ denotes the element $4\zeta_4\beta_0\beta_1\beta_2 \in K_1$, a square root of Δ. The field K_2 is the field obtained by adjoining c_0 to K_1, that is, $K_2 := K(\zeta_{12}, \beta_0, \beta_1, \beta_2, c_0)$.

Lemma 3.4.2. *The polynomial h splits completely over K_2, that is, c_0, c_1, c_2 are elements of K_2.*

Proof. By definition of K_2 we have that c_0 is in there. The discriminant of h is $\Delta = -16(D^3 + 27) = (4\zeta_4\beta_0\beta_1\beta_2)^2$, that is, Δ is a square inside K_1 and $\delta = 4\zeta_4\beta_0\beta_1\beta_2$ is one of its square roots. This means that adjoining a root of h to K_1 we get a splitting field for h. □

Remark 3.4.3. It is possible to explicitly write the roots c_1, c_2 in terms of the generators of K_2, the elements $\zeta_{12}, \beta_0, \beta_1, \beta_2, c_0$. Namely, one can then see that the other two roots of h are

$$\frac{-t - c_0 \pm \epsilon}{2},$$

where $\epsilon = \frac{\delta}{c_0(3c_0+2t)}$ and $\delta = 4\zeta_4\beta_0\beta_1\beta_2$, the square root of the discriminant Δ of h.

Let $E := K(\delta, c_0) \subset K_2$ be the field obtained by adjoining the elements $\delta, c_0 \in K_2$ to K.

Let $F := K(\beta_0) \subset K_2$ be the field obtained by adjoining $\beta_0 \in K_2$ to K.

Let $L := K(\beta_1, \beta_2) \subset K_2$ be the field given by adjoining $\beta_1, \beta_2 \in K_2$ to K.

Lemma 3.4.4. *The following statements hold.*

1. *The extension E/K is a Galois extension of degree* 6 *with Galois group* $\mathrm{Gal}(E/K) \cong S_3$.

2. *The extension F/K is a Galois extension of degree* 2 *with Galois group* $\mathrm{Gal}(F/K) \cong C_2$.

3. *The extension L/K is a Galois extension of degree* 8 *with Galois group* $\mathrm{Gal}(L/K) \cong D_4$.

4. *The fields* E, F, *and* L *intersects pairwise trivially, that is, the intersection of any two of them equals* K.

5. *The compositum field* $E \cdot F \cdot L$ *equals* K_2.

Proof. 1. By construction, the field E is the splitting field of the cubic polynomial $h = v^3 + tv^2 + 4$, that is irreducible over K and whose discriminant is not a square in K. The statement follows.

2. The field F is the splitting field of the second degree polynomial $v^2 - (3+t)$. The statement trivially follows.

3. The field L is the splitting field of the polynomial

$$l = v^4 + (-2t+3)v^2 + t^2 - 3t + 9,$$

and so L/K is a Galois extension. The roots of l are $\pm\beta_1, \pm\beta_2$, therefore the Galois group $\mathrm{Gal}(L/K)$ is generated by $\gamma_1, \gamma_2, \gamma$, where γ_1 changes the sign of β_1, γ_2 changes the sign of β_2, and γ switches β_1 and β_2. Since L/K is Galois, we have the following chain of equalities: $\#\,\mathrm{Gal}(L/K) = [L:K] = 8$. One can easily check that $\gamma\gamma_1 \neq \gamma_1\gamma$, and that these two are the only elements of order 4 of $\mathrm{Gal}(L/K)$. Summarising, $\mathrm{Gal}(L/K)$ is a non-abelian group of order 8 with exactly two elements of order 4. Then $\mathrm{Gal}(L/K)$ must be isomorphic to D_4.

4. By explicit computations.

5. The compositum field $E \cdot F \cdot L$ is by construction contained in K_2, since E, F, and L are all defined as subsets of K_2. Then, we only need to show that other inclusion. The field K_2 is obtained by adjoining $\zeta_{12}, \beta_0, \beta_1, \beta_2, c_0$ to K. From the definition of E, F, and L, the elements $\beta_0, \beta_1, \beta_2, c_0$ are inside $E \cdot F \cdot L$, as well as $4\zeta_4\beta_0\beta_1\beta_2 = \delta \in E$ (see Remark 3.4.3) and $\frac{1}{3}(\beta_1^2 - t) = \zeta_3 \in L$. Then the element

$$\alpha = \frac{4\zeta_3\beta_0\beta_1\beta_2}{4\zeta_4\beta_0\beta_1\beta_2} = \frac{\zeta_3}{\zeta_4}$$

is also inside $E \cdot F \cdot L$. Recall that $\zeta_3 = \zeta_{12}^4$ and $\zeta_4 = \zeta_{12}^3$, then

$$\alpha = \frac{\zeta_3}{\zeta_4} = \frac{\zeta_{12}^4}{\zeta_{12}^3} = \zeta_{12}.$$

This proves that $K_2 \subseteq E \cdot F \cdot L$ and, therefore, $K_2 = E \cdot F \cdot L$. ☐

