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Chapter 4

Simple CM curves of genus 3 over $\mathbb{Q}$

ABSTRACT. In this chapter, we will study the isomorphism classes of principally polarized simple CM abelian threefolds with field of moduli $\mathbb{Q}$*. In Section 4.3, we will determine the sextic CM fields corresponding to simple CM curves of genus* 3 *with field of moduli* $\mathbb{Q}$*.*

4.1 Introduction

It is known that there are only finitely many CM elliptic curves over $\mathbb{Q}$, up to isomorphism over $\overline{\mathbb{Q}}$. An equivalent formulation of this statement is that there are only finitely many class number one imaginary quadratic fields. The complete list of such fields is given by Heegner [16] (1952), Baker [2] (1966) and Stark [41] (1967). A list of simple CM curves of genus 2 defined over $\mathbb{Q}$, up to isomorphism over $\overline{\mathbb{Q}}$, is given by van Wamelen [45] and the completeness of the list of van Wamelen is shown by Murabayashi and Umegaki [31].

In this chapter, we determine the sextic CM fields corresponding to simple CM curves of genus 3 with field of moduli $\mathbb{Q}$. One way to do this would be to compute the curves corresponding to the CM fields of Chapter 3. However, the current techniques are not sufficient to compute all CM curves of genus 3, see Section 4.4. Instead, in this chapter, we will use an alternative method inspired by Murabayashi [30] and Shimura [38].

It is known that every *principally polarized simple abelian variety* over $\mathbb{C}$ of dimension g with $g \leq 3$ is isomorphic to the *Jacobian variety* of a curve of genus g (see Theorem 1.3.6). By the fact that the *Torelli map* is injective on $\mathbb{C}$-points, *the field of moduli* of $J(C)$ is the same as the field of moduli of the curve C. Therefore, we will be interested in finding the CM fields corresponding to the *principally polarized simple CM abelian threefolds* with rational field of moduli.

Theorem 4.1.1. *There exist exactly* 37 *CM points over* $\mathbb{Q}$ *in the moduli space of principally polarized abelian threefolds that are simple over* $\overline{\mathbb{Q}}$ *and have CM by a maximal order. In fact, it is exactly one point for each field in Table 3.1.*

Remark 4.1.2. For some of these abelian threefolds, we know that they can be defined over $\mathbb{Q}$, for others we do not know. See Section 4.4.

4.2 Polarized CM abelian varieties over $\mathbb{Q}$

Let K be a CM field and Φ be a primitive CM type of K. Let $P := (A, \theta, \varphi)$ be a polarized abelian variety over $\mathbb{C}$ of type $(K, \Phi, t, \mathfrak{m})$ (for the definition see page 14) such that $\theta^{-1}(\mathrm{End}(A)) = \mathcal{O}_K$. Let M be the field of moduli of (A, φ). By the first main theorem of CM (Theorem 1.5.6), we can determine MK^r as a class field over K^r. This theorem does not provide information about the size of M. We will see that sometimes the field M depends on the *isomorphim class* of (A, φ) whereas MK^r is determined only by (K, Φ).

To say something about M requires investigation of the behavior of P under an automorphism of $\mathbb{C}$ that does not necessarily fix K^r.

Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. By Proposition 1.4.13, the abelian variety $\sigma(A, \theta)$ is of type $(K, \sigma\Phi)$ and if σ is the identity map on K^r, then $\sigma(A, \theta)$ is of type (K, Φ). In this chapter, we broaden our perspective from only the automorphisms in $\mathrm{Aut}(\mathbb{C}/K^r)$, to all the ones $\sigma \in \mathrm{Aut}(\mathbb{C})$ for which there exists $[\sigma] \in \mathrm{Aut}(K)$ such that we have $\sigma \circ \Phi = \Phi \circ [\sigma]$ as sets.

Note that if Φ is primitive, then $[\sigma]$ is uniquely determined by σ and Φ if it exists.

For a given $P := (A, \theta, \varphi)$, we denote by $\theta|_J$, for any subfield J of K, the restriction of θ to J and by M_J the field of moduli of $(A, \theta|_J, \varphi)$. Throughout this chapter, every P of CM type (K, Φ) is assumed to have CM by $\mathcal{O}_K$.

