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Chapter 1

Background

In this chapter we introduce some basic notions that will come in handy later. In Section 1.1 we introduce lattices, focusing on integral lattices and giving some properties that will be mostly used in Chapter 3; in Section 1.2 we introduce some basic notions of algebraic geometry, together with some well and less well known results that are needed to state and prove the results contained in the next chapters.

1.1 Lattice theory warm up

In this section we introduce the notion of lattices together with some basic results for later use. In the first part we follow [vL05, Section 2.1].

For any two abelian groups A and G, a symmetric bilinear map $A \times A \to G$ is said to be *non-degenerate* if the induced homomorphism $A \to \mathrm{Hom}(A, G)$ is injective.

A *lattice* is a free $\mathbb{Z}$-module L of finite rank endowed with a non-degenerate symmetric, bilinear form $b_L \colon L \times L \to \mathbb{Q}$, called the *pairing* of the lattice. If x, y are two elements of L, the notation $x \cdot y$ may be used instead of $b_L(x, y)$, if no confusion arises.

A lattice is called *integral* if the image of its pairing is contained in $\mathbb{Z}$.

An integral lattice L is called *even* if $b_L(x, x) \in 2\mathbb{Z}$ for every x in L.

A sublattice of L is a submodule L' of L such that b_L is non-degenerate on L'.

A sublattice L' of L is called *primitive* if the quotient L/L' is torsion free.

The *signature* of L is the signature of the vector space $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$ together with the inner product induced by the pairing b_L.

Let E and L be two lattices. We define $E \oplus L$ to be the lattice whose underlying $\mathbb{Z}$-module is $E \times L$ and whose pairing $b_{E\oplus L}$ is defined as follows. Let $(e,l), (e',l')$ be two elements of $E \times L$; then we set

$$b_{E\oplus L}\big((e,l),(e',l')\big) := b_E(e,e') + b_L(l,l').$$

Remark 1.1.1. The natural embeddings of E and L into $E \oplus L$ defined by

$$e \mapsto (e,0)$$

and

$$l \mapsto (0,l)$$

respectively, both respect the intersection pairings on E, L and $E \oplus L$.

If S is a sublattice of a lattice L, then we define its *orthogonal complement*, denoted by $S^{\perp}$, to be the sublattice of L given by

$$S^{\perp} = \{x \in L \mid \forall y \in S,\ b_L(x,y) = 0\ \}.$$

Lemma 1.1.2. *Let S be a sublattice of a lattice L. The following statements hold.*

1. *The orthogonal complement $S^{\perp}$ of S is a primitive sublattice of L and its rank equals* $\mathrm{rk}(L) - \mathrm{rk}(S)$*;*
2. *$S \oplus S^{\perp}$ is a finite-index sublattice of L;*
3. *$(S^{\perp})^{\perp} = S_{\mathbb{Q}} \cap L$.*

Proof. This is a well known result. For a proof, see for example [vL05, Lemma 2.1.5]. □

Let L be a lattice with pairing b_L. With $L(n)$ we denote the lattice with the same underlying module and pairing given by $n \cdot b_L$.

Let L be a lattice of rank n with pairing b_L and fix a basis $(e_1, ..., e_n)$ of L. Then the *Gram matrix* of L with respect to the basis $(e_1, ..., e_n)$ is the $n \times n$ matrix $[b_L(e_i, e_j)]_{1\le i,j\le n}$.

The *determinant*, also called *discriminant*, of the lattice L, denoted by $\det L$, is the determinant of any Gram matrix of L. One can easily see that the determinant of a lattice is independent of the choice of the basis, and hence of the Gram matrix.

Remark 1.1.3. Let M be an $r \times r$ symmetric $\mathbb{Q}$-matrix with maximal rank. Then $(\mathbb{Z}^r, M)$ denotes the lattice whose underlying $\mathbb{Z}$-module is $\mathbb{Z}^r$ and whose intersection pairing is defined by

$$e_i \cdot e_j := M[i,j]$$

where $e_1 = (1,0,...,0), ..., e_r = (0,...,0,1)$ is the standard basis of $\mathbb{Z}^r$ and $M[i,j]$ is the (i,j)-th entry of the matrix M.

A lattice L is called *unimodular* if $\det L = \pm 1$.

Lemma 1.1.4. *Let E and L be two lattices of rank m and n, and signature (e_+, e_-) and (l_+, l_-), respectively. Then the lattice $E \oplus L$ has*

1. *rank equal to $m+n$,*
2. *determinant equal to $\det E \cdot \det L$,*
3. *signature equal to $(e_+ + l_+, e_- + l_-)$.*

Proof. Fix the bases $(e_1, ..., e_m)$ and $(l_1, ..., l_n)$ for E and L respectively, and let M and N be the the associated Gram matrices. By the definition of the pairing $b_{E\oplus L}$ it follows that the Gram matrix of $E \oplus L$ with respect to the basis $(e_1, ..., e_m, l_1, ..., l_n)$ is the block matrix

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}.$$

The statements follow. □

Lemma 1.1.5. *Let S be a finite-index sublattice of a lattice L. Then the determinant of S equals $[L:S]^2 \cdot \det(L)$.*

Proof. [BHPVdV04, Lemma I.2.1]. □

Let L be an integral lattice. We define the *dual lattice* of L to be the lattice

$$L^* = \{x \in L_{\mathbb{Q}} \mid \forall y \in L,\ b_L(x,y) \in \mathbb{Z}\ \}.$$

The pairing on L^* is given by linearly extending b_L to L^*; we will use b_L to also denote the pairing on L^*.

Remark 1.1.6. Sometimes the dual lattice L^* of an integral lattice L is also defined as $\mathrm{Hom}(L, \mathbb{Z})$. The two definitions are equivalent, in fact L^* and $\mathrm{Hom}(L, \mathbb{Z})$ are isomorphic as abelian groups, and the map $\Psi\colon L^* \to \mathrm{Hom}(L, \mathbb{Z})$ defined by $x \mapsto (x^*\colon y \mapsto b_L(x, y))$ is an isomorphism. In order to see it, let $(e_1, ..., e_r)$ be a basis of L, then there exists a basis $(x_1, ..., x_r)$ of L^* such that $x_i \cdot e_j = \delta_{i,j}$; analogously, there is a basis $(y_1, ..., y_r)$ of $\mathrm{Hom}(L, \mathbb{Z})$ such that $y_i(e_j) = \delta_{i,j}$. Obviously $x_i^* = y_i$, and so it follows that Ψ is an isomorphism.

Given an integral lattice L, it is easy to see that L is a sublattice of the dual lattice L^*; nevertheless, the dual lattice L^* does not need to be integral, since there is no condition on $b_L(x, y)$ to be integral for any x, y inside $L^* - L$.

Lemma 1.1.7. *Let L be an integral lattice. Then L is a finite index sublattice of L^* and $|\det L| = [L^* : L]$.*

Proof. Well known result. For a proof we refer to [vL05, Lemma 2.1.13]. □

Remark 1.1.8. From Lemma 1.1.7 it follows that if L is a unimodular lattice, then L is equal to its dual lattice L^*.

Let L be an integral lattice, let $S \subset L$ be a sublattice and let $T = S^\perp$ be its orthogonal complement inside L. We can naturally embed $S \oplus T$ into L, by sending $(s, t) \in S \oplus T$ to $s + t \in L$.

Let x be an element of L. By Lemma 1.1.2, the lattice $S \oplus T$ has finite-index inside L; let m be the index $[L : S \oplus T]$. Then $mx \in S \oplus T$; write $mx = s + t$, for some $s \in S, t \in T$. Consider the element $s/m \in L_\mathbb{Q}$ and let y be an element of S. Since $t \in T = S^\perp$, one has that $y \cdot s = y \cdot (s + t)$. Then $y \cdot s = y \cdot (s + t) = y \cdot (mx) = m(y \cdot x)$, that is, $y \cdot s$ is divisible by m. It follows that $y \cdot (s/m)$ is an integer and so, by the generality of y, the element $s/m \in S_\mathbb{Q}$ is contained in S^*. The same argument holds to show that $t/m \in T^*$.

Then we define a map $L \to S^* \oplus T^*$ by sending $x \in L$ to the element $(s/m, t/m) \in S^* \oplus T^*$. The next lemma shows that this map is a finite-index embedding.

Lemma 1.1.9. *Let L be an integral lattice, and S a sublattice of L. Let $T = S^\perp$ be the orthogonal complement of S inside L. Then the maps* *defined before are finite-index embeddings.*

$$S \oplus T \hookrightarrow L \hookrightarrow S^* \oplus T^*$$

Proof. The first map is trivially an embedding and, by Lemma 1.1.2, $S \oplus T$ has the same rank as L, so the embedding is finite-index.

Also the second map is trivially injective.

The lattice L has finite index inside $S^* \oplus T^*$ since $S^* \oplus T^*$ has, by Lemma 1.1.7, the same rank as $S \oplus T$, that in turn has the same rank as L, as we have seen before. □

Let L be an even lattice with pairing b_L. We define the *discriminant group* of L to be the quotient

$$A_L := L^*/L.$$

The pairing b_L of L induces a map $q_L \colon A_L \to \mathbb{Q}/2\mathbb{Z}$, called *the discriminant quadratic form of* L, defined by $[x] \mapsto b_L(x,x) + 2\mathbb{Z}$. The discriminant group is a finite group, and the minimal number of generators is denoted by $\ell(A_L)$.

Lemma 1.1.10. *The map q_L is well defined and quadratic. The cardinality of A_L equals $|\det L|$.*

Proof. This is a standard result. For a proof see [vL05, Lemma 2.1.17]. □

Lemma 1.1.11. *Let L be an even lattice of rank r, and let A_L denote its discriminant group. Then $\ell(A_L) \leq r$.*

Proof. The group A_L is generated by the classes of the generators of L^*, and L^* has the same rank as L, namely r. □

Let L be a unimodular lattice, and $S \subset L$ a primitive sublattice of L; let T denote the orthogonal complement $S^\perp$ of S inside L. Recall that $\mathrm{Hom}(L,\mathbb{Z})$ and $\mathrm{Hom}(S,\mathbb{Z})$ are isomorphic to L^* and S^*, respectively (cf. Remark 1.1.6); since L is unimodular, then $L = L^*$ (cf. Remark 1.1.8). The restriction map $\mathrm{Hom}(L,\mathbb{Z}) \to \mathrm{Hom}(S,\mathbb{Z})$ induces a map $L \to A_S$.

$$L = L^* \xrightarrow{\cong} \mathrm{Hom}(L,\mathbb{Z}) \longrightarrow \mathrm{Hom}(S,\mathbb{Z}) \xrightarrow{\cong} S^* \longrightarrow S^*/S = A_S$$

The kernel of this map is $S \oplus T$, and so it induces an isomorphism

$$\psi_S \colon L/(S \oplus T) \to A_S.$$

The analogous construction for L and T induces an isomorphism

$$\psi_T \colon L/(S \oplus T) \to A_T.$$

Let $\delta_S \colon A_S \to A_T$ be the isomorphism given by the composition $\psi_T \circ \psi_S^{-1}$.

Proposition 1.1.12. *Let L, S, T and δ_S be defined as before. Then the following diagram commutes.*

$$\begin{CD} A_S @>{\cong}>{\delta_S}> A_T \\ @V{q_S}VV @VV{q_{S^T}}V \\ \mathbb{Q}/2\mathbb{Z} @>>{[-1]}> \mathbb{Q}/2\mathbb{Z} \end{CD}$$

Proof. [Nik79, Proposition 1.6.1] or [BHPVdV04, Lemma I.2.5]. □

Let L be a lattice. With $\mathcal{O}(L)$ we denote the group of isometries of L.