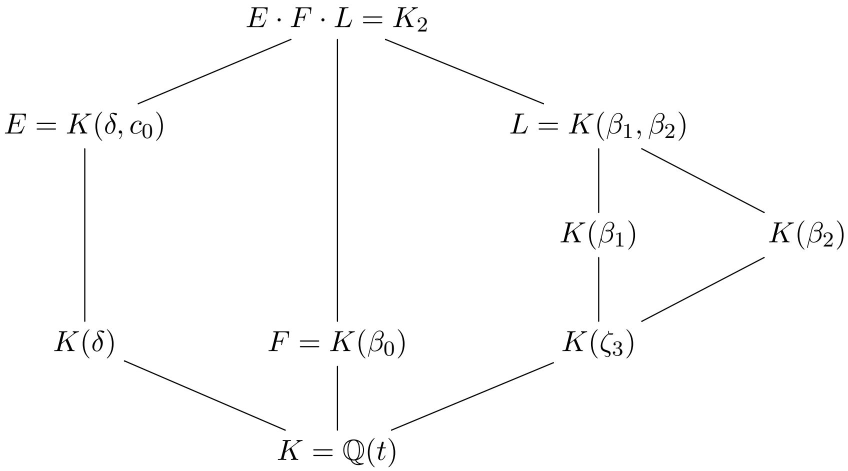

Figure 3.2: An alternative description of K_2.

Theorem 3.4.5. *The field extension K_2/K is a Galois extension of degree $2^5 \cdot 3$. The Galois group $\mathrm{Gal}(K_2/K)$ is isomorphic to the group*

$$S_3 \times C_2 \times D_4.$$

Proof. By Lemma 3.4.4.(5) and (4) we have that $K_2 = E \cdot F \cdot L$ and that E, F, and L intersect pairwise trivially. It follows that

$$\mathrm{Gal}(K_2/K) \cong \mathrm{Gal}(E/K) \times \mathrm{Gal}(F/K) \times \mathrm{Gal}(L/K).$$

From Lemma 3.4.4.(1)–(3) we know that

$$\mathrm{Gal}(E/K) \cong S_3,$$
$$\mathrm{Gal}(F/K) \cong C_2,$$
$$\mathrm{Gal}(L/K) \cong D_4.$$

The statement follows. ☐

Remark 3.4.6. Let t_0 be an element of $\overline{\mathbb{Q}}$, and let K_{t_0} and K_{2,t_0} the specializations to t_0 of K and K_2, respectively. Trivially, we have that $[K_{2,t_0} : K_{t_0}] \leq [K_2 : K]$; sometimes, this inequality can be strict. This is the case, for example, if $t_0 = \zeta_{12}$: in this case, in fact, we have that $[K_{2,t_0} : K_{t_0}] = 2^3\,3$.

One might then ask whether there exists a t_0 such that K_{2,t_0} is exactly K_{t_0}. The answer to this question is positive: with the help of Maarten Derickx, we have been able to find an element $\alpha \in \overline{\mathbb{Q}}$, living in a number field of degree 64, such that $[K_{2,\alpha} : K_\alpha] = 1$. See [Fes16] for the explicit computations.

Remark 3.4.7. For future reference it can be useful to explicitly describe an isomorphism between $\operatorname{Gal}(K_2/K)$ and $S_3 \times C_2 \times D_4$. In order to do so, we will present five automorphisms $\tau_i \in \operatorname{Gal}(K_2/K)$, with $i = 1, 2, 3, 4, 5$, such that:

$$\begin{aligned} \operatorname{Gal}(E/K) &= \langle \tau_1, \tau_2 \rangle \cong S_3; \\ \operatorname{Gal}(F/K) &= \langle \tau_3 \rangle \cong C_2; \\ \operatorname{Gal}(L/K) &= \langle \tau_4, \tau_5 \rangle \cong D_4. \end{aligned}$$

The field K_2 is generated by $c_0, \zeta_{12}, \beta_0, \beta_1, \beta_2$ over K, so to describe an element $\tau \in \operatorname{Gal}(K_2/K)$ it is enough to describe its action on those elements. The action of τ_i on those generators of K_2 over K is listed in the table below. For the convenience of the reader, the table also lists the action of τ_i, for $i = 1, ..., 5$, on other interesting elements of K_2.

	c_0	c_1	c_2	δ	ζ_{12}	ζ_4	ζ_3	β_0	β_1	β_2
τ_1	c_0	c_2	c_1	$-\delta$	ζ_{12}^7	$-\zeta_4$	ζ_3	β_0	β_1	β_2
τ_2	c_1	c_2	c_0	δ	ζ_{12}	ζ_4	ζ_3	β_0	β_1	β_2
τ_3	c_0	c_1	c_2	δ	ζ_{12}^7	$-\zeta_4$	ζ_3	$-\beta_0$	β_1	β_2
τ_4	c_0	c_1	c_2	δ	ζ_{12}^{11}	$-\zeta_4$	ζ_3^2	β_0	$-\beta_2$	β_1
τ_5	c_0	c_1	c_2	δ	ζ_{12}^7	$-\zeta_4$	ζ_3	β_0	β_1	$-\beta_2$

Recalling the notation introduced in Section 1.2, we say that K_2 is the field of definition of a class $D \in \operatorname{Pic} X_\eta$ if $\operatorname{Gal}(\overline{K}/K_2)$ is the stabilizer of D inside $G_K := \operatorname{Gal}(\overline{K}/K)$; we say that $D \in \operatorname{Pic} X_\eta$ can be defined over K_2 if $\operatorname{Gal}(\overline{K}/K_2)$ is contained in the stabilizer of D inside G_K.