Proposition 4.2.1. *(Shimura) Let (K, Φ) be a primitive CM pair, and let (K^r, Φ^r) be its reflex. Let $P = (A, \theta, \varphi)$ be of CM type (K, Φ) and defined over $\mathbb{C}$. Let J be a subfield of K, $\theta|_J$ be the restriction of θ to J and M_J be the field of moduli of $(A, \theta|_J, \varphi)$. Then the following assertions hold.*

(i) *$M_J K^r$ is the field of moduli of P.*

(ii) *K^r is normal over $M_J \cap K^r$.*

(iii) *$M_J K^r$ is normal over M_J and $\mathrm{Gal}(M_J K^r / M_J)$ is isomorphic to a subgroup of $\mathrm{Aut}(K/J)$ via the map*

$$\sigma|_{M_J K^r} \mapsto [\sigma]$$

for $\sigma \in \mathrm{Aut}(\mathbb{C}/M_J)$, where $[\sigma] \in \mathrm{Aut}(K/J)$ is such that $\sigma\Phi = \Phi[\sigma]$.

Proof. This is Shimura [39, Proposition 5.17] except that Shimura does not explicitly give the map in *(iii)*. Therefore, we reprove the second part of *(iii)*.

Let $\sigma \in \mathrm{Aut}(\mathbb{C}/M_J)$ and let λ be an isomorphism from $(A, \varphi, \theta|_J)$ to $(\sigma A, \sigma\varphi, {}^\sigma\theta|_J)$, where ${}^\sigma\theta(\alpha) = \sigma\theta([\sigma]^{-1}\alpha)$ for all $\alpha \in K$. Since A is simple, we have $\theta(K) = \mathrm{End}_0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$. Let $[\sigma] \in \mathrm{Aut}(K/J)$ be given by $[\sigma](\alpha) = \theta^{-1}(\lambda^{-1}\sigma(\theta(\alpha))\lambda)$. Then we get that $\sigma\theta$ and $\theta[\sigma]$ have the same CM type, hence $\sigma\Phi = \Phi[\sigma]$. Recall that $[\sigma]$ is uniquely defined by $\sigma\Phi = \Phi[\sigma]$ as Φ is primitive.

We have $[\sigma] = \mathrm{id}_K$ if and only if $\sigma\Phi = \Phi$ if and only if $\sigma|_{M_J K^r} = \mathrm{id}_{M_J K^r}$. Hence the map $\sigma|_{M_J K^r} \mapsto [\sigma]$ is a well-defined injection, so $\mathrm{Gal}(M_J K^r / M_J)$ is isomorphic to a subgroup of $\mathrm{Aut}(K/J)$. □

Remark 4.2.2. If K is abelian over $\mathbb{Q}$ and embedded inside $\mathbb{C}$ and Φ is a primitive CM type of K, then we have $[\sigma] = \sigma|_K$ as $\sigma\Phi = \Phi\sigma$.

Proposition 4.2.3. *Let $P = (A, \theta, \varphi)$ be of a primitive CM type (K, Φ). Let J be a subfield of K such that $M_J = \mathbb{Q}$. Then $J = \mathbb{Q}$ and $M_K = K^r \cong K$ is Galois over $\mathbb{Q}$.*

Proof. Suppose that J is a subfield of K and $M_J = \mathbb{Q}$. By Proposition 4.2.1-*(ii)*, the reflex field K^r is normal over $\mathbb{Q}$; and by *(iii)*, the Galois group $\mathrm{Gal}(K^r/\mathbb{Q})$ is isomorphic to a subgroup of $\mathrm{Aut}(K/J)$. In particular, we get

$$[K^r : \mathbb{Q}] = \#\mathrm{Gal}(K^r/\mathbb{Q}) \leq \#\mathrm{Aut}(K/J) \leq [K : J]. \tag{4.2.1}$$

On the other hand, since Φ is primitive, we get $K^{rr} = K$ by Lemma 1.2.5. Furthermore, since $K^r/\mathbb{Q}$ is normal, the reflex field K^{rr} is isomorphic to a subfield of K^r. Hence we have $K^{rr} = K \subset K^r$, and therefore, by (4.2.1) we get $J = \mathbb{Q}$ and $K^r \cong K$ and by Proposition 4.2.1-(i), we get $M_K = M_J K^r = K^r$. □