Let S be a sublattice of L. With $\mathcal{O}(L)_S$ we denote the group of isometries of L sending S to itself.

An isometry σ of a lattice L extends by linearity to an isometry of L^*. It therefore induces an automorphism $\bar{\sigma}$ of the discriminant group A_L. In this way we define the map $\rho_L \colon \mathcal{O}(L) \to \mathrm{Aut}(A_L)$.

Corollary 1.1.13. *Let L be an even unimodular lattice and S a primitive sublattice of L. Let $T = S^\perp$ denote the orthogonal complement of S inside L. There is an isomorphism ϱ_S between* $\mathrm{Aut}(A_S)$ *and* $\mathrm{Aut}(A_T)$ *making the following diagram commute.*

$$\begin{array}{ccccc} & & \mathcal{O}(L)_S & & \\ & \swarrow^{\mathrm{res}_S} & & \searrow^{\mathrm{res}_T} & \\ \mathcal{O}(S) & & & & \mathcal{O}(T) \\ \downarrow{\scriptstyle \rho_S} & & & & \downarrow{\scriptstyle \rho_T} \\ \mathrm{Aut}(A_S) & & \xrightarrow[\varrho_S]{\cong} & & \mathrm{Aut}(A_T) \end{array}$$

Proof. Let $\delta_S\colon A_S \to A_T$ be the isomorphism as in Proposition 1.1.12. Define $\varrho_S\colon \operatorname{Aut}(A_S) \to \operatorname{Aut}(A_T)$ by

$$\phi \mapsto \delta_S \circ \phi \circ \delta_S^{-1}.$$

First notice that ϱ is bijective, since the map $\operatorname{Aut}(A_S) \to \operatorname{Aut}(A_T)$ defined by

$$\phi \mapsto \delta_S^{-1} \circ \phi \circ \delta_S$$

serves as its inverse.

The commutativity of the diagram follows from the fact that we use δ_S to identify A_S and A_T. See also [Huy15, Lemma 14.2.5]. □

Lemma 1.1.14. *Let L be a unimodular lattice and S a primitive sublattice of L and keep the notation as in Corollary 1.1.13.*

Let $\operatorname{res}_{S,T}\colon \mathcal{O}(L)_S \to \mathcal{O}(S) \times \mathcal{O}(T)$ be the map defined by

$$\alpha \mapsto (\alpha_{|S}, \alpha_{|T}).$$

Then the map $\operatorname{res}_{S,T}$ is well defined, injective, and its image is

$$\{(\beta, \gamma) \in \mathcal{O}(S) \times \mathcal{O}(T) \mid \varrho_S(\rho_S(\beta)) = \rho_T(\gamma)\}.$$

Proof. See [Huy15, Proposition 14.2.6] or [Nik79, Theorem 1.6.1, Corollary 1.5.2]. □

Proposition 1.1.15. *Let L be an even indefinite lattice of signature (m, n) and rank $m+n$, with discriminant lattice A_L. If $\ell(A_L) \leq m+n-2$, then any other lattice with the same rank, signature and discriminant group is isomorphic to L.*

Proof. See [Nik79, Corollary 1.13.3] or [HT15, Proposition 5]. □

Let L be an even lattice, $S \subseteq L$ a finite-index sublattice, and $\iota\colon S \hookrightarrow L$ the inclusion map.

Let $p \in \mathbb{Z}$ be a prime and consider the quotient group L/pL. If x is an element of L, we denote with $[x]_L = x + pL$ its class inside L/pL. The same construction and notation holds if we substitute L with S. When clear from the context, we will drop the subscripts L or S, and we will write simply $[x]$ for $[x]_L$ or $[x]_S$, respectively.

The inclusion map ι induces the homomorphism $\iota_p\colon S/pS \to L/pL$, defined by

$$\iota_p\colon [x]_S \mapsto [x]_L.$$

Remark 1.1.16. Notice that if p is a prime, then S/pS and L/pL are $\mathbb{F}_p$-vector spaces and the homomorphism ι_p is a homomorphism of $\mathbb{F}_p$-vector spaces.

We define S_p to be the kernel of ι_p.

Lemma 1.1.17. *The following equality holds:*

$$S_p = \frac{S \cap pL}{pS}.$$

Proof. The inclusion $\frac{S\cap pL}{pS} \subseteq S_p$ is trivial.

In order to see the other inclusion, let λ be an element of S such that $[\lambda] \in S_p$, that is, $\iota_p([\lambda]) \in pL$. From this it follows that $\lambda = p\lambda'$, for some $\lambda' \in L$. Then $\lambda \in S \cap pL$ and the statement follows. □

Lemma 1.1.18. *Let x, y, x', y' be elements of L such that $[x]_L = [x']_L$ and $[y]_L = [y']_L$. Then $b_L(x, y) \equiv b_L(x', y') \bmod p$.*

Proof. From the hypothesis it follows that there exist two elements $\lambda, \mu \in L$ such that $x' = x + p\lambda$ and $y' = x + p\lambda$. Then

$$\begin{aligned} b_L(x', y') &= b_L(x + p\lambda, y + p\mu) = \\ &= b_L(x, y) + pb_L(x, \mu) + pb_L(\lambda, y) + p^2 b_L(\lambda, \mu) \\ &\equiv b_L(x, y) \bmod p. \end{aligned}$$

□

Using the pairing b_L on L and Lemma 1.1.18, we can define symmetric, bilinear forms on L/pL and S/pS, denoted by

$$b_{L,p} \colon (L/pL)^2 \to \mathbb{Z}/p\mathbb{Z}$$

and

$$b_{S,p} \colon (S/pS)^2 \to \mathbb{Z}/p\mathbb{Z}$$

respectively, both defined by sending $([x], [y])$ to $b_L(x, y) \mod p$.

Lemma 1.1.19. *The following diagram commutes.*

$$\begin{array}{ccc} (S/pS)^2 & \xrightarrow{b_{S,p}} & \mathbb{Z}/p\mathbb{Z} \\ {\scriptstyle \iota_p^2}\downarrow & & \Vert \\ (L/pL)^2 & \xrightarrow[b_{L,p}]{} & \mathbb{Z}/p\mathbb{Z} \end{array}$$

Proof. Let x, y be two elements of S. Then

$$b_{L,p}(\iota_p([x]_S), \iota_p([y]_S)) = b_{L,p}([x]_L, [y]_L) = b_L(x, y) \bmod p.$$

By definition

$$b_{S,p}([x]_S, [y]_S) = b_L(x, y) \bmod p.$$

$\square$

Let $[x]_L$ be an element of L/pL, and define the homomorphism

$$[x]^* \colon S/pS \to \mathbb{Z}/p\mathbb{Z}$$

by sending $[y]_P \in S/pS$ to $b_{L,p}([x], [y])$. In this way we get the morphism

$$\phi_{L,p} \colon L/pL \to \mathrm{Hom}(S/pS, \mathbb{Z}/p\mathbb{Z}),$$

defined by sending $[x]_L$ to $[x]^*$. In the same way, we define the morphism

$$\phi_{S,p} \colon S/pS \to \mathrm{Hom}(S/pS, \mathbb{Z}/p\mathbb{Z}).$$

Let k_p denote the kernel of $\phi_{S,p}$.

Lemma 1.1.20. *The subspace k_p contains S_p and it is fixed by all the isometries of S.*

Proof. First we show $S_p \subseteq k_p$. Let $\mathfrak{x}$ be an element of S_p and fix a representative $x \in S$ of $\mathfrak{x}$, that is $\mathfrak{x} = [x]_S$. By Lemma 1.1.17, there is a $x' \in L$ such that $x = px'$. It follows that

$$[x]^*([y]_S) = [px']^*([y]_S) = b_{L,p}(px', y) = p\, b_{L,p}(x', y) \equiv 0 \bmod p,$$

for any $y \in S$. So $\phi_{S,p}([x]_S) = [x]^* = 0$ and hence $[x]_S \in k_p$.

In order to show that k_p is fixed by the isometries of S, let $[x]_S$ be an element of k_p and σ any isometry of S. Then we have that $[\sigma x]^*([y]_S) = b_L(\sigma x, y) = b_L(x, \sigma^{-1} y)$. Since $[x]_S \in k_p$ we have that $[x]^* = 0$, and so $b_L(x, \sigma^{-1} y) \equiv 0 \mod p$. It follows that, for any $y \in L$, $b_L(\sigma x, y) \equiv 0 \mod p$, and therefore $[\sigma x] \in k_p$. $\square$

Lemma 1.1.21. *The following diagram commutes.*

$$
\begin{array}{ccccccc}
0 & \longrightarrow & S_p & \hookrightarrow & S/pS & \xrightarrow{\iota_p} & L/pL \\
 & & \downarrow & & \| & & \downarrow \phi_{L,p} \\
0 & \longrightarrow & k_p & \hookrightarrow & S/pS & \xrightarrow[\phi_{S,p}]{} & \mathrm{Hom}(S/pS, \mathbb{Z}/p\mathbb{Z})
\end{array}
$$

Proof. The left square is trivially commutative, since all the maps involved are inclusions.

The right square is also commutative since ι_p preserves the pairing on L/pL (cf. Lemma 1.1.19). □

Remark 1.1.22. Let S be a lattice of rank r and fix a basis $(e_1, ..., e_r)$. Let M be the Gram matrix of S associated to the fixed basis. Then we have that S is isometric to the lattice $(\mathbb{Z}^r, M)$; the isometry is given by sending e_i to the i-th element of the canonical basis of $\mathbb{Z}^r$.

Using this notation, k_p is the subspace of $S/pS \cong (\mathbb{Z}/p\mathbb{Z})^r$ given by the classes of the elements $\underline{x} \in \mathbb{Z}^r$ such that $\underline{x} \cdot M \equiv \underline{0} \mod p$.

Keeping the notation introduced before, let $x \in S$ be such that $[x]_S \in k_p$ and $x^2 \equiv 0 \mod 2p^2$. Let y be another element of S such that $[x]_S = [y]_S$, that is, there is an element $z \in L$ such that $y = x + pz$. It follows that $y^2 = (x + pz)^2 = x^2 + 2px \cdot z + p^2 z^2$. By hypothesis $x^2 \equiv 0 \mod 2p^2$; since $[x]_S \in k_p$, the product $x \cdot z$ is divisible by p, and so $2px \cdot z \equiv 0 \mod 2p^2$; since L, and therefore S, is an even lattice, z^2 is even, and so $p^2 z^2 \equiv 0 \mod 2p^2$; hence $y^2 \equiv 0 \mod 2p^2$. We can then define $k'_p \subset S/pS$ to be the following subset of k_p:

$$k'_p := \{[x]_S \in k_p \mid x^2 \equiv 0 \mod 2p^2\}.$$

Lemma 1.1.23. *The subset $k'_p \subset k_p$ contains S_p and it is invariant under all the isometries of S.*

Proof. First we show that S_p is contained in k'_p. Let $\mathfrak{x}$ be an element of S_p. By Lemma 1.1.17, there is an element $y \in L$ such that $\mathfrak{x} = [py]$. It follows that $\mathfrak{x} = [py + px']$, for any $x' \in S$. Then $\mathfrak{x}^2 = p^2 y^2 + 2p^2 y \cdot x' + p^2 x'^2$. Recall that L is an even lattice, and so $y \cdot x' \in \mathbb{Z}$ and $y^2, x'^2 \in 2\mathbb{Z}$. Then, $\mathfrak{x}^2 \equiv 0 \mod 2p^2$ and thus we have proved $S_p \subseteq k'_p$.