We say that K_2 is *the field of definition of* $\operatorname{Pic}\overline{X_\eta}$ if $\operatorname{Gal}(K_2/K)$ acts freely on $\operatorname{Pic}\overline{X_\eta}$; we say that $\operatorname{Pic}\overline{X_\eta}$ *can be defined over* K_2 if all the elements of $\operatorname{Pic}\overline{X_\eta}$ can be defined over K_2.

Proposition 3.4.8. *The lattice* $\operatorname{Pic}\overline{X_\eta}$ *can be defined over* K_2.

Proof. First we claim that the lattice Λ' can be defined over K_2. In order to see this just notice that all the divisors in $\mathcal{D}'$ can be defined over K_2, as well as the automorphisms in G. Therefore we can conclude that all the divisors in $G\mathcal{D}'$ are fixed by $\operatorname{Gal}(\overline{K}/K_2)$ and hence, the lattice Λ' can be defined over K_2.

By Corollary 3.3.23, we know that Λ' is a finite-index sublattice in $\operatorname{Pic}\overline{X_\eta}$. Let $m \geq 1$ be the index $[\operatorname{Pic}\overline{X_\eta} : \Lambda']$. Now let N be an element of $\operatorname{Pic}\overline{X_\eta}$. Then mN is in Λ', that is, it can be written as linear combination of elements of $G\mathcal{D}'$. It follows that mN can be defined over K_2 and so $\operatorname{Gal}(\overline{K}/K_2)$ fixes mN. Since the Galois action is linear and $\operatorname{Pic}\overline{X_\eta}$ is torsion-free, it follows that $\operatorname{Gal}(\overline{K}/K_2)$ fixes N too, i.e., N can be defined over K_2. The statement follows from the generality of N. □

Remark 3.4.9. The natural action of the absolute Galois group G_K on the geometric Picard group induces a map from G_K to the group of isometries of $\operatorname{Pic}\overline{X_\eta}$. Let H_K be the kernel of this map.

$$0 \to H_K \to G_K \to \mathcal{O}(\operatorname{Pic}\overline{X_\eta})$$

Then Proposition 3.4.8 can be rephrased by saying that $\operatorname{Gal}(\overline{K}/K_2)$ is contained in H_K. Later we will see that in fact $H_K = \operatorname{Gal}(\overline{K}/K_2)$, that is, K_2 is the field of definition of $\operatorname{Pic}\overline{X_\eta}$. (cf. Remark 3.7.4).

3.5 More divisors

By Corollary 3.3.23 we know that Λ' is not the full geometric Picard lattice of X_η. In order to generate $\operatorname{Pic}\overline{X_\eta}$ we then need more divisors on $\overline{X_\eta}$. Combining different techniques (cf. Remark 3.5.1) we managed to find more plane conics with splitting pull-back on $\overline{X_\eta}$.

Remark 3.5.1. If we add all the divisors coming from the del Pezzo surfaces of degree 1 of which X_η is a double cover (cf. Subsection 3.3.2) to the ones in the set $\mathcal{D}'$, and we take the union of their G-orbits, then this set generates a sublattice of $\operatorname{Pic}\overline{X_\eta}$ that is bigger than Λ', but that can still be proven not to be the full geometric Picard lattice, using an argument as in Corollary 3.3.23.

Failing in finding other six-tangent conics with particular symmetric equations, we decided to go for an extensive search. A generic conic inside $\mathbb{P}^2_K$ is given by a linear combination of the six monomials of degree 2 in x, y, z. The field K is a function field over an infinite field, and performing computations on K or over some algebraic extension of K, like K_2, requires a lot of computational power. Therefore an extensive search for six-tangent plane conics, running through all the possible 6-tuples of coefficients, looks infeasible over K_2 or K. This is why, once again, we reduced our computations to a finite field.

Fix an integer $t_0 \in \mathbb{Z}$, and an integral model for X_{t_0}. Let p be a prime of good reduction for the fixed integral model of X_{t_0} and recall the notation introduced after stating Proposition 3.3.14. Let K_{2,t_0} be the number field obtained by specializing K_2 to t_0, and let $\mathfrak{p}$ be a prime of $\mathcal{O}_{t_0}$ above p. Let $G\mathcal{D}'_{t_0,\mathfrak{p}} := \{(D)_{t_0,\mathfrak{p}} \;:\; D \in G\mathcal{D}'\}$ be the set given by first specializing to t_0 and then reducing modulo $\mathfrak{p}$ the divisors in $\mathcal{D}'$. Let Λ_0 be the sublattice generated by the divisors of $G\mathcal{D}'_{t_0,\mathfrak{p}}$. Notice that using the specialization and the reduction maps, we get an isometry between Λ' and Λ_0. From Proposition 3.4.8, it follows that Λ_0 can be defined over $\kappa(\mathfrak{p})$.