Note that $M_{\mathbb{Q}} := M$. For $\sigma \in \mathrm{Aut}(\mathbb{C}/M_{\mathbb{Q}})$, we define

$$^{\sigma}P = (\sigma A, {}^{\sigma}\theta, \sigma\varphi), \tag{4.2.2}$$

where ${}^{\sigma}\theta(\alpha) = \sigma\theta([\sigma]^{-1}\alpha)$ for all $\alpha \in K$.

Proposition 4.2.4. *(Shimura [38, page 69]) For $\sigma \in \mathrm{Aut}(\mathbb{C}/M_{\mathbb{Q}})$, the polarized simple abelian variety ${}^{\sigma}P = (\sigma A, {}^{\sigma}\theta, \sigma\varphi)$ is of type (K, Φ).*

Proof. Proposition 4.2.1-*(iii)* tells us $\Phi = \sigma\Phi[\sigma]^{-1}$ and then the result follows from the definition of ${}^{\sigma}\theta$. □

Recall the notation $(P' : P) \in \mathfrak{C}_K$ from (1.5.5).

Proposition 4.2.5. *(Shimura [38, Proposition 2]) Let K be a CM field and Φ be its CM type. Let P and P' be polarized simple abelian varieties of CM type (K, Φ) and defined over $k \subset \mathbb{C}$. For any σ, $\gamma \in \mathrm{Aut}(\overline{k}/\mathbb{Q})$, the following holds.*

(i) *If $(P' : P) = [(b, \mathfrak{c})]$, then we have $({}^{\sigma}P' : {}^{\sigma}P) = [([\sigma]b, [\sigma]\mathfrak{c})]$.*

(ii) *If $({}^{\sigma}P : P) = [(b, \mathfrak{c})]$ and $({}^{\gamma}P : P) = [(d, \mathfrak{e})]$, then we have*

$$({}^{\gamma\sigma}P : P) = [(([\gamma]b)d, ([\gamma]\mathfrak{c})\mathfrak{e})].$$

(iii) If $(P' : P) = [(b, \mathfrak{c})]$ *and* $({}^{\sigma}P : P) = [(d, \mathfrak{e})]$*, then we have*

$$({}^{\sigma}P' : P') = [(([\sigma]b)b^{-1}d, ([\sigma]\mathfrak{c})\mathfrak{c}^{-1}\mathfrak{e})]. \qquad \square$$

Proof. (i) See Proposition 2-(i) in Shimura [38].

(ii) We have $({}^{\gamma\sigma}P : P) = ({}^{\gamma\sigma}P : {}^{\gamma}P)({}^{\gamma}P : P)$ and by *(i)*, we have $({}^{\gamma\sigma}P : {}^{\gamma}P) = [([\gamma]b, [\gamma]\mathfrak{c})]$. It follows

$$({}^{\gamma\sigma}P : P) = [([\gamma]b, [\gamma]\mathfrak{c})][(d, \mathfrak{e})].$$

(iii) We have $({}^{\sigma}P' : P') = ({}^{\sigma}P' : {}^{\sigma}P)({}^{\sigma}P : P)(P' : P)^{-1}$. By *(i)*, we have $({}^{\sigma}P' : {}^{\sigma}P) = [([\sigma]b, [\sigma]\mathfrak{c})]$. Hence the result follows from the following

$$({}^{\sigma}P' : P') = [([\sigma]b, [\sigma]\mathfrak{c})][(d, \mathfrak{e})][(b, \mathfrak{c})]^{-1}.$$

$\square$

4.3 Principally polarized simple CM abelian threefolds

In this chapter, our interest will be determining the sextic CM fields that correspond to principally polarized CM abelian threefolds with rational field of moduli.

In this section, we will prove the following.

Theorem 4.3.1.