In order to show that k_p' is invariant under the isometries of S consider a class $[x] \in k_p'$ and let σ be an isometry of S. By Lemma 1.1.20 $\sigma[x] \in k_p$. Since σ is an isometry, $(\sigma x)^2 = x^2 \equiv 0 \bmod 2p^2$, and so also σx is an element of k_p'. $\square$

Corollary 1.1.24. *The equality $S = L$ holds if and only if ι_p is injective for every prime p.*

Proof. We only need to prove that if ι_p is injective for every prime p then $S = L$, as the other implication is trivial.

Assume then that ι_p is injective for every prime p. Let λ be an element of L. Since S has finite index inside L, there is a minimal $m \in \mathbb{Z}_{>0}$ such that $m\lambda \in S$.

If $m = 1$ we are done. So assume $m > 1$. Then m can either be a prime or not a prime.

If m is a prime, say q, let $S_q = \frac{S \cap qL}{qS}$ be the kernel of the map $\iota_q \colon S/qS \to L/qL$ (cf. Lemma 1.1.17). Then it follows that $[q\lambda]$ is inside S_q. By assumption, $S_q = \{0\}$. This means that $[q\lambda] = 0$ or, equivalently, that $q\lambda \in qS$. Since S is a torsion-free group (it is a lattice), we can conclude that $\lambda \in S$. But then, by the minimality of m, we get $m = 1$, contradicting the assumption of m to be greater than 1.

If m is not a prime, let p be a prime divisor of m and write $m = pm'$, for some $m' \in \mathbb{Z}$. Using the same argument as before, we show that $m'\lambda$ is in S. In this way we got a $m' < m$ such that $m'\lambda \in S$, contradicting the minimality of m.

This shows that $m = 1$ and so, by generality of λ, we have proved that $S = L$. $\square$

Let L be an integral lattice, and let $S \subset L$ be a finite-index sublattice of L. Let p be a prime, and let e_p denote the dimension of $S_p = \frac{S \cap pL}{pS}$ as $\mathbb{F}_p$-vector space. Let $([y_1], ..., [y_{e_p}])$ be an $\mathbb{F}_p$-basis of S_p. Then there exist $x_1, ..., x_{e_p} \in L - S$ such that $[y_i] = [px_i]$, for $i = 1, ..., e_p$. Let S' be the sublattice of L generated by $S \cup \{x_1, ..., x_{e_p}\}$. Obviously S is a finite-index sublattice of S' and, by construction, we have that pS' is contained in S.

Lemma 1.1.25. *Let L, S, S', e_p and $x_1, ..., x_{e_p} \in L - S$ be defined as before. Then S'/S is an $\mathbb{F}_p$ vector space of dimension e_p.*

Proof. Since pS' is contained in S, the quotient S'/S is an $\mathbb{F}_p$-vector space. We claim that the classes $[x_1], ..., [x_{e_p}]$ form an $\mathbb{F}_p$-basis for S'/S. Clearly, they generate it, since they are the only generators of S' not contained in S. To show that they are linearly independent, assume by contradiction that there are $a_1, ..., a_{e_p} \in \mathbb{F}_p$ such that $a_1[x_1] + ... + a_{e_p}[x_{e_p}] = 0$. This means that if we lift the classes $a_1, ..., a_{e_p} \in \mathbb{F}_p$ to the integers $b_1, ..., b_{e_p} \in \mathbb{Z}$, then $b_1 x_1 + ... + b_{e_p} x_{e_p}$ is inside S; so, multiplying by p, it follows that $b_1 y_1 + ... + b_{e_p} y_{e_p} \in pS$. This last statement implies that $a_1[y_1] + ... + a_{e_p}[y_{e_p}] = 0 \in S/pS$, contradicting the hypothesis on $([y_1], ..., [y_{e_p}])$ to be an $\mathbb{F}_p$-basis of S_p. Then $([x_1], ..., [x_{e_p}])$ is an $\mathbb{F}_p$-basis for S'/S and the statement follows. □

Corollary 1.1.26. *S_p and S'/S are isomorphic as $\mathbb{F}_p$-vector spaces.*

Proof. By Lemma 1.1.25, S'/S is an $\mathbb{F}_p$-vector space of dimension e_p; the $\mathbb{F}_p$-vector space S_p has dimension e_p by definition. So S_p and S'/S are two $\mathbb{F}_p$-vector spaces of the same dimension, hence they are isomorphic. □

Remark 1.1.27. A more direct way to show that S_p and S'/S are isomorphic is given by considering the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & S_p & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S & \xrightarrow{[p]} & S & \longrightarrow & S/pS & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S' & \xrightarrow{[p]} & S' & \longrightarrow & S'/pS' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & S'/S & \xrightarrow{[p]} & S'/S & & & &
\end{array}
$$

Then, applying the snake lemma, we have the exact sequence

$$0 \longrightarrow S_p \longrightarrow S'/S \xrightarrow{[p]} S'/S.$$

Since $pS' \subseteq S$, the map $[p]$ given by the multiplication by p is the zero map. The map $S_p \to S'/S$ is then an isomorphism.

Proposition 1.1.28. *Let p be a prime, and let L, S, S' and e_p be defined as before. Then* $\det S' = p^{-2e_p} \det S$.

Proof. Since S is a finite-index sublattice of S', it follows that the index $[S' : S]$ equals the cardinality of S'/S; by Lemma 1.1.25, the $\mathbb{F}_p$-vector space S'/S has dimension e_p, and so

$$[S' : S] = \#(S'/S) = p^{e_p}.$$

Then, by Lemma 1.1.5, we have that $\det S = p^{2e_p} \det S'$ or, equivalently, $\det S' = p^{-2e_p} \det S$. □

Remark 1.1.29. Since L is an integral lattice, so are S and S', and therefore $\det S$ and $\det S'$ are both integers. It follows that, for any prime p, if p^m is the maximal power of p dividing $\det S$, then $2e_p \leq m$.

As immediate consequence, we have that the map ι_p is injective for all the primes p whose square does not divide $\det S$.

Remark 1.1.30 (Some classic lattices). Here we introduce the notation for some notable lattices. These lattices will be useful later.

With U we denote the lattice of rank 2 and Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let n be a positive integer.

With A_n we denote the lattice associated to the root system A_n. It is an even, positive definite lattice of rank n and determinant $n+1$. See [CS99, Section 4.6.1] for more information.

With E_8 we denote the lattice associated to the root system E_8. It is an even, positive definite lattice of rank 8 and determinant 1. See [CS99, Section 4.8.1] for more information.

With Λ_{K3} we denote the lattice given by

$$\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

One can immediately notice that Λ_{K3} is an even unimodular lattice of rank 22, determinant -1, and signature $(3, 19)$.

1.2 Geometric background

In this section we give some general definitions and results in algebraic geometry. We focus on the study of surfaces. After giving the definition of surface, we present some well-known results about the Picard group of a surface, double covers, K3 surfaces, and del Pezzo surfaces.

Let k be a field. A *variety* over k is a separated, geometrically reduced scheme X that is of finite type over $\operatorname{Spec} k$.

We say that a variety X is *smooth* if the morphism $X \to \operatorname{Spec} k$ is smooth.

A variety has *pure dimension d* if all its irreducible components have dimension d.
A *curve* is a variety of pure dimension 1.
A *surface* is a variety of pure dimension 2.
A *three-fold* is a variety of pure dimension 3.

Let X be a variety over a field k, and let K be any extension of k. Then we denote by X_K the base-change of X to K. Let $\overline{k}$ be a fixed algebraic closure of k. Then we denote by $\overline{X} := X_{\overline{k}}$ the base-change of X to $\overline{k}$.

1.2.1 The Picard lattice

In this subsection we introduce the notion of *Picard lattice* of a surface. In doing so we basically follow [Har77, Section II.6] and [vL05, Section 2.2].

Let X be a scheme. We define the *Picard group* of X, denoted by $\operatorname{Pic} X$, to be the group of isomorphism classes of invertible sheaves of X (see [Har77, p.143]).

Remark 1.2.1. Equivalently, one can define the Picard group of X as the group $H^1(X, \mathcal{O}^*)$. In fact [Har77, Exercise III.4.5] shows that $\operatorname{Pic} X \cong H^1(X, \mathcal{O}^*)$.

Let X be an irreducible variety over a field k. We define the *Cartier divisor group*, denoted by $\operatorname{CaDiv} X$ to be the group $H^0(X, \mathcal{K}^*/\mathcal{O}^*)$, where $\mathcal{K}$ is the sheaf of total quotient rings of $\mathcal{O}$. A Cartier divisor is *principal* if it is in the image $\operatorname{PCaDiv} X$ of the natural map $H^0(X, \mathcal{K}^*) \to H^0(X, \mathcal{K}^*/\mathcal{O}^*)$. We define the *Cartier divisor class group*, denoted by $\operatorname{CaCl} X$, to be the quotient $\operatorname{CaDiv} X / \operatorname{PCaDiv} X$. For more details about these definitions, see [Har77, p.141], or also [HS00, A.2.2].

Assume X to be smooth, and let $K(X)$ denote the function field of X. We define the *(Weil) divisor group*, denoted by $\operatorname{Div} X$, to be free abelian group generated by all the prime Weil divisors of X. The *group of principal divisors* of X, denoted by $\operatorname{PDiv} X$, is the image of the map $K(X)^* \to \operatorname{Div} X$, defined by sending a function f to the divisor $(f) = \sum_Y v_Y(f)Y$, where the sum is over all the prime Weil divisors Y and $v_Y(f)$ is the valuation of f in the discrete valuation ring associated to the generic point of Y. We define the *(Weil) divisor class group*, denoted by $\operatorname{Cl} X$, to be the quotient $\operatorname{Div} X / \operatorname{PDiv} X$. For more details about these definitions, see [Har77, p.130], or also [HS00, A.2.1].

Proposition 1.2.2. *Let X be an irreducible, smooth variety over a field k. Then there are natural isomorphisms*

$$\operatorname{Div} X \cong \operatorname{CaDiv} X,$$

and

$$\operatorname{Pic} X \cong \operatorname{CaCl} X \cong \operatorname{Cl} X.$$

Proof. See [Har77, Proposition II.6.11] for the proof of $\operatorname{Div} X \cong \operatorname{CaDiv} X$.
See [Har77, Proposition II.6.15] for the proof of $\operatorname{Pic} X \cong \operatorname{CaCl} X$.
See [Har77, Corollary II.6.16] for the proof of $\operatorname{Pic} X \cong \operatorname{Cl} X$. □

Remark 1.2.3. If X is a smooth, irreducible variety, then we can identify Weil divisors and Cartier divisors. We will then simply talk about divisors, without specifying 'Weil' or 'Cartier'. In general, if we leave out this specification, a divisor is intended to be a Weil divisor.

From now on, let X be a projective, smooth, geometrically irreducible surface over a field k. Fix an algebraic closure $\overline{k}$ of k and let $\overline{X} = X_{\overline{k}}$ denote the base-change of X to $\overline{k}$.

Theorem 1.2.4. *There is a unique pairing* $\operatorname{Div} \overline{X} \times \operatorname{Div} \overline{X} \to \mathbb{Z}$, *denoted by $C \cdot D$ for any two divisors C, D, such that*

1. *if C and D are nonsingular curves meeting transversally, then $C \cdot D = \#(C \cap D)$, the number of points of $C \cap D$;*
2. *$C \cdot D = D \cdot C$;*
3. *$(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$;*
4. *if D is a principal divisor then $D \cdot C = 0$, for any divisor C.*

Proof. [Har77, Theorem V.1.1]. □

We call this unique pairing on $\operatorname{Div}\overline{X}$ the *intersection pairing* of X. Let k_1 be an extension of k such that $k \subseteq k_1 \subseteq \overline{k}$. Then the intersection pairing of X restricts to a pairing on $\operatorname{Div} X_{k_1}$; in particular, it restricts to a pairing on $\operatorname{Div} X$.