Let m be the positive integer for which $\kappa(\mathfrak{p}) = \mathbb{F}_{p^m}$. Then we run through all the 6-tuples $a = (a_0, ..., a_5) \in (\mathbb{F}_{p^m})^6$ such that the conic

$$C_a\colon a_0x^2 + a_1y^2 + a_2z^2 + a_3xy + a_4xz + a_5yz = 0$$

intersects $B_{t_0,p}$ with even multiplicity everywhere. Notice that in this case we have a finite number of 6-tuples to run through, 'only' p^{6m} (or $6\,p^{5m}$, if we assume at least one coefficient to always be non-zero), and that for a computer performing computations over a finite field is much easier than performing computations over (an algebraic extension of) a function field.

For each conic C_a found in this way, we compute the lattice Λ_a inside $\operatorname{Pic}\overline{X_{t_0,p}}$, generated by the irreducible components of the pull-back of C_a on $X_{t_0,p}$ and the divisors of $G\mathcal{D}'_{t_0,\mathfrak{p}}$.

Then the rank of Λ_a is greater than or equal to the rank of Λ_0 and, if equality holds, then $\det(\Lambda_a) \leq \det(\Lambda_0)$. Since the specialization map and the reduction map are both injective and both have torsion-free cokernel (cf. Propositions 1.2.51 and 1.2.52), if $\operatorname{rk}\Lambda_a = \operatorname{rk}(\Lambda_0)$ and $\det(\Lambda_a) < \det(\Lambda_0)$ then we know that C_a lifts to a curve $C \subset \mathbb{P}^2_{\overline{K}}$ such that: the pull-back D of C on $\overline{X_\eta}$ splits into two irreducible components (by Proposition 1.2.27); the classes inside $\operatorname{Pic}\overline{X_\eta}$ of the two irreducible components of D are not in Λ'. Then the components D, together with the divisors on $\overline{X_\eta}$ we already have, generate a bigger sublattice of $\operatorname{Pic}\overline{X_\eta}$.

Lifting the conic $C_a \subset \mathbb{P}^2_{\mathbb{F}_{p^m}}$ to a conic $C \subset \mathbb{P}^2_{K_2}$ was hard, since the coefficients of the equation defining C are roots of polynomials over K to be computed inside K_2. We divided this process into two steps:

1. first we lift C_a to a conic C_{t_0} defined over some number field;

2. then we lift C_{t_0} to a conic C defined over K_2.

The first step is the hardest one, and we accomplished it by looking at the equation of C_a, looking for symmetries and vanishing coefficients, hoping that these phenomena would reflect a symmetry or a vanishing coefficient also in characteristic 0 (this was not always the case). Such assumption would make the computations over K_2 easier, possibly easy enough for a computer to be handled.

The second step was accomplished by considering the coefficients of C_{t_0} for different values of t_0, and then interpolating these values in terms of t.

Let $\mathcal{D} = \{D_1, ..., D_5\}$ be the set of divisors on $\overline{X_\eta}$ given by

$$D_1 = D_1', D_2 = D_2', D_3 = D_3',$$

and

$$D_4\colon \begin{cases} 2xy - c_1 z^2 &= 0 \\ x^3 - y^3 - w &= 0 \end{cases}$$

$$D_5\colon \begin{cases} a_5x^2 + c_5(y^2+z^2) + yz &= 0 \\ r_5x^3 + v_5xyz - w &= 0 \end{cases}$$

where

$$a_5 = \frac{\zeta_{12}(-\zeta_6+2)}{9}(\beta_0\beta_1+\beta_0\beta_2+\beta_1\beta_2+t),$$
$$c_5 = \frac{\zeta_{12}(\zeta_6-2)}{3},$$
$$r_5 = \frac{\zeta_{12}(\zeta_6-2)}{9}(2\beta_0\beta_1\beta_2+(2t-3)\beta_0+(2t-3\zeta_3)\beta_1+(2t+3\zeta_6)\beta_2),$$
$$v_5 = -\beta_0-\beta_1-\beta_2.$$

Remark 3.5.2. The divisor D_4 was obtained by considering the generators of the geometric Picard lattice of a del Pezzo surface of which X_η is a double cover (see Subsection 3.3.2).

The divisor D_5 was found using the technique described in Remark 3.5.1.

For every $i = 1, ..., 5$, the divisor $D_i \subset \overline{X_\eta}$ is defined by two equations, namely $f_i = w - g_i = 0$, where f_i and g_i are two homogeneous polynomial in x, y, z of degree 2 and 3 respectively. Since the polynomial f_i has no w-term, we denote by C_i the conic of $\mathbb{P}^2_K$ it defines.

Proposition 3.5.3. *For every $i \in \{1, ..., 5\}$, the following statements hold:*

1. *the conic $C_i \subset \mathbb{P}^2_K$ intersects the branch locus B_η of π with even multiplicity everywhere;*
2. *the divisor D_i on $\overline{X_\eta}$ is an irreducible component of the pull-back of C_i via π;*
3. *the curve $D_i \subset \overline{X_\eta}$ is isomorphic to the conic C_i.*

Proof. Analogous to Proposition 3.3.13. □

Let $G\mathcal{D} := \{\, sD_i \ : \ s \in G,\ i \in \{1, ..., 5\}\,\}$ denote the set obtained by letting the automorphisms of G act on the elements of $\mathcal{D}$ and let Λ be the sublattice of $\operatorname{Pic}\overline{X_\eta}$ generated by the elements of $G\mathcal{D}$.