(i) If a principally polarized simple CM abelian threefold over $\mathbb{C}$ *with CM by the ring of integer of a CM field* K *has field of moduli* $\mathbb{Q}$*, then* K *is one of the cyclic sextic CM fields in Table 3.1; in particular* $h_K^* = 2^{t_K - 1} = 1$ *or* 4*, where* t_K *is the number of primes in* F *that are ramified in* K*.*

(ii) For every cyclic sextic CM field K *with* $h_K^* = 1$*, up to isomorphism, there exists a* unique *principally polarized simple abelian threefold over* $\mathbb{C}$ *that has CM by* $\mathcal{O}_K$*, and such principally polarized abelian threefolds have field of moduli* $\mathbb{Q}$*.*

(iii) *For every cyclic sextic CM field K with $h_K^* = 4$ and a primitive CM type Φ that satisfies $I_0(\Phi^r) = I_{K^r}$, there are four isomorphism classes of principally polarized simple abelian threefolds over $\mathbb{C}$ that have CM by $\mathcal{O}_K$. Among these four isomorphism classes, one of them has rational field of moduli and the other three have field of moduli F, which is the cubic subfield of K.*

Lemma 4.3.2. *Let K be a CM field of degree $2g$ with an odd prime g and let Φ be a CM type of K. Let $P = (A, \theta, \varphi)$ be a g-dimensional polarized abelian variety of type (K, Φ) with CM by $\mathcal{O}_K$.*

If A is simple over $\mathbb{C}$ and the field of moduli $M_{\mathbb{Q}}$ of (A, φ) is $\mathbb{Q}$ then K is cyclic over $\mathbb{Q}$, the CM type Φ is primitive and we have $I_0(\Phi^r) = I_{K^r}$.

Proof. Suppose that A is simple over $\mathbb{C}$. Then by Theorem 1.4.1 the CM type Φ of A is primitive. By Proposition 4.2.3, we have that K is Galois over $\mathbb{Q}$. Hence the maximal totally real subfield F is also normal over $\mathbb{Q}$ with an odd prime degree g. Every group with a prime order is cyclic, hence $\mathrm{Gal}(F/\mathbb{Q})$ is cyclic. Let k be the subfield of K that is fixed by the order-g cyclic subgroup of $\mathrm{Gal}(K/\mathbb{Q})$. Since K is a CM field of order $2g$ and F is totally real, we have $\rho \in \mathrm{Gal}(K/F)$ and $\mathrm{Gal}(K/F) = 2$. Therefore, the subfield k is imaginary quadratic over $\mathbb{Q}$. Moreover, since ρ commutes with every element in $\mathrm{Gal}(K/k)$, the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is abelian hence cyclic of degree $2g$.

Moreover, recall that the first main theorem of Complex Multiplication (Theorem 1.5.6) says that $M_{\mathbb{Q}}K^r$ is the unramified class field over K^r corresponding to the ideal group $I_0(\Phi^r)$. Hence if $M_{\mathbb{Q}} = \mathbb{Q}$, then (K, Φ) satisfies $I_0(\Phi^r) = I_{K^r}$. □

Lemma 4.3.2 proves *(i)* of Theorem 4.3.1 as follows. If $M_{\mathbb{Q}} = \mathbb{Q}$, then K is cyclic over $\mathbb{Q}$, the CM type Φ is primitive and (K, Φ) satisfies $I_0(\Phi^r) = I_{K^r}$. By Theorem 3.1.2, the cyclic sextic CM fields that satisfy $I_0(\Phi^r) = I_{K^r}$ for a primitive CM type are listed in Table 3.1. Therefore, we have $h_K^* \in \{1,\ 4\}$.

Remark 4.3.3. We can also see $h_K^* \in \{1,\ 4\}$ directly as follows. Since K is cyclic over $\mathbb{Q}$, the totally real cubic subfield F is also *cyclic* over $\mathbb{Q}$. Moreover, Proposition 3.3.3 tells us that K contains a *class number one* imaginary quadratic field. So there is only one rational prime that is ramified in K/F (see Proposition 4.8-(ii) in II of Lang [21]). Since F is a cubic field, the number t_K of primes in F that are ramified in K is at most 3. Furthermore, since K is cyclic, we have $t_K \neq 2$. Hence $t_K \in \{1,3\}$ and so $h_K^* = 2^{t_K-1} \in \{1,\ 4\}$.