Remark 1.2.5. From Theorem 1.2.4.(4), it immediately follows that the intersection pairing of X induces a pairing on $\operatorname{Cl} X \cong \operatorname{Pic} X$.

Let $D, E \in \operatorname{Div} X$ be two divisors of X. We say that D and E are *linearly equivalent*, denoted by $D \sim_{\text{lin}} E$, if and only if they have the same class inside $\operatorname{Cl} X \cong \operatorname{Pic} X$.

Remark 1.2.6. Trivially, $\operatorname{Div} X/\sim_{\text{lin}} = \operatorname{Cl} X$.

Let T be a non-singular curve. We define an *algebraic family of effective divisors on X parametrised by T* to be an effective Cartier divisor $\mathcal{D}$ on $X \times T$, flat over T (cf. [Har77, Example III.9.8.5]).

Let D and E be two divisors of X. We say that D and E are *prealgebraically equivalent* if and only if there are two non singular curves T_1, T_2 defined over $\overline{k}$, two algebraic families $\mathcal{D}_1$ and $\mathcal{D}_2$ of effective divisors on $\overline{X}$ parametrised by T_1 and T_2 respectively, two closed fibers D_1, E_1 of $\mathcal{D}_1$ and two closed fibers D_2, E_2 of $\mathcal{D}_2$, such that $D = D_1 - D_2,\ E = E_1 - E_2$. We say that D and E are *algebraically equivalent*, denoted by $D \sim_{\text{alg}} E$, if there is a chain of divisors $D = C_0, C_1, ..., C_n = E$ in $\operatorname{Div}\overline{X}$ such that C_i and C_{i+1} are prealgebraically equivalent, for $i = 0, ..., n-1$. Let $\operatorname{Div}^0_{\text{alg}} X$ be the group of divisors of X that are algebraically equivalent to 0, and let $\operatorname{Pic}^0_{\text{alg}} X$ be its image inside $\operatorname{Pic} X$. We define the *Néron–Severi group* of X, denoted by $\operatorname{NS} X$, to be the quotient $\operatorname{Div} X/\operatorname{Div}^0_{\text{alg}} X$. For more details about these definitions see [Har77, Exercise V.1.7].

Theorem 1.2.7 (Néron–Severi)**.** *Let X be defined as before. Then $\operatorname{NS} X$ is a finitely generated abelian group.*

Proof. See [LN59] or [Nér52] for a proof with k arbitrary. See [Har77, Appendix B.5] for a proof with $k = \mathbb{C}$. □

Remark 1.2.8. By Theorem 1.2.7, we have that $\operatorname{NS} X \cong \mathbb{Z}^\rho \oplus (\operatorname{NS} X)_{\text{tors}}$, for some integer $\rho \in \mathbb{Z}_{\geq 0}$. We define this $\rho = \rho(X)$ to be the *Picard number* of X. Note that $\rho = \dim_{\mathbb{Q}} \operatorname{NS}(X) \otimes \mathbb{Q}$.

We say that D and E are *numerically equivalent*, using the notation $D \sim_{\text{num}} E$, if and only if $D \cdot C = E \cdot C$ for every divisor $C \in \operatorname{Div} X$. Let $\operatorname{Div}^0_{\text{num}} X$ be the group of divisors of X that are numerically equivalent to 0, and let $\operatorname{Pic}^0_{\text{num}} X$ be its image inside $\operatorname{Pic} X$. We define $\operatorname{Num} X$ to be the quotient $\operatorname{Div} X / \operatorname{Div}^0_{\text{num}}$.

Remark 1.2.9. Let D and E two divisors of X. From Theorem 1.2.4.(4) it immediately follows that if D and E are linearly equivalent, they are numerically equivalent too (cf. Proposition 1.2.11).

Proposition 1.2.10. *The group* $\operatorname{Num} X$ *is a torsion free abelian group.*

Proof. The group $\operatorname{Num} X$ is abelian since it is a quotient of $\operatorname{Div} X$, which is abelian by definition.

In order to see that $\operatorname{Num} X$ is torsion free let D be a divisor of X and let $[D]_{\text{num}}$ its class inside $\operatorname{Num} X$. Assume $m[D]_{\text{num}} = [mD]_{\text{num}} = 0$. This means that $(mD) \cdot C = 0$, for every divisor $C \in \operatorname{Div} X$. It follows that, for every divisor $C \in \operatorname{Div} X$

$$0 = (mD) \cdot C = m(D \cdot C),$$

and so either $m = 0$, or $(D \cdot C) = 0$ for every $C \in \operatorname{Div} X$, i.e., $[D]_{\text{num}} = 0$. Hence $\operatorname{Num} X$ is torsion free. □

Proposition 1.2.11. *Let* D, E *be two divisors of* X*. If* $D \sim_{\text{lin}} E$*, then* $D \sim_{\text{alg}} E$*. If* $D \sim_{\text{alg}} E$*, then* $D \sim_{\text{num}} E$*.*

Proof. See [Har77, Exercise V.1.7.(b) and (c)]. □

Remark 1.2.12. The previous proposition tells us that there are two natural surjections:

$$\operatorname{Pic} X \to \operatorname{NS} X \to \operatorname{Num} X.$$

Remark 1.2.13. From Proposition 1.2.11, we trivially get that:

$$\operatorname{Pic}^0_{\text{alg}} X \subseteq \operatorname{Pic}^0_{\text{num}} X,$$

and that

$$\operatorname{Pic} X / \operatorname{Pic}^0_{\text{alg}} X \cong \operatorname{NS} X,$$
$$\operatorname{Pic} X / \operatorname{Pic}^0_{\text{num}} X \cong \operatorname{Num} X.$$

Proposition 1.2.14. *The natural map* $\operatorname{Num} X \to \operatorname{NS} X/(\operatorname{NS} X)_{\text{tors}}$ *is an isomorphism.*

Proof. It follows from [vL05, Proposition 2.2.17] and Remark 1.2.13. □

Remark 1.2.15. From Proposition 1.2.14, it follows that $\operatorname{Num} X$ is a free $\mathbb{Z}$-module of rank $\rho(X)$; also note that the intersection pairing of X naturally induces a pairing on $\operatorname{Num} X$. Then $\operatorname{Num} X$, endowed with the pairing induced by the intersection pairing, is a lattice of rank $\rho(X)$.

Also, using the surjection $\operatorname{NS} X \to \operatorname{Num} X$, the pairing on $\operatorname{Num} X$ induces a pairing on $\operatorname{NS} X$.

We can summarize the previous definitions and results with the following commutative diagrams with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{PDiv} X & \hookrightarrow & \operatorname{Div} X & \longrightarrow & \operatorname{Pic} X & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Div}^0_{\text{alg}} X & \hookrightarrow & \operatorname{Div} X & \longrightarrow & \operatorname{NS} X & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Div}^0_{\text{num}} X & \hookrightarrow & \operatorname{Div} X & \longrightarrow & \operatorname{Num} X & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Pic}^0_{\text{alg}} X & \hookrightarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{NS} X & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Pic}^0_{\text{num}} X & \hookrightarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{Num} X & \longrightarrow & 0
\end{array}$$

Remark 1.2.16. If the adjective *'geometric'* precedes any of the operators of this subsection introduced so far, then we mean the operator acting on $\overline{X}$ instead of X. For example, the *geometric Picard group* of X is the Picard group of $\overline{X}$, that is, $\operatorname{Pic} \overline{X}$.

Assume k is perfect and let $G_k := \operatorname{Gal}(\overline{k}/k)$ be the absolute Galois group of k, and fix an embedding of $Xbar$ inside a projective space over $\overline{k}$; then G_k acts on the set of prime divisors of $\overline{X}$, by acting on the coefficients of the equations defining them. This induces an action of

G_k on $\operatorname{Div} \overline{X}$ and, since G_k sends principal divisors to principal divisors, it also induces an action of G_k on $\operatorname{Pic} \overline{X}$.

Let $k_1 \subset \overline{k}$ be an algebraic extension of k. Let D be an element of $\operatorname{Div} \overline{X}$. We say that k_1 is *the field of definition of* D if $\operatorname{Gal}(\overline{k}/k_1)$ is the stabilizer of D inside G_k; we say that D *can be defined* over k_1 if $\operatorname{Gal}(\overline{k}/k_1)$ is contained in the stabilizer of D inside G_k.

Analogously, if $[D]$ is an element of $\operatorname{Pic} \overline{X}$, we say that k_1 is *the field of definition of* $[D]$ if $\operatorname{Gal}(\overline{k}/k_1)$ is the stabilizer of $[D]$ inside G_k; we say that $[D]$ *can be defined* over k_1 if $\operatorname{Gal}(\overline{k}/k_1)$ is contained in the stabilizer of $[D]$ inside G_k.

Remark 1.2.17. Let $k_1 \subset \overline{k}$ an algebraic extension of k. Let D be an element of $\operatorname{Div} \overline{X}$ and let $[D]$ denote its class inside $\operatorname{Pic} \overline{X}$. The fact k_1 is the field of definition of $[D]$ does not imply that k_1 is the field of definition of D: there might be an element $\sigma \in \operatorname{Gal}(\overline{k}/k)$ sending D to $D' = \sigma D$, such that $D' \neq D$ but $[D] = [D']$. For the same reason, the fact that $[D]$ can be defined over k_1 does not imply that D can be defined over k_1.

Let X be a surface over $k = \mathbb{C}$. Then we can consider the complex analytic space X_h associated to X. The topological space of X_h has $X(\mathbb{C})$ as underlying set. Let $\mathcal{O}_{X_h}$ denote structure sheaf of X_h. The exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X_h} \to \mathcal{O}^*_{X_h} \to 0$$

of sheaves induces an exact sequence of (cohomology) groups

$$0 \to H^1(X_h, \mathbb{Z}) \to H^1(X_h, \mathcal{O}_{X_h}) \to H^1(X_h, \mathcal{O}^*_{X_h}) \to H^2(X_h, \mathbb{Z}) \to \dots .$$

Serre, in [Ser56], showed that $H^i(X_h, \mathcal{O}_{X_h}) \cong H^i(X, \mathcal{O}_X)$ for every i. Since $\operatorname{Pic} X \cong H^1(X, \mathcal{O}^*_X)$ (cf. Remark 1.2.1), we have the following exact sequence of groups.

$$0 \to H^1(X_h, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \operatorname{Pic} X \to H^2(X_h, \mathbb{Z}) \to \dots . \quad (1.1)$$

Proposition 1.2.18. *The Néron–Severi group* $\operatorname{NS} X$ *is isomorphic to a subgroup of* $H^2(X_h, \mathbb{Z})$ *and the second Betti number* $b_2 = \dim H^2(X_h, \mathbb{C})$ *is an upper bound for the Picard number of* X*.*

Proof. The image of $H^1(X, \mathcal{O}_X)$ inside $\operatorname{Pic} X$ is exactly $\operatorname{Pic}^0_{\mathrm{alg}} X$ (see [Har77, Appendix B, p. 447]). Recalling that $\operatorname{NS} X \equiv \operatorname{Pic} X / \operatorname{Pic}^0_{\mathrm{alg}} X$ (cf. Remark 1.2.13), the statement immediately follows from the exact sequence (1.1). □

Remark 1.2.19. The pairing of $\operatorname{NS} X$ induced by the intersection pairing of X (cf. Remark 1.2.15) corresponds to the cup-product of $H^2(X_h, \mathbb{Z})$.

