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Preface

This thesis has four chapters and is organized as follows.

Chapter 1 is an introduction to abelian varieties and complex multiplication theory. It also contains facts from unramified class field theory. We present facts that we will use in later chapters, including the main theorem of complex multiplication. The results in this chapter are not new and most are due to Shimura and Taniyama [40].

Chapter 2 is a joint work with Marco Streng that appeared as *The CM class number one problem for curves of genus* 2 [18]. In Sections 2.3 and 2.4, we give a solution to the *CM class number one problem* for curves of genus 2 (Theorems 2.3.15 and 2.4.5).

Chapter 3 deals with the *CM class number one problem* for curves of genus 3 with a simple Jacobian. We give a partial solution to this problem. We restrict ourselves to the case where the sextic CM field corresponding to such a curve contains an imaginary quadratic subfield. We give the complete list of such sextic CM fields in Table 3.1 (unconditional) and Tables 3.3–3.12 (under GRH).

Chapter 4 gives the complete list of sextic CM fields K for which there exist principally polarized simple abelian threefolds with CM by $\mathcal{O}_K$ with rational field of moduli.

List of Notation

$\mathcal{O}_K$	the ring of integers (maximal order) of a number field K
I_K	the group of fractional ideals of a number field K
P_K	the group of principal fractional ideals of a number field K
Cl_K	the quotient I_K/P_K
h_K	the order of Cl_K
$F_{\gg 0}$	the group of totally positive elements of a totally real number field F
P_K^+	the group of principal ideals that are generated by the elements of $F_{\gg 0}$
W_K	the group of roots of unity of K
μ_K	the order of the group of roots of unity W_K
$\mathcal{O}_K^\times$	the unit group of $\mathcal{O}_K$
d_K	the discriminant of K
K/F	a number field extension of finite degree
$\mathrm{Gal}(K/F)$	the Galois group of K over F
$f_{K/F}$	the finite part of the conductor of K/F
$\mathfrak{D}_{K/F}$	the different of K over F, an ideal of $\mathcal{O}_K$
$N_{K/F}$	the ideal norm from K to F
$\mathrm{tr}_{K/F}$	the trace from K to F

t_K	the number of primes in F that are ramified in K
h_K^*	the relative class number h_K/h_F of K/F
$\mathrm{D}_\mathfrak{p}$	the decomposition group of a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$
$\mathrm{I}_\mathfrak{p}$	the inertia group of a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$
$\left(\frac{K/F}{\mathfrak{P}}\right)$	the Frobenius automorphism in $\mathrm{Gal}(K/F)$ corresponding to a prime ideal $\mathfrak{P} \subset \mathcal{O}_K$, page 2
Q_K	the Hasse unit index $[\mathcal{O}_K^\times : W_K\mathcal{O}_F^\times]$ of K/F, page 24
K_1K_2	the smallest field in $\overline{\mathbb{Q}}$ that contains the fields K_1 and K_2 (for K_1, $K_2 \subset \overline{\mathbb{Q}}$)
(K,Φ)	a CM pair, where K is a CM field and Φ is a CM type of K
(K^r,Φ^r)	the reflex of (K,Φ)
F^r	the maximal totally real subfield of K^r
N_Φ	the type norm map from K to K^r, page 5
$I_0(\Phi)$	an ideal group generated by the elements $\mathfrak{a}$ of I_K such that $\mathrm{N}_\Phi(\mathfrak{a}) = (\alpha) \in P_{F^r}$ and $\alpha\overline{\alpha} \in \mathbb{Q}$, page 17
A^*	the dual abelian variety of an abelian variety A
$\mathrm{End}(A)$	the ring of endomorphisms of A/k over $\overline{k}$
$\mathrm{End}_0(A)$	$\mathrm{End}(A) \otimes_\mathbb{Z} \mathbb{Q}$
P	a polarized abelian variety of a CM type (K,Φ), page 17
M_J	the field of moduli of $P\|_J = (A,\theta\|_J,\mathcal{C})$, page 75
$\mathfrak{f}(P)$	an ideal in $\mathcal{O}_F$ determined by P, page 15