Proposition 3.5.4. *The lattice Λ is an even lattice of rank 19, signature $(1, 19)$, discriminant $2^5 3^3$ and discriminant group isomorphic to $C_6 \times C_{12}^2$.*

Proof. The proof goes as the proof of Proposition 3.3.14. See [Fes16]. □

Remark 3.5.5. The set $\mathcal{D}$ is minimal in order to obtain a lattice of rank 19 and discriminant $2^5\,3^3$. For any $j \in \{1, ..., 5\}$, the set

$$G(\mathcal{D} - \{D_j\}) = \{sD_i : s \in G,\ i \in \{1, ..., 5\} - \{j\}\}$$

generates either a lattice of rank less than 19 or a lattice with rank 19 and discriminant at least $2^5\,3^5$. See [Fes16] for the explicit computations in `MAGMA`.

Corollary 3.5.6. *The lattice* Λ *is isometric to the lattice* (3.4) *in Theorem 3.1.4.*

Proof. Let Σ be the lattice given in (3.4). Notice that both Λ and Σ are indefinite even lattices. From Proposition 3.5.4 we know that $\ell(A_\Lambda) = 3 < 17 = \mathrm{rk}(\Lambda) - 2$. Then, by Proposition 1.1.15, we have that Λ and Σ are isometric if and only if they have the same rank, signature, and discriminant group. One can easily see that these invariants of Σ are the same as the invariants of Λ given in Proposition 3.5.4. □

Corollary 3.5.7. *The lattice* Λ *is a finite-index sublattice of* $\operatorname{Pic}\overline{X_\eta}$*. The index* $[\operatorname{Pic}\overline{X_\eta} : \Lambda]$ *divides* 12.

Proof. In order to show that Λ is a finite-index sublattice it is enough to recall that the rank of $\operatorname{Pic}\overline{X_\eta}$ is 19 (cf. Corollary 3.3.23) and that Λ has indeed rank 19.

The second statement follows from Lemma 1.1.5, recalling that $\det\Lambda = 2^5\,3^3$. □

3.6 The proof of the main theorem

In this section we show the proof of Theorem 3.1.4. The strategy of the proof is the same used by Michael Stoll and Damiano Testa in proving [ST10, Theorem 7].

Using the same notation as before, let $\Lambda \subseteq \operatorname{Pic}\overline{X_\eta}$ be the sublattice of $\operatorname{Pic}\overline{X_\eta}$ generated by the divisors inside $G\mathcal{D} := \{sD\ :\ s \in G,\ D \in \mathcal{D}\}$.

In what follows, for the sake of easy notation, we will denote the geometric Picard lattice of X_η by simply P.

In the first part of the section we restate some results proved in Section 1.1, keeping in mind that $P = \operatorname{Pic} \overline{X_\eta}$ is an even lattice and Λ is a finite-index sublattice of P (cf. Corollary 3.5.7).

Let $p \in \mathbb{Z}$ be a prime and consider the quotient groups $\Lambda/p\Lambda$ and P/pP. They also have the structure of $\mathbb{F}_p$-vector spaces. As in Section 1.1, if x is an element of Λ, we denote by $[x]_\Lambda$ and $[x]_P$ its class inside $\Lambda/p\Lambda$ and P/pP respectively.

The inclusion map $\Lambda \hookrightarrow P$ induces the group homomorphism

$$\iota_p \colon \Lambda/p\Lambda \to P/pP$$

defined by $[x]_\Lambda \to [x]_P$.

Let Λ_p denote the kernel of ι_p.

Lemma 3.6.1. *The following equality holds:*

$$\Lambda_p = \frac{\Lambda \cap pP}{p\Lambda}.$$

Proof. Lemma 1.1.17. □

Let $[x]_P$ be an element of P/pP, and define the homomorphism

$$[x]^* \colon \Lambda/p\Lambda \to \mathbb{Z}/p\mathbb{Z}$$

by sending $[y]_\Lambda \in \Lambda/p\Lambda$ to $b_{P,p}([x]_P, [y]_P)$, where $b_{P,p}$ is defined as in Section 1.1. By lemmas 1.1.18 and 1.1.19 we can define the following morphism:

$$\phi_{P,p} \colon P/pP \to \operatorname{Hom}(\Lambda/p\Lambda, \mathbb{Z}/p\mathbb{Z}),$$

defined by sending $[x]_P$ to $[x]^*$. In the same way, we define the morphism

$$\phi_{\Lambda,p} \colon \Lambda/p\Lambda \to \operatorname{Hom}(\Lambda/p\Lambda, \mathbb{Z}/p\mathbb{Z}).$$

Let k_p denote the kernel of $\phi_{\Lambda,p}$.

Lemma 3.6.2. *Let p be any prime. The following diagram is commutative.*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Lambda_p & \hookrightarrow & \Lambda/p\Lambda & \xrightarrow{\iota_p} & P/pP \\
 & & \downarrow & & \| & & \downarrow \phi_{P,p} \\
0 & \longrightarrow & k_p & \hookrightarrow & \Lambda/p\Lambda & \xrightarrow[\phi_{\Lambda,p}]{} & \operatorname{Hom}(\Lambda, \mathbb{Z}/p\mathbb{Z})
\end{array}$$

Proof. See Lemma 1.1.21. □

Since Λ is an even lattice, we can define k_p' to be the subset of k_p given by

$$\{[\lambda]_\Lambda \in k_p \mid \lambda^2 \equiv 0 \bmod 2p^2\}.$$

Lemma 3.6.3. *Then k_p' contains Λ_p and it is fixed by all the isometries of Λ.*

Proof. See Lemma 1.1.23. □

Remark 3.6.4. Notice that while Λ_p also depends on $\operatorname{Pic} X_\eta$, the sets k_p and k_p' depend only on Λ.