1.2.2 Weighted projective spaces

In the next sections, we will use the notion of *weighted projective space*. In introducing it we follow [Dol82].

Let $Q = (q_0, ..., q_r)$ be a $r+1$-tuple of positive integers. Let k be any field and let $S(Q)$ be the polynomial algebra $k[T_0, ..., T_r]$ over the field k graded by the conditions

$$\deg T_i = q_i,$$

for $i = 0, ..., r$. We define the *weighted projective space of type* Q, or *weighted projective space with weights* Q, the projective scheme given by $\operatorname{Proj}(S(Q))$, denoted by $\mathbb{P}_k(Q)$.

If $k = \mathbb{Q}$, we might drop the subscript and write $\mathbb{P}(Q)$ for $\mathbb{P}_{\mathbb{Q}}(Q)$.

Example 1.2.20. If $Q = (\underbrace{1, ..., 1}_{r+1})$, then the weighted projective space with weights Q is simply the projective space $\mathbb{P}^r$.

Let $f(T_0, ..., T_1)$ be a homogeneous polynomial of weighted degree d. Then the equation $f(T_0, ..., T_1) = 0$ defines an *hypersurface* of degree d in $\mathbb{P}_k(Q)$. We say that an hypersurface of $\mathbb{P}_k(Q)$ is an *hyperplane* if it has degree $d = 1$.

Example 1.2.21. The equation $T_i = 0$ defines an hyperplane if and only if $q_i = 1$.

Remark 1.2.22. For more theory and results about weighted projective spaces we refer to [Dol82] and [Kol96, V.1.3].

1.2.3 Double covers

In this subsection we introduce the notion of *double cover* of a surface, focusing on double covers of the projective plane. Given a double cover

X of the projective plane, the main goal of the subsection is to give a characterization of the plane curves whose pull-back on X splits into two irreducible components. A large deal of what is described in this subsection is a part of a joint work with Ronald van Luijk, and can be found in [FvL15, Section 5].

Let k be a field with characteristic different from 2, and fix an algebraic closure $\overline{k}$ of k.

Let X, Y be two smooth, projective surfaces defined over k. We say that X is a *double cover* of Y if there is a morphism $f\colon X \to Y$ that is surjective, finite and of degree 2 (see [Har77, Section II.3] for these definitions).

Let $\pi\colon X \to Y$ be a double cover of Y.

By definition of the double cover, the pre-image inside $X(\overline{k})$ of a point of $Y(\overline{k})$ has at most 2 elements. We define the *branch locus* of π, denoted by $B \subset Y(\overline{k})$, to be the subset of $Y(\overline{k})$ defined by

$$\{x \in Y(\overline{k}) \mid \#\pi^{-1}(x) = 1\}.$$

Proposition 1.2.23. *The branch locus of π is a divisor of Y.*

Proof. It follows from [Zar58]. □

We define the *ramification locus* of π, denoted by $R \subset X$, to be the preimage $\pi^{-1}(B)$ on X of the branch locus B.

The double cover $\pi\colon X \to Y$ induces the involution ι_X on X, defined by sending each point $P \in X$ to the unique other point of the fiber $\pi^{-1}(\pi(P))$, unless $P \in R$; if $P \in R$, then ι_X fixes P.

The following definitions are given as in [FvL15, Section 5.1]. Given a curve C over k, the normalisation map $\vartheta\colon \tilde{C} \to C$ is unique up to isomorphism; the curve $\tilde{C}$ is regular and both $\tilde{C}$ and ϑ are called the normalisation of C. For more details, see [Mum99, Theorem III.8.3] for the case that C is irreducible; for the general case, take the disjoint unions of the normalisations of the irreducible components. If P is a singular point of C, we say that P is an *ordinary singular point* if, when we consider the blow up of C at P, all the points above P are smooth.

Recall that the geometric genus $g(C)$ of a geometrically integral curve C over k is defined to be the geometric genus of the unique regular projective geometrically integral model of C; for the definition of geometric genus, see [Har77, p. 181]. If C is itself projective, then this model is the normalisation $\tilde{C}$ of C. Note that we have $g(C_{\overline{k}}) \leq g(C)$ with equality if and only if $\tilde{C}$ is smooth (see [Tat52]). In particular, we have $g(C) = 0$ if and only if $\tilde{C}$ is smooth and C is geometrically rational.

Let $C \subset \mathbb{P}^2_k$ be a curve over k and $S \in C$ a closed point of C with local ring $\mathcal{O}_{S,C}$; let $V \subset \mathbb{P}^2$ be an open neighbourhood of S, and let $C' \subset \mathbb{P}^2_k$ be a curve that in V is given by $h = 0$ for some $h \in \mathcal{O}_{\mathbb{P}^2}(V)$; assume that S does not lie on a common component of C and C'; then the intersection multiplicity $\mu_S(C, C')$ of C and C' at S is the length of the $\mathcal{O}_{S,C}$-module $\mathcal{O}_{S,C}/(h)$. If S is a smooth point of C, then the local ring $\mathcal{O}_{S,C}$ is a discrete valuation ring, say with valuation v_S, and $\mu_S(C, C')$ equals $v_S(h)$.

We extend the notion of intersection multiplicity, replacing the point S on the curve C by a *branch* of C, that is, a point of the normalisation of C.

Let $C \subset \mathbb{P}^2$ be a curve and let $\vartheta \colon \tilde{C} \to C$ be the normalisation of C. Let $T \in \tilde{C}$ be a closed point with local ring $\mathcal{O}_{T,\tilde{C}}$. Let $C' \subset \mathbb{P}^2_k$ be a curve that is given in an open neighbourhood $V \subset \mathbb{P}^2$ of $\vartheta(T)$ by $h = 0$ for some $h \in \mathcal{O}_{\mathbb{P}^2_k}(V)$. If the curves C and C' have no irreducible components in common, then the intersection multiplicity $\mu_T(\tilde{C}, C')$ of $\tilde{C}$ and C' at T is defined to be the length of the $\mathcal{O}_{T,\tilde{C}}$-module $\mathcal{O}_{T,\tilde{C}}/(\vartheta^* h)$.

With the same notation as above, the quantity $\mu_T(\tilde{C}, C')$ is the same as $\mathrm{ord}_T(h)$ as defined in [Ful98, Section 1.2]. Since $\tilde{C}$ is regular, the local ring $\mathcal{O}_{T,\tilde{C}}$ is a discrete valuation ring, say with valuation v_T, and we have $\mu_T(\tilde{C}, C') = v_T(\vartheta^* h)$. If k is algebraically closed, then we have $\mu_T(\tilde{C}, C') = \dim_k \mathcal{O}_{T,\tilde{C}}/(\vartheta^* h)$.

Lemma 1.2.24. *Let $C, C' \subset \mathbb{P}^2_k$ be curves with no common irreducible components, and let $\vartheta \colon \tilde{C} \to C$ be the normalisation of C. Then for every $S \in C$ we have*

$$\mu_S(C, C') = \sum_{T \mapsto S} \mu_T(\tilde{C}, C') \cdot [k(T) : k(S)],$$

where the summation runs over all closed points $T \in \tilde{C}$ with $\vartheta(T) = S$ and where $[k(T) : k(S)]$ denotes the degree of the residue field extension.

Proof. This follows immediately from [Ful98, Example 1.2.3]. □

Let $C, C' \subset \mathbb{P}^2_k$ be curves over k that do not have any components in common. Let Γ denote either C or its normalisation $\tilde{C}$. Then we define the subset $b(\Gamma, C')$ of $\Gamma(\overline{k})$ as

$$b(\Gamma, C') = \{T \in \Gamma(\overline{k}) \ : \ \mu_T(\Gamma, C') \text{ is odd}\}.$$

From now on let X be a smooth, projective, irreducible surface over k, and let $\pi\colon X \to \mathbb{P}^2_k$ be a double cover of the projective plane.

Lemma 1.2.25. *Let D be a geometrically integral curve on X, let $C = \pi(D)$ be its image under π, and assume C is not equal to the branch locus B. Let $\tilde{D}_{\overline{k}}, \tilde{C}_{\overline{k}}$ be the normalisations of $D_{\overline{k}}$ and $C_{\overline{k}}$ respectively. The restriction of π to D induces a morphism $\tilde{\pi}\colon \tilde{D}_{\overline{k}} \to \tilde{C}_{\overline{k}}$. The branch locus of $\tilde{\pi}$ is exactly $b(\tilde{C}, B) \subset \tilde{C}(\overline{k})$.*

Proof. We present the proof as in [FvL15, Lemma 5.4]. Without loss of generality, we assume $k = \overline{k}$. Let ϑ denote the normalisation map $\tilde{C} \to C$. Let $T \in \tilde{C}(k)$ be a point. Since $\tilde{C}$ is regular, the local ring $\mathcal{O}_{T,\tilde{C}}$ is a discrete valuation ring, say with valuation v_T. As the characteristic of k is not equal to 2, there is an open neighbourhood $V \subset \mathbb{P}^2$ of $\vartheta(T)$ and an element $h \in \mathcal{O}_{\mathbb{P}^2}(V)$ such that the double cover $\pi^{-1}(V)$ of V is isomorphic to the subvariety of $V \times \mathbb{A}^1(u)$ given by $u^2 = h$. We denote the image of h in the local ring $\mathcal{O}_{T,\tilde{C}}$ and the function field $k(\tilde{C}) = k(C)$ by h as well. The extension $k(C) \subset k(D)$ of function fields is obtained by adjoining a square root $\eta \in k(D)$ of h to $k(C)$. Note that the degree of the restriction of π to D is 1 if and only if this extension is trivial, i.e., h is a square in $k(C)$. The intersection $B \cap V$ is given by $h = 0$, so we have $\mu_T(\tilde{C}, B) = v_T(h)$. Suppose $T' \in \tilde{D}(k)$ is a point with $\tilde{\pi}(T') = T$. Since the characteristic of k is not equal to 2, the extension $\mathcal{O}_{T,\tilde{C}} \subset \mathcal{O}_{T',\tilde{D}}$ of discrete valuation rings of $k(C)$ and $k(D) = k(C)(\eta)$, respectively, is ramified if and only if $v_T(h)$ is odd, that is, T is contained in $b(\tilde{C}, B)$, which proves the lemma. □

Proposition 1.2.26. *Let D be a geometrically integral projective curve on X, let $C = \pi(D)$ be its image under π, and assume $g(C) = 0$. Assume also that C is not equal to the branch locus B. Let $\tilde{C}$ denote the normalisation of C and set $n = \#b(\tilde{C}, B)$. The following statements hold.*

1. *If $n = 0$, then π restricts to a birational morphism $D \to C$ and $g(D) = 0$.*

2. *If $n > 0$, then π restricts to a double cover $D \to C$ and we have that $g(D) = g(D_{\overline{k}}) = \frac{1}{2}n - 1$.*

Proof. From $g(C) = 0$, we find that the normalisation $\tilde{C}$ is smooth. Since the characteristic of k is not 2 and for any finite separable field extension ℓ of k we have $g(D_\ell) = g(D)$ (see [Tat52, Corollary 2]), we may (and do) replace k, without loss of generality, by a quadratic extension ℓ for which $\tilde{C}(\ell) \neq \emptyset$. Then $\tilde{C}$ is isomorphic to $\mathbb{P}^1$. Let $\tilde{D}$ denote the normalisation of D. The morphism π induces a morphism $\tilde{\pi}\colon \tilde{D} \to \tilde{C} \cong \mathbb{P}^1$ of degree at most 2. We claim that $\tilde{D}$ is smooth. Indeed, if $\deg(\tilde{\pi}) = 1$, then this is clear. If $\deg(\tilde{\pi}) = 2$, then because the characteristic of k is not 2, the curve $\tilde{D}$ can be covered by open affine curves that are given by $y^2 = f(x)$ for some polynomial $f \in k[x]$; the regularity of $\tilde{D}$ implies that each polynomial f is separable, which implies that $\tilde{D}$ is smooth. This shows that $g(D) = g(D_{\overline{k}})$, so we may (and do) replace k, without loss of generality, by $\overline{k}$.

