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Summary

Let E be an elliptic curve over $\mathbb{C}$ with *complex multiplication (CM)* by the maximal order $\mathcal{O}_K$ of an imaginary quadratic field K. The first main theorem of complex multiplication for elliptic curves then states that the field extension $K(j(E))$, obtained by adjoining the j-invariant of E to K, is equal to the *Hilbert class field* of K, see Theorem 11.1 in Cox [11]. Note that if E is defined over $\mathbb{Q}$, then the Hilbert class field $K(j(E))$ is equal to K, which implies that the class group Cl_K is trivial.

We can ask for which imaginary quadratic fields K the corresponding elliptic curve with CM by $\mathcal{O}_K$ is defined over $\mathbb{Q}$. This is equivalent to asking to find all imaginary quadratic fields with trivial class group Cl_K. This problem is known as Gauss' class number one problem, which was solved by Heegner in 1952 [16], Baker in 1967 [2], and Stark in 1967 [41]. The imaginary quadratic fields with trivial class group are the fields $\mathbb{Q}(\sqrt{-d})$ with $d \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$.

In the 1950's, Shimura and Taniyama [39] generalized the first main theorem of CM for elliptic curves to *abelian varieties.* We say that an abelian variety A of dimension g has CM if the endomorphism ring of A contains an order of a *CM field* of degree $2g$. Let K be a CM field of degree $2g$ with maximal order $\mathcal{O}_K$, and let Φ be a *CM type* of K. Let A be a polarized simple abelian variety over $\mathbb{C}$ of dimension g that has CM by $\mathcal{O}_K$. Then the first main theorem of CM says that the field of moduli M of the polarized simple abelian variety A gives an unramified class field H over the *reflex field* K^r of K. Moreover, the class field H corresponds to the ideal group $I_0(\Phi^r)$ (see page 17), which only depends on (K, Φ), see Theorem 1.5.6. Note that the first main theorem of CM implies that if the polarized abelian variety A is defined over K^r, then *the CM class group* $I_{K^r}/I_0(\Phi^r)$ is trivial.

As in the elliptic curve case, we can ask for which CM pairs (K, Φ) the corresponding CM abelian varieties are defined over K^r. Equivalently, we can ask for which CM pairs (K, Φ) the *CM class group* $I_{K^r}/I_0(\Phi^r)$ is trivial. In this thesis we give an answer to this problem for quartic CM fields (see Chapter 2), and for sextic CM fields containing an imaginary quadratic field (see Chapter 3).

Furthermore, we can ask for which CM fields the corresponding simple CM abelian varieties have field of moduli $\mathbb{Q}$. Murabayashi and Umegaki [31] determined the quartic CM fields that correspond to a simple CM abelian surface with field of moduli $\mathbb{Q}$. In Chapter 4, we determine the sextic CM fields that correspond to a simple CM abelian threefold with field of moduli $\mathbb{Q}$.