Lemma 3.6.5. *The sublattice $\Lambda \subseteq \operatorname{Pic} \overline{X_\eta}$ is equal to $\operatorname{Pic} \overline{X_\eta}$ if and only if the map ι_p is injective for every prime p whose square divides $\det \Lambda$.*

Proof. See Proposition 1.1.24 and Remark 1.1.29. □

In Section 3.2 we provided the subgroup G of automorphisms of X_η and we showed that it embeds into the isometry group of $\operatorname{Pic} \overline{X_\eta}$. In Section 3.4 we computed the Galois group $\operatorname{Gal}(K_2/K)$, where K_2 is the splitting field of $\operatorname{Pic} \overline{X_\eta}$. The group $\operatorname{Gal}(K_2/K)$ acts on the classes of $\operatorname{Pic} \overline{X_\eta}$, by acting on the coefficients of a representative of a class inside $\operatorname{Pic} \overline{X_\eta}$. This action induces a group homomorphism

$$\operatorname{Gal}(K_2/K) \to \mathcal{O}(\operatorname{Pic} \overline{X_\eta}).$$

Let $\tilde{G}$ be the group generated by G and the image of $\operatorname{Gal}(K_2/K)$ inside $\mathcal{O}(\operatorname{Pic} \overline{X_\eta})$, and let $\tilde{\mathcal{D}}$ be the set of divisors obtained by letting $\tilde{G}$ act on $\mathcal{D}$, namely

$$\tilde{\mathcal{D}} := \{\, sD_i \;:\; s \in \tilde{G},\; i \in \{1, ..., 5\}\,\}.$$

Let $\tilde{\Lambda}$ be the sublattice of $\operatorname{Pic} \overline{X_\eta}$ generated by the divisors inside $\tilde{\mathcal{D}}$.

Proposition 3.6.6. *The following equality holds:*

$$\Lambda = \tilde{\Lambda}.$$

Proof. Since the group G embeds inside $\tilde{G}$, the lattice Λ is a sublattice of $\tilde{\Lambda}$. As in Proposition 3.3.14, one can see that $\tilde{\Lambda}$ has same rank and determinant as Λ. See [Fes16] for the explicit computations. The statement follows. □

Corollary 3.6.7. *The action of $\tilde{G}$ on $\operatorname{Pic} \overline{X_\eta}$ induces an action on Λ.*

Proof. The lattice Λ is the lattice generated by the divisors inside $\tilde{\mathcal{D}}$, and $\tilde{\mathcal{D}}$ is stable under the action of $\tilde{G}$. □

We now have all the ingredients to prove Theorem 3.1.4. The proof consists of two main steps. Let p be either 2 or 3. First we explicitly compute k'_p, a subset of $\Lambda/p\Lambda$ that contains Λ_p but that is independent of $\operatorname{Pic} \overline{X_\eta}$; then we show that every non-zero element inside k'_p is not an element of Λ_p.

Proof of Theorem 3.1.4. By Corollary 3.5.7 we know that the lattice Λ is of finite index inside $\operatorname{Pic} \overline{X_\eta}$. We want to show that Λ is the full geometric Picard lattice.

By Lemma 3.6.5 we have that to prove the statement it is enough to prove that the map ι_p is injective for every prime p whose square divides $\det \Lambda$. By Proposition 3.5.4 we have that Λ has discriminant $2^5\, 3^3$ and, therefore, to prove the statement it suffices to prove the injectivity of ι_p for $p = 2, 3$.

Let p be equal to 2 or 3. From Corollary 3.6.7 we know that $\tilde{G}$ acts on Λ. Since the $\mathbb{F}_p$-vector space Λ_p is the kernel of a $\tilde{G}$-equivariant homomorphism, it is $\tilde{G}$-invariant. So if an element $[x]$ is in Λ_p, its whole $\tilde{G}$-orbit $\tilde{G}[x]$ is contained in Λ_p. Since the discriminant of Λ is $2^5\, 3^3$, by Proposition 1.1.28, the $\mathbb{F}_p$-vector space Λ_p can have dimension at most 2 and 1, for $p = 2, 3$ respectively. Since Λ_p is stable under the action of $\tilde{G}$, it follows that the $\tilde{G}$-orbit of every element in Λ_p spans an $\mathbb{F}_p$-vector space of dimension at most 2 or 1, for $p = 2, 3$ respectively.

Analogous statements hold if we consider the action of just G, instead of the whole $\tilde{G}$.