By hypotheses, C does not equal the branch locus B, so we may apply Lemma 1.2.25. The Riemann-Hurwitz formula then yields

$$2g(D) - 2 = 2g(\tilde{D}) - 2 = \deg(\tilde{\pi}) \cdot (2g(\tilde{C}) - 2) + n = n - 2\deg(\tilde{\pi}).$$

If $n = 0$, then we find $\deg(\tilde{\pi}) = 1$ and $g(D) = 0$. If $n > 0$, then $\tilde{\pi}$ is not unramified, so $\deg(\tilde{\pi}) = 2$ and we obtain $g(D) = \frac{1}{2}n - 1$. □

Corollary 1.2.27. *Let $C \subset \mathbb{P}^2$ be a geometrically integral projective curve with $g(C) = 0$ that is not equal to the branch locus B. Let $\tilde{C}$ denote its normalisation and set $n = \#b(\tilde{C}, B)$. The following statements hold.*

1. *If $n = 0$, then there exists a field extension ℓ of k of degree at most 2 such that the preimage $\pi^{-1}(C_\ell)$ consists of two irreducible components that are birationally equivalent with C_ℓ.*

2. *If $n > 0$, then the preimage $\pi^{-1}(C)$ is geometrically integral and has geometric genus $\frac{1}{2}n - 1$.*

Proof. Let $A = \pi^*(C)$ the pullback of the curve C on the surface X. Since C is geometrically integral and $C \neq B$, the curve A is geometrically reduced. The morphism $A \to C$ induced by π has degree 2, and so $\overline{A} = A \times_k \overline{k}$ consists of at most two components. Then there is an extension ℓ of k of degree at most 2 such that the components of A_ℓ are geometrically irreducible. Let ℓ be such an extension and let D be an irreducible component of A_ℓ.

Suppose $n = 0$. Applying Proposition 1.2.26 to D_ℓ and $C_\ell = \pi(D_\ell)$ shows that the morphism $D_\ell \to C_\ell$ induced by π is a birational map. Since $D_\ell \to C_\ell$ has degree 2, there is a unique second component of A_ℓ, which equals $\iota(D_\ell)$. This proves the first statement.

Suppose $n > 0$. By Proposition 1.2.26, the morphism $D_{\overline{k}} \to C_{\overline{k}}$ induced by π has degree 2, so $D_{\overline{k}}$ is the only component of $A_{\overline{k}}$ and therefore A is geometrically integral. Its genus follows from Proposition 1.2.26. □

Remark 1.2.28. In Corollary 1.2.27 the hypotheses do not involve only the curve C, but also its normalisation $\tilde{C}$. In particular, in (1) we assume that $\#b(\tilde{C}, B) = 0$. Even though the cardinalities of $b(\tilde{C}, B)$ and $b(C, B)$ are not always the same, in some cases the equality holds: for example, if C is smooth, then $C \cong \tilde{C}$, and so $\#b(\tilde{C}, B) = \#b(C, B)$; if C is singular, but all the singularities lie outside $C \cap B$, then again the equality holds. For more details about the relation between $\#b(\tilde{C}, B)$ and $\#b(C, B)$, see Propositions 1.2.29 and 1.2.31.

In the previous results, we described the preimage $\pi^{-1}(C) \subset X$ of a curve $C \subset \mathbb{P}^2$, by looking at the intersection points of the branch locus B with the normalisation $\tilde{C}$ of C. It is possible to give an analogous description of $\pi^{-1}(C)$ by looking at the intersection points of B and C itself, if we assume that all singular points of C that lie on B are ordinary singular points.

The following proposition describes the integer n used in Proposition 1.2.26 in terms of C directly.

Proposition 1.2.29. *Let $C, C' \subset \mathbb{P}^2$ be two projective plane curves with no components in common. Let $\tilde{C}$ be the normalisation of C. Assume also that C' is smooth and that all singular points of C that lie on C' are ordinary singularities of C. For each point $S \in C(\overline{k})$, let m_S denote* *the multiplicity of S on C. Then we have*

$$\#b(\tilde{C}, C') = \sum_{S \in C(\overline{k}) \cap C'(\overline{k})} c_S(C, C')$$

with

$$c_S(C, C') = \begin{cases} m_S & \text{if } m_S \equiv \mu_S(C, C') \pmod 2, \\ m_S - 1 & \text{if } m_S \not\equiv \mu_S(C, C') \pmod 2. \end{cases}$$

Proof. Let $\vartheta \colon \tilde{C} \to C$ be the normalisation map. Then we may write $b(\tilde{C}, C') = \bigcup_S b_S(\tilde{C}, C')$ with

$$b_S(\tilde{C}, C') = \{T \in \vartheta^{-1}(S) \ : \ \mu_T(\tilde{C}, C') \text{ is odd}\}$$

and where the disjoint union runs over all $S \in C(\overline{k}) \cap C'(\overline{k})$. Suppose $S \in C(\overline{k}) \cap C'(\overline{k})$. Since C' is smooth and the point S is either smooth or an ordinary singularity on C, at most one of the m_S points $T \in \vartheta^{-1}(S)$ satisfies $\mu_T(\tilde{C}, C') > 1$. Hence, there is a point $T_0 \in \vartheta^{-1}(S)$ such that for all $T \in \vartheta^{-1}(S)$ with $T \neq T_0$ we have $\mu_T(\tilde{C}, C') = 1$ and thus $T \in b_S(\tilde{C}, C')$. Since we are working over an algebraically closed field, Lemma 1.2.24 yields $\mu_S(C, C') = \mu_{T_0}(\tilde{C}, C') + m_S - 1$. Hence, we have $T_0 \in b_S(\tilde{C}, C')$ if and only if m_S and $\mu_S(C, C')$ have the same parity. It follows that $\#b_S(\tilde{C}, C') = c_S(C, C')$. The proposition follows. □

We will continue to use the notation $c_S(C, C')$ of Proposition 1.2.29, which we call the *contribution* of S with respect to C'. We set $c_S(C, C')$ equal to 0 for $S \in C(\overline{k})$ with $S \notin C'$.

Remark 1.2.30. Let $C \subset \mathbb{P}^2$ be a geometrically integral projective curve. The points of contribution 0 with respect to C' are the points of $C(\overline{k})$ that are not on C', together with the smooth points $S \in C(\overline{k})$ for which $\mu_S(C, C')$ is even. The points of contribution 1 are the smooth and double points S of $C(\overline{k})$ with $S \in C'$ for which $\mu_S(C, C')$ is odd. The points of type $m > 1$ are the ordinary singular points S of $C(\overline{k})$ of multiplicity m or $m+1$ with $S \in C'$ for which $\mu_S(C, C') \equiv m \pmod 2$.

Proposition 1.2.31. *Let C and C' be two geometrically integral projective curves in $\mathbb{P}^2$. Let $\tilde{C}$ denote the normalisation of C and let $C^{\mathrm{s}} \subset C(\overline{k})$ denote the set of singular points of C. Assume that C' is smooth and that all singular points of C that lie on C' are ordinary. Then the following statements hold.*

1. *The set $b(\tilde{C}, C')$ is empty if and only if the sets $b(C, C')$ and $C^{\mathrm{s}} \cap C'$ are.*

2. *We have $\#b(\tilde{C}, C') = 2$ if and only if either*

   (a) *$b(C, C') = \emptyset$ and there exists a point $S \in C(\overline{k})$ such that $m_S \in \{2, 3\}$ and $C^{\mathrm{s}} \cap C' = \{S\}$, or*

   (b) *there exist two points of C, say $S_1, S_2 \in C(\overline{k})$, with $S_1 \neq S_2$, such that $b(C, C') = \{S_1, S_2\}$ and $m_{S_1}, m_{S_2} \in \{1, 2\}$ and $C^{\mathrm{s}} \cap C' \subset \{S_1, S_2\}$.*

Proof. Given that the contributions $c_S(C, C')$ are nonnegative, this follows easily from Proposition 1.2.29 and Remark 1.2.30. □

1.2.4 K3 surfaces

In this subsection we briefly introduce the notion of $K3$ surface, giving the definition, some basic properties and some results that will be needed in the following of the thesis. For an extensive study of the topic, see [Huy15]; for more details about K3 surfaces over $\mathbb{C}$, see [BHPVdV04].

Let k be any field. A *K3 surface* over k is a smooth, projective, geometrically irreducible surface X with canonical divisor $K_X \sim_{\mathrm{lin}} 0$ and $H^1(X, \mathcal{O}_X) = 0$. A *complex* K3 surface is a K3 surface defined over $k = \mathbb{C}$.

Let X be a K3 surface over a field k. For $p, q \in \{0, 1, 2\}$, we define the (p, q)-Hodge number as

$$h^{p,q} := \dim H^q(X, \Omega_X^p),$$

where $\Omega_X^q = \bigwedge^q \Omega_X$ is the sheaf of regular q-forms on X.

Remark 1.2.32. Let X be a complex K3 surface. Then one can consider the Hodge structure on $H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X)$, where $H^{p,q}(X)$ denotes the group $H^p(X, \Omega_X^q)$.

For an introduction to Hodge theory on complex surfaces, we refer to [BHPVdV04, Section IV.2]; for an extensive study of Hodge theory on complex manifolds (and not only), see [Voi07].

We present now some basic results about K3 surfaces over any field k first, and then for complex K3 surfaces in particular.

Proposition 1.2.33. *Let X be a K3 over a field k. The the following statements hold.*

1. *Linear, algebraic and numeric equivalences are all equivalent, that is,* $\operatorname{Pic} X \cong \operatorname{NS} X \cong \operatorname{Num} X$.
2. *The Hodge diamond of X is the following.*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

3. *The Picard number of X is at most* 22*, that is,*

$$\rho(X) \leq 22.$$

4. *The arithmetic genus p_a of X is*

$$p_a(X) = 1.$$

Proof. 1. [Huy15, Proposition 1.2.4].

2. [Huy15, Subsection 1.2.4].
3. [Huy15, Remark 1.3.7].
4. By definition $p_a = \dim H^2(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X)$. Since X is a K3 surface, $H^1(X, \mathcal{O}_X) = 0$ and $\mathcal{O}_X \cong \omega_X = \Omega_X^2$, and so it follows that $\dim H^1(X, \mathcal{O}_X) = 0$ and

$$\dim H^2(X, \mathcal{O}_X) = \dim H^2(X, \Omega_X^2) = h^{2,2} = 1,$$

using point (2). Then $p_a = 1 - 0 = 1$. See also [Har77, Exercise III.5.3] and [Huy15, Subsection 1.2.3].

□

From Proposition 1.2.33.(1) it follows that $\operatorname{Pic} X$, endowed with the pairing induced by the intersection pairing of X, is a lattice of rank $\rho(X)$ (cf. Remark 1.2.15), called the *Picard lattice* of X.

Proposition 1.2.34. *The lattice* Pic X *is an even lattice of signature* $(1, \rho(X) - 1)$.

Proof. The parity of Pic X follows from the adjunction formula for surfaces (see [Har77, Proposition V.1.5]), recalling that X is a K3 surface and so $K = 0$.

The signature immediately follows from the Hodge index theorem (cf. [Har77, Theorem V.1.9]). □

Lemma 1.2.35. *Let X be a K3 surface over a field k, and let $D \in$ Div X be such that $D^2 = -2$. Then either D or $-D$ is linearly equivalent to an effective divisor.*

Proof. Let $\mathcal{L}(D)$ be the sheaf associated to the divisor D, and set $h^i(D) = \dim H^i(X, \mathcal{L}(D))$. By the Riemann–Roch formula we have

$$h^0(D) - h^1(D) + h^2(D) = \frac{1}{2} D \cdot (D - K) + 1 + p_a,$$

(cf. [Har77, Theorem V.1.6]). By Serre duality (cf. [Har77, Theorem III.7.7]), $h^2(D) = h^0(K - D)$. Since X is a K3 surface, $K = 0$ (by definition) and $p_a = 1$ (cf. Proposition 1.2.33); since, by initial assumption, $D^2 = -2$, we have

$$h^0(D) - h^1(D) + h^0(-D) = 1.$$

Since the terms on the left-hand side of the equation are all non-negative integers,

$$h^0(D) + h^0(-D) \geq 1.$$

It follows that $h^0(D) \geq 1$ or $h^0(-D) \geq 1$, that is, D or $-D$ is linearly equivalent to an effective divisor, respectively. □

If X is a K3 surface over $k = \mathbb{C}$, then we have some more results.

Proposition 1.2.36. *Let X be a complex K3 surface. Then the following statements hold.*

1. *The cohomology group $H^2(X, \mathbb{Z})$, endowed with the cup product, is a lattice isometric to the lattice Λ_{K3} (cf. Remark 1.1.30).*

2. *There is a primitive embedding of lattices* $\operatorname{Pic} X \hookrightarrow H^2(X, \mathbb{Z})$*. The image of the embedding is* $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$*.*

3. $\rho(X) \leq 20$*.*

4. *Let* X' *be another complex K3 surface, and assume there is a dominant rational map* $X' \dashrightarrow X$*. Then* $\rho(X) = \rho(X')$*.*

Proof.

1. [BHPVdV04, Proposition VIII.3.3.(ii)].

2. It follows from Lefschetz $(1, 1)$ Theorem (cf. [BHPVdV04, Theorem IV.2.13]).

3. It directly follows from point (2) of this proposition, Remark 1.2.32, and Proposition 1.2.33.(2).

4. [Sch13, Proposition 10.2].

□

Remark 1.2.37. If X is a K3 surface over $\mathbb{C}$, Proposition 1.2.36.(2) tells us that there is a primitive embedding of lattices $\operatorname{Pic} X \hookrightarrow H^2(X, \mathbb{Z})$. If we consider the étale cohomology instead of the singular one, a similar statement holds also for K3 surfaces defined over finite fields, as follows.

Let X be a K3 surface over a finite field k of characteristic p. Let ℓ be a prime different from p and define the étale cohomology groups $H^i_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ and the *Tate twist* $H^i_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$ as in [Mil80].

It turns out that $H^i_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$ is a $\mathbb{Z}_\ell$-module of rank $1, 0, 22, 0, 1$ for $i = 0, 1, 2, 3, 4$ (cf. [Băd01, Section 8.4 and Theorem 10.3]). In particular, $H^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$ has rank 22, it is endowed with a perfect pairing with values in $\mathbb{Z}_\ell$, and there is a primitive embedding of lattices $\operatorname{Pic} \overline{X} \otimes \mathbb{Z}_\ell \hookrightarrow H^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$, respecting the given pairings (see [Mil80, Remark V.3.29.(d)]).

Remark 1.2.38. The isometry $H^2(X, \mathbb{Z}) \cong \Lambda_{K3}$ in Proposition 1.2.36.(1) is not unique, nor canonical. Fixing such an isometry ϕ is called a *marking* of X. The pair (X, ϕ) is called a *marked K3 surface.*

Remark 1.2.39. Let X be a K3 surface over a field k. If $\rho(\overline{X}) = 22$, then X is said to be *supersingular.*

If $k \hookrightarrow \mathbb{C}$ and $\rho(\overline{X}) = 20$, then X is said to be *singular.*

Remark 1.2.40. If X is a complex K3 surface, then we define the *transcendental lattice* of X, denoted by $T = T(X)$, to be the orthogonal complement of the image of $\operatorname{Pic} X$ inside $H^2(X, \mathbb{Z})$. Note that from Proposition 1.2.36.(2) one has $H^{2,0}(X) \oplus H^{0,2}(X) \subseteq T(X) \otimes \mathbb{C}$.

In what follows, if X is a complex K3 surface, we will identify $\operatorname{Pic} X$ with its image inside $H^2(X, \mathbb{Z})$.

After giving some basic definitions, we state the Global Torelli Theorem for K3 surfaces, and we show how it can be used to obtain some information about the automorphism group of a complex K3 surface.

Let X and Y be two complex K3 surfaces. A lattice homomorphism ϕ between $H^2(Y, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$ is called a *Hodge isometry* if it preserves the lattice pairing and its $\mathbb{C}$-linear extension $\phi_{\mathbb{C}}$ preserves the Hodge structure, that is, $\phi_{\mathbb{C}}(H^{p,q}(Y)) = H^{p,q}(X)$.

A Hodge isometry is called *effective* if it sends ample classes to ample classes.

Proposition 1.2.41. *Let $f\colon X \to Y$ be an isomorphism between two K3 surfaces. The isomorphism f induces, by pull-back, a lattice homomorphism $f^*\colon H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$. The homomorphism f^* is an effective Hodge isometry.*

Proof. Let $f^*\colon H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ be the homomorphism induced by f, by pull-back. Since f is an isomorphism, f^* is an isometry of lattices. The pull-back of a holomorphic 2-form of Y is a holomorphic form of X, hence the $\mathbb{C}$-linear extension $f^*_{\mathbb{C}}$ of f^* sends $H^{2,0}(Y)$ to $H^{2,0}(X)$; since $H^{0,2}(Y) = \overline{H^{2,0}(Y)}$, we also have that $f^*_{\mathbb{C}}$ sends $H^{0,2}(Y)$ to $H^{0,2}(X)$; hence $f^*_{\mathbb{C}}$ sends $H^{2,0}(Y) \oplus H^{0,2}(Y)$ to $H^{2,0}(X) \oplus H^{0,2}(X)$ and therefore also $H^{1,1}(Y)$ to $H^{1,1}(X)$. Thus, f^* is an Hodge isometry.

To show that f^* is also effective, let $D \in \operatorname{Pic} Y$ be a very ample class. Then D gives an embedding $\phi_D\colon Y \to \mathbb{P}^n$, for some integer n, determined by a basis $(s_0, ..., s_n)$ of $H^0(Y, D)$. If f is an isomorphism, then the composition $f \circ \phi_D$ is an embedding of X, given by the elements $f \circ s_i = f^* s_i \in H^0(X, f^*D)$. Thus, also f^*D is a very ample class. Using the linearity of f^*, it follows that f^* sends ample divisor classes to ample divisor classes, proving the statement. □

The previous proposition states that every isomorphism $X \to Y$ of complex K3 surfaces gives an effective Hodge isometry from $H^2(Y, \mathbb{Z})$

to $H^2(X,\mathbb{Z})$. The converse is also true, as shown by the following theorem.

Theorem 1.2.42 (Torelli theorem for K3 surfaces)**.** *Let X, Y be two complex K3 surfaces and let $\phi\colon H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ be an effective Hodge isometry. Then there is a (unique) isomorphism $f\colon X \to Y$ such that $\phi = f^*$.*

Proof. It follows from [BHPVdV04, Theorem VIII.11.1 and Corollary VIII.11.4]. □

Let X be a K3 surface and let f be an automorphism of X. Then f induces, by pull-back, an isometry, say ϕ, of $\operatorname{Pic} X$. Define the map $(\cdot)^*_{\operatorname{Pic}}\colon \operatorname{Aut}(X) \to \mathcal{O}(\operatorname{Pic} X)$ by sending any f to the corresponding ϕ. In general, $(\cdot)^*_{\operatorname{Pic}}$ does not need to be injective, but we will show that in some cases it is so (cf. Proposition 1.2.47).

Remark 1.2.43. We have seen that every automorphism f of X induces an effective Hodge isometry f^* of $H^2(X,\mathbb{Z})$ (cf. Proposition 1.2.41). Let $\mathcal{O}_H(H^2(X,\mathbb{Z}))$ denote the subgroup of effective Hodge isometries of $H^2(X,\mathbb{Z})$, and let us identify $\operatorname{Pic} X$ with its image inside $H^2(X,\mathbb{Z})$. Then, by Proposition 1.2.36.(2), $\operatorname{Pic} X = H^{1,1}(X) \cap H^2(X,\mathbb{Z})$, and so an effective Hodge isometry sends the Picard lattice to itself. Then, we can define the restriction map

$$|_{\operatorname{Pic}}\colon \mathcal{O}_H(H^2(X,\mathbb{Z})) \to \mathcal{O}(\operatorname{Pic} X)$$

sending an effective Hodge isometry of $H^2(X,\mathbb{Z})$ to the isometry it induces on $\operatorname{Pic} X$. Note that if f is an automorphism of X, then the isometry of $\operatorname{Pic} X$ it induces equals the map $(f^*)|_{\operatorname{Pic}}$. In other words, the following diagram is commutative.

$$\begin{array}{ccc} \operatorname{Aut}(X) & \xrightarrow{(\cdot)^*_{\operatorname{Pic}}} & \mathcal{O}(\operatorname{Pic} X) \\ & {\scriptstyle (\cdot)^*}\searrow \quad \nearrow{\scriptstyle |_{\operatorname{Pic}}} & \\ & \mathcal{O}_H(H^2(X,\mathbb{Z})) & \end{array}$$

Thanks to Theorem 1.2.42, we know that the automorphisms of X are in a 1-to-1 correspondence with the effective Hodge isometries of $H^2(X,\mathbb{Z})$.