Let $p = 2$. In [Fes16] we computed the subset k'_p. Inside k'_p we found only one non-trivial G-orbit spanning a vector subspace of dimension at most 2. Let W denote this subspace. The subspace W has dimension

2, and it admits a basis $\{w_1, w_2\}$ such that

$$w_1 = [E_1]_\Lambda$$
$$w_2 = [E_2]_\Lambda,$$

where $E_1 := \psi_{0,3,0}D_4 - D_4$ and $E_2 := \tau_2^2\bar{\sigma}_{(x,y)}(\psi_{0,3,0}D_3 - D_3)$. Using the same technique used in Proposition 3.3.14 one is able to check that $E_1^2 = E_2^2 = -8$. Assume w_1 is an element of Λ_p, then E_1 is an element of Λ that is 2-divisible in P, say $E_1 = 2C$, for some $C \in P$. Since $E_1^2 = -8$, the class C is a -2-class, and then either C' or $-C'$ is effective (cf. Lemma 1.2.35). By construction $E_1 = E_{1,1} - E_{1,2}$, where $E_{1,1} = \psi_{0,3,0}D_4$ and $E_{1,2} = D_4$. Note that both $E_{1,1}$ and $E_{1,2}$ are elements of $\tilde{G}\mathcal{D}$, so $E_{1,1}^2 = E_{1,2}^2 = -2$. Let H be the hyperplane class in $\operatorname{Pic} X_\eta$, and notice that it is ample (in fact $3H$ is very ample). Since $E_1 = 2C$ with C a -2-class and H is ample, we have that the intersection number $H \cdot E_1 = 2H \cdot C$ is either positive or negative (according to whether C or $-C$ is effective); on the other hand, $E_1 = E_{1,1} - E_{1,2}$, and so $H \cdot E_1 = H \cdot E_{1,1} - H \cdot E_{1,2} = 2 - 2 = 0$, yielding a contradiction. Therefore E_1 cannot be 2-divisible. The same argument holds for E_2 as well as for any other element of W, since the orbit of every element of W spans the whole W, as in k'_p there are no 1-dimensional subspaces generated by $\tilde{G}$-orbits. So we have shown that ι_2 is injective.

Let $p = 3$. We computed the subset k'_p. Among the vectors in k'_p, we looked for those whose orbit under $\tilde{G}$ spans a 1-dimensional $\mathbb{F}_3$-vector space. There are no such vectors. See [Fes16] for the explicit computations.

In this way we proved that ι_3 is also injective, and therefore the injective morphism

$$\iota\colon \Lambda \to \operatorname{Pic} \overline{X_\eta}$$

is an isomorphism. □

3.7 Some consequences

Theorem 3.1.4 can be useful for gaining additional information about the geometric Picard lattice of every fiber of $\overline{\mathfrak{X}}$. Also, using the computations done to generate the lattice $\tilde{\Lambda}$, it is possible to compute the Galois module structure of $\operatorname{Pic} \overline{X_\eta}$.

Corollary 3.7.1. *Let $t_0 \in \overline{\mathbb{Q}}$ be an algebraic number. Then the surface X_{t_0} has either geometric Picard lattice isomorphic to* (3.4) *or geometric Picard number 20.*

Proof. By Lemma 1.2.51 we know that the specialization map

$$\mathrm{sp}_{t_0}\colon \operatorname{Pic}\overline{X_\eta} \to \operatorname{Pic}\overline{X_{t_0}}$$

is injective and has torsion free cokernel.

This implies that the rank of $\operatorname{Pic}\overline{X_{t_0}}$ is greater than or equal to 19 and that

$$\operatorname{Pic}\overline{X_{t_0}}/\operatorname{Pic}\overline{X_\eta} = \operatorname{coker}(\mathrm{sp}_{t_0}) \cong \mathbb{Z}^{\rho(\overline{X_{t_0}})-\rho(\overline{X_\eta})}.$$

Since X_{t_0} is a K3 surface, the rank of $\operatorname{Pic}\overline{X_{t_0}}$ can be at most 20. So $\rho(\overline{X_{t_0}}) \in \{19, 20\}$.

Assume that $\rho(\overline{X_{t_0}}) = 19$. Then

$$\operatorname{coker}(\mathrm{sp}_{t_0}) = \operatorname{Pic}\overline{X_{t_0}}/\operatorname{Pic}\overline{X_\eta} \cong \mathbb{Z}^{19-19} = \{1\}$$

and therefore $\operatorname{Pic}\overline{X_\eta} \cong \operatorname{Pic}\overline{X_{t_0}}$. □

This result gives us some information about the rational points on each smooth fiber of $\mathfrak{X}$. Recall that if X is a $K3$ surface defined over a number field K, we say that X *has potentially dense rational points* if there is a finite field extension K'/K such that the set $X(K')$ of K'-rational points is Zariski dense inside $X(\mathbb{C})$.

Corollary 3.7.2. *Let $t_0 \in \mathbb{Q}$ be an algebraic number such that X_{t_0} is smooth. Then the K3 surface X_{t_0} defined over the number field $\mathbb{Q}(t_0)$ admits an elliptic fibration. Also, X_{t_0} has potentially dense rational points.*

Proof. By Corollary 3.7.1 we have that $\rho(\overline{X_{t_0}}) \geq 19 > 5$. Then, by [Huy15, Proposition 11.1.3.(ii)], X_{t_0} admits an elliptic fibration. The second statement immediately follows from [BT00, Theorem 1.1] or [BT00, Theorem 1.4 and Remark 1.5]. □

Proposition 3.7.3. *The Galois group* $\operatorname{Gal}(K_2/K)$ *acts faithfully on* $\operatorname{Pic}\overline{X_\eta}$*, that is, K_2 is the field of definition of* $\operatorname{Pic}\overline{X_\eta}$*.*

Proof. By definition, the list $\tilde{\mathcal{D}}$ of divisors on $\overline{X_\eta}$ is stable under the action of $\tilde{G}$, and so it is stable under the action of $\mathrm{Gal}(K_2/K)$. Combining Proposition 3.6.6 and Theorem 3.1.4, the lattice generated by $\tilde{\Lambda}$ is $\mathrm{Pic}\,\overline{X_\eta}$. Using the action of $\mathrm{Gal}(K_2/K)$ on $\tilde{\Lambda}$, in [Fes16] we explicitly computed the 19×19 matrices representing the action of $\mathrm{Gal}(K_2/K)$ on $\mathrm{Pic}\,\overline{X_\eta}$. One can then check that none of these matrices is the identity matrix. □

Remark 3.7.4. Let G_K denote the absolute Galois group of K, and let H_K be the kernel of the map from G_K to $\mathcal{O}(\overline{X_\eta})$ induced by the action of G_K on $\mathrm{Pic}\,\overline{X_\eta}$.

$$0 \to H_K \to G_K \to \mathcal{O}(\mathrm{Pic}\,\overline{X_\eta})$$

In Remark 3.4.9 we have seen that $\mathrm{Gal}(\overline{K}/K_2)$ is contained in H_K. Then from Proposition 3.7.3 it follows that $\mathrm{Gal}(\overline{K}/K_2) = H_K$. From this we also get that $\mathrm{Gal}(K_2/K)$ embeds into $\mathcal{O}(\mathrm{Pic}\,\overline{X_\eta})$, in fact

$$\mathrm{Gal}(K_2/K) \cong \mathrm{Gal}(\overline{K}/K)/\,\mathrm{Gal}(\overline{K}/K_2) \cong G_K/H_K \hookrightarrow \mathcal{O}(\mathrm{Pic}\,\overline{X_\eta}).$$

Theorem 3.7.5. *Considering the action of* $\mathrm{Gal}(K_2/K)$ *on* $\mathrm{Pic}\,\overline{X_\eta}$*, the following statements hold.*

1. $H^0(\mathrm{Gal}(K_2/K), \mathrm{Pic}\,\overline{X_\eta})$ *is isomorphic to* $\mathbb{Z}$ *and it is generated by the class of the hyperplane section of* X_η*;*

2. $H^1(\mathrm{Gal}(K_2/K), \mathrm{Pic}\,\overline{X_\eta})$ *is isomorphic to* C_2^3*;*

3. *for every non-trivial subgroup* $H \subseteq \mathrm{Gal}(K_2/K)$*, we have*

$$H^1(H, \mathrm{Pic}\,\overline{X_\eta}) \cong C_2^i,$$

   with $i \in \{0, 1, 2, 3, 4, 5, 6, 8, 10, 12\}$*;*

4. *there are* 49 *normal subgroups* N *of* $\mathrm{Gal}(K_2/K)$ *for which the group* $H^1(N, \mathrm{Pic}\,\overline{X_\eta})$ *is trivial;*

5. *there are* 47 *normal subgroups* N *of* $\mathrm{Gal}(K_2/K)$ *for which the group* $H^1(N, \mathrm{Pic}\,\overline{X_\eta})$ *is non-trivial.*

Proof. By explicit computations. See [Fes16]. □

Using Theorem 3.1.4, it is also possible to deduce information about the transcendental lattice.

Proposition 3.7.6. *The transcendental lattice $T(\overline{X_\eta})$ is isometric to a sublattice of $U(3) \oplus A_2(4)$ of rank 3, signature $(2,1)$, determinant $2^5\,3^3$, and discriminant group isomorphic to $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$.*

Proof. The lattice $T(\overline{X_\eta})$ is the orthogonal complement of $\operatorname{Pic} \overline{X_\eta}$ inside $H^2(X, \mathbb{Z})$. Then from Theorem 3.1.4 and Proposition 1.2.36 it immediately follows that $T(\overline{X_\eta})$ has rank 3 and signature $(2,1)$.

From Theorem 3.1.4 and Proposition 1.1.12 it follows that $T(\overline{X_\eta})$ has determinant $2^5\,3^3$, and discriminant group $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$.

We only need to show that $T(\overline{X_\eta})$ embeds into $U(3) \oplus A_2(4)$. In order to see this, recall that at the end of Section 3.2 we have seen that there is a subgroup G_s acting symplectically (and faithfully) on $H^2(\overline{X_\eta}, \mathbb{Z})$. Then, by Lemma 1.2.48, we have that $T(\overline{X_\eta})$ is contained in $H^2(\overline{X_\eta}, \mathbb{Z})^{G_s}$, the sublattice of $H^2(\overline{X_\eta}, \mathbb{Z})$ invariant under G_s. In [Has12], Hashimoto gives a complete list of abstract groups acting symplectically on the second cohomology group of a K3 surface. For each such group, he also computes the sublattice fixed by the group, that depends only on the abstract group and not on the surface. Keeping in mind that G_s is isomorphic to the group $\mathfrak{A}_{4,3}$ (cf. Remark 3.2.12), the statement follows from the tables in [Has12, Subsections 10.2 and 10.3]. □