Let $T(X)$ be the transcendental lattice of X. By Corollary 1.1.13, there is an isomorphism $\varrho\colon \operatorname{Aut}(A_P) \to \operatorname{Aut}(A_T)$ between the automorphism groups of the discriminant groups $A_P = A_{\operatorname{Pic}(X)}$ and $A_T = A_{T(X)}$, making the following diagram commute.

$$
\begin{array}{ccccc}
 & & \operatorname{Aut}(X) & & \\
 & & \downarrow{\scriptstyle (\cdot)^*} & & \\
 & & \mathcal{O}_H(H^2(X,\mathbb{Z})) & & \\
 & \swarrow{\scriptstyle \operatorname{res}_P = |_{\operatorname{Pic}}} & & \searrow{\scriptstyle \operatorname{res}_T} & \\
\mathcal{O}(\operatorname{Pic} X) & & & & \mathcal{O}(T(X)) \\
{\scriptstyle \rho_P}\downarrow & & & & \downarrow{\scriptstyle \rho_T} \\
\operatorname{Aut}(A_P) & & \xrightarrow[\varrho_P]{\hspace{6em}} & & \operatorname{Aut}(A_T)
\end{array}
$$

Proposition 1.2.44. *The group $\mathcal{O}_H(H^2(X,\mathbb{Z}))$ is isomorphic to a subgroup of the group*

$$\{(\beta,\gamma) \in \mathcal{O}(\operatorname{Pic} X) \times \mathcal{O}(T(X)) \mid \varrho_P(\rho_P(\beta)) = \rho_T(\gamma)\}. \tag{1.2}$$

Proof. Let $\mathcal{O}_{\operatorname{Pic}}(H^2(X,\mathbb{Z}))$ be the subgroup of $\mathcal{O}(H^2(X,\mathbb{Z}))$ given by all the isometries sending $\operatorname{Pic} X$ to itself. Then, by definition, $\mathcal{O}_H(H^2(X,\mathbb{Z}))$ is contained in $\mathcal{O}_{\operatorname{Pic}}(H^2(X,\mathbb{Z}))$. Also, from Lemma 1.1.14, we have that $\mathcal{O}_{\operatorname{Pic}}(H^2(X,\mathbb{Z}))$ is isomorphic to the group (1.2). The statement follows. □

Corollary 1.2.45. *The group* $\operatorname{Aut}(X)$ *embeds into the group* (1.2) *in Proposition 1.2.44.*

Proof. By the Torelli theorem for K3 surfaces (cf. Theorem 1.2.42), the group $\operatorname{Aut}(X)$ is in 1-to-1 correspondence with $\mathcal{O}_H(H^2(X,\mathbb{Z}))$. By Proposition 1.2.44, $\mathcal{O}_H(H^2(X,\mathbb{Z}))$ is isomorphic to a subgroup of the group (1.2). □

Proposition 1.2.46. *Let X be a K3 surface with odd Picard number. Then $\mathcal{O}(T(X)) = \{\pm \operatorname{id}_T\}$.*

Proof. See Corollary [Huy15, 3.3.5]. □

Proposition 1.2.47. *Let X be a complex K3 surface, and assume that its Picard lattice has odd rank and discriminant not a power of* 2*. Then the map* $(\cdot)^*_{\mathrm{Pic}}\colon \mathrm{Aut}(X) \to \mathcal{O}(\mathrm{Pic}\, X)$ *is injective.*

Proof. We have seen that the map $(\cdot)^*_{\mathrm{Pic}}$ equals the composition of the pull-back $(\cdot)^*\colon \mathrm{Aut}(X) \to \mathcal{O}_H(H^2(X,\mathbb{Z}))$ and the restriction map $|_{\mathrm{Pic}}\colon \mathcal{O}_H(H^2(X,\mathbb{Z})) \to \mathcal{O}(\mathrm{Pic}\, X)$ (cf. Remark 1.2.43). It follows that if $\phi \in \mathcal{O}(\mathrm{Pic}\, X)$ is an element in the image of $\mathrm{Aut}(X)$, then there is an automorphism f of X such that $\phi = f^*_{|\,\mathrm{Pic}\, X}$. Now assume that there is also another such automorphism, say f', such that $\phi = (f')^*_{\mathrm{Pic}\, X}$ or, equivalently, such that $\phi = f'^*_{|\,\mathrm{Pic}\, X}$. By Proposition 1.2.44 we have that the automorphisms f and f' respectively correspond to the elements (ϕ, ρ) and (ϕ, ρ') in $\mathcal{O}(\mathrm{Pic}\, X) \times \mathcal{O}(T(X))$, with ϕ inducing the same automorphism on $A_P = A_T$ as ρ and ρ' respectively.

By Proposition 1.2.46 we have that $\rho, \rho' \in \{\pm\, \mathrm{id}_{T(X)}\}$. If $\rho = \rho'$, then f and f' correspond to the same element in $\mathcal{O}(\mathrm{Pic}\, X) \times \mathcal{O}(T(X))$ and therefore they must be equal (cf. Corollary 1.2.45). Then assume, without any loss of generality, that $\rho = \mathrm{id}_T$ and $\rho' = -\mathrm{id}_T$. It follows that ρ induces the identity on A_P and ρ' the multiplication by -1, and they both must be equal to the morphism induced by ϕ. The identity and the multiplication by -1 can be the same map only if A_P is isomorphic to a power of the group $\mathbb{Z}/2\mathbb{Z}$. Since the cardinality of the discriminant group of a lattice equals the determinant of the lattice, and by the initial hypothesis the determinant of $\mathrm{Pic}\, X$ is not a power of 2, then A_P cannot be isomorphic to a power of $\mathbb{Z}/2\mathbb{Z}$ and therefore ρ and ρ' do not induce the same automorphism of A_P. This way we get a contradiction, coming from the assumption that $\rho \neq \rho'$. Hence, $\rho = \rho'$ and this concludes the argument. □

After talking about automorphism of K3 surfaces in general, we introduce the notion of symplectic automorphisms. Let X be a K3 surface and let f be an automorphism of X. We say that f is *symplectic* if the induced action on $H^0(X, \Omega^2) = H^{2,0}(X)$ is the identity. The symplectic automorphisms of X form a subgroup of $\mathrm{Aut}(X)$, denoted by

$$\mathrm{Aut}_s(X) \subset \mathrm{Aut}(X).$$

Lemma 1.2.48. *Let X be a K3 surface and let f be an automorphism of X. Then f is symplectic if and only if f^* acts as the identity on $T(X)$.*

Proof. See [Huy15, Remark 15.1.2]. □

Proposition 1.2.49. *Let X be a complex K3 surface. Let f be a symplectic automorphism of X, and assume f has finite order n. Then f fixes a finite number of points of X. In particular, if $\#\operatorname{Fix}(f)$ denotes the number of points fixed by f, we have that only the following tuples $(n, \#\operatorname{Fix}(f))$ can and do occur.*

n	2	3	4	5	6	7	8
$\#\operatorname{Fix}(f)$	8	6	4	4	2	3	2
$\rho(X) \geq$	9	13	15	17	17	19	19

The table has been completed by a lower bound for $\rho(X)$ coming from the existence of a symplectic automorphism of order n.

Proof. See [Huy15, Section 15.1.2]. □

We conclude the section by giving some results about families of K3 surfaces.

Theorem 1.2.50. *The family of marked complex K3 surfaces with Picard number at least ρ is parametrised by the union of countably many complex manifolds of dimension $20 - \rho$.*

Proof. It follows from [Dol96, Corollary 3.2]. □

Lemma 1.2.51. *Let $\mathfrak{X} \to \mathbb{A}^1_k$ be a flat proper morphism over a field k, such that its fibers are K3 surfaces. Assume the characteristic of k to be 0. Let η and t be the generic point and a closed point of $\mathbb{A}^1$, respectively, and let X_η and X_t denote the fibers above η and t respectively. Then the specialization map*

$$\operatorname{sp}_t \colon \operatorname{Pic} X_\eta \to \operatorname{Pic} X_t$$

preserves the intersection pairing, is injective and has torsion-free cokernel.

The same holds also if we base-extend X_η and X_t to an algebraic closure of their field of definition.

Proof. It follows from [MP12, Proposition 3.6]. See also [Huy15, Proposition 17.2.10]. □

Lemma 1.2.52. *Let k be a number field and let $\mathcal{O}_k$ be its ring of integers. Let X be a K3 surface over k, and let $\mathfrak{X} \to \operatorname{Spec}(\mathcal{O}_k)$ be an integral model of X. Let $\mathfrak{p}$ be a prime of good reduction for $\mathfrak{X}$, that is, the fiber $\mathfrak{X}_\mathfrak{p}$ is a K3 surface. Then the reduction map*

$$\mathrm{sp}_\mathfrak{p} \colon \operatorname{Pic} X \to \operatorname{Pic} \mathfrak{X}_\mathfrak{p}$$

preserves the intersection pairing, is injective and has torsion-free cokernel.

The same holds also if we base-extend X and $\mathfrak{X}_\mathfrak{p}$ to an algebraic closure of their field of definition.

Proof. Just note that if η is the generic point of $\operatorname{Spec}(\mathcal{O}_k)$, and $\mathfrak{X}_\eta$ denotes the fiber of $\mathfrak{X}$ above η, then $\mathfrak{X}_\eta \cong X$. The result then follows from [MP12, Proposition 3.6]. See also [EJ11, Theorem 3.4] and [Huy15, Remark 17.2.11]. □

1.2.5 Del Pezzo surfaces

In this section we introduce the notion of del Pezzo surface and some basic results about these surfaces, focusing on del Pezzo surfaces of degree 1 and 2. For a general introduction to del Pezzo surfaces we refer to [Man86, Sections IV.24-26] and [Kol96, Section III.3]; another standard reference is also [Dem80].

Let X be a smooth, projective, geometrically irreducible surface over a field k. We say that X is a *del Pezzo surface* if its anti-canonical divisor $-K_X$ is ample. We define the *degree* of X to be the self intersection K_X^2 of its (anti-)canonical divisor.

From now until the end of the subsection, let X denote a del Pezzo surface over k, and let d denote the the degree of X.

Lemma 1.2.53. *Keeping the notation introduced before, one has the following inequality: $1 \leq d \leq 9$.*

Proof. [Man86, Theorem IV.24.3.(i)]. □

Remark 1.2.54. A set of closed points on the plane is said to be in *general position* if no three points lie on a line; no six points lie on a conic; no eight points lie on a singular cubic, with one of the points at the singularity.

Theorem 1.2.55. *Keeping the notation as before, the following statements hold, under the assumption that k is algebraically closed.*

1. *If $d = 9$, then X is isomorphic to $\mathbb{P}^2$.*
2. *If $d = 8$, then X is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or to the blow-up of $\mathbb{P}^2$ at one point.*
3. *If $7 \geq d \geq 1$, then X is isomorphic to the blow-up of $\mathbb{P}^2$ at $9 - d$ points in general position.*

If $d \geq 3$, then the converse of the above statements is also true, that is, the blow up of $\mathbb{P}^2$ at $9 - d$ points in general position is a del Pezzo surface of degree d.

Proof. [Man86, Theorem IV.24.4]. □

Remark 1.2.56. For $d \in \{1, 2\}$, stricter conditions on the points are required in order for the converse of Theorem 1.2.55.(3) to hold. For the details, see [Man86, Theorem IV.26.2].

Corollary 1.2.57. *Let X be a del Pezzo surface over an algebraically closed field k. Then X is birational to $\mathbb{P}^2_k$.*

Proof. Trivial using Theorem 1.2.55. □

Corollary 1.2.58. *Let X be a del Pezzo surface over k, assume that X is not birational to $\mathbb{P}^1 \times \mathbb{P}^1$ and that k is algebraically closed. Set $r := 9 - d$. Then $\operatorname{Pic} X$, endowed with the intersection pairing of X, is a lattice of rank $r + 1$, admitting a basis $(E_0, E_1, ..., E_r)$ such that*

- $E_0^2 = 1$;
- $E_i^2 = -1$, *for $i = 1, ..., r$;*
- $E_i \cdot E_j = 0$, *for every $i \neq j$.*

Proof. [Man86, Proposition IV.25.1]. □

Proposition 1.2.59. *Let X be a del Pezzo surface of degree d over k.*

If $d = 1$, then X is isomorphic to a hypersurface of degree 6 inside $\mathbb{P}_k(1,1,2,3)$. Conversely, any smooth hypersurface of degree 6 inside $\mathbb{P}_k(1,1,2,3)$ is a del Pezzo surface of degree 1.

If $d = 2$ then X is isomorphic to a hypersurface of degree 4 inside $\mathbb{P}_k(1,1,1,2)$. Conversely, any smooth hypersurface of degree 4 inside $\mathbb{P}_k(1,1,1,2)$ is a del Pezzo surface of degree 2.

Proof. See [Kol96, Theorem III.3.5]. □