Lemma 4.3.4. *Let K be a cyclic sextic CM field in Table 3.1 and let F be the totally real cubic subfield of K. Let $(\mathcal{O}_F^\times)^+$ be the group of totally positive units in $\mathcal{O}_F$. Then we have $(\mathcal{O}_F^\times)^+ = (\mathcal{O}_F^\times)^2$. Moreover, we have $\mathrm{Cl}_F = \mathrm{Cl}_F^+$.*

Proof. For each field in Table 3.1, using Sage [36], we check that the map

$$\mathrm{sign} : \mathcal{O}_F^\times \to (C_2)^3 \tag{4.3.1}$$

is surjective. On the other hand, since F is a totally real cubic field, we have $[\mathcal{O}_F^\times : (\mathcal{O}_F^\times)^2] = 8$ by the Dirichlet unit theorem. Hence the kernel of the map (4.3.1) is $(\mathcal{O}_F^\times)^2$, and therefore the first equality follows.

Moreover, since the map (4.3.1) is surjective, we have $\mathrm{Cl}_F = \mathrm{Cl}_F^+$. □

Proposition 4.3.5. *(Shimura [38, Proposition 1]) Let K be a CM field and let Φ be a primitive CM type of K. If $\mathfrak{D}_{K/F} \neq \mathcal{O}_K$, then there is a principally polarized abelian variety $P = (A, \theta, \varphi)$ of type (K, Φ).* □

Corollary 4.3.6. *For every CM class number one sextic cyclic CM field K and every primitive CM type Φ of K, there exists a principally polarized abelian threefold of type (K, Φ).*

Proof. Theorem 3.1.2 proves that all CM class number one cyclic sextic CM fields K are in Table 3.1 and we can see that all the fields in this table satisfy $\mathfrak{D}_{K/F} \neq \mathcal{O}_K$. Therefore, the result follows from Proposition 4.3.5. □

The following is a variant of Proposition 4.4 in Streng [43].

Proposition 4.3.7. *Let K be a CM field of degree $2g$ and let F be the maximal totally real subfield of K. Let Φ be a primitive CM type of K. Suppose $(\mathcal{O}_F^\times)^+ = N_{K/F}(\mathcal{O}_K^\times)$. If there exists a principally polarized abelian variety over $\mathbb{C}$ of type (K, Φ), then there are exactly h_K^* isomorphism classes of such principally polarized abelian varieties.*

Proof. Set

$$\mathrm{H}_\Phi := \{(t,\mathfrak{m}) \in K^\times \times I_K : \ \mathrm{Im}(\phi(t)) > 0 \text{ for all } \phi \in \Phi,\ \bar{t} = -t, \\ \text{and } t^{-1}\mathcal{O}_K = \mathfrak{D}_{K/\mathbb{Q}}\mathfrak{m}\overline{\mathfrak{m}}\}.$$

The group $K^\times$ acts on H_Φ via $x(t,\mathfrak{m}) = ((x\overline{x})^{-1}t, x\mathfrak{m})$ for $x \in K^\times$. Let $\mathrm{H}'_\Phi := \mathrm{H}_\Phi/\{((x\overline{x})^{-1}, x\mathcal{O}_K) : x \in K^\times\}$.

Let CM_Φ be the set of isomorphism classes of principally polarized g-dimensional abelian varieties of type (K,Φ). Let $P_i = (A_i, \theta_i, \varphi_i)$ be a principally polarized g-dimensional abelian varietiy of type (K,Φ). Then there is a $\mathbb{Z}$-lattice $\mathfrak{m}_i$ in K and $t_i \in K^\times$ such that $A_i \cong \mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}_i)$, and $\phi(t_i) > 0$ for all $\phi \in \Phi$ and $\bar{t}_i = -t_i$, see page 14. By Proposition 1.5.5, it holds that P_1 and P_2 are isomorphic if and only if $(P_1 : P_2) := [(t_1^{-1}t_2, \mathfrak{m}_1\mathfrak{m}_2^{-1})] = [(1,\mathcal{O}_K)]$, which holds if and only if there exists $x \in K^\times$ such that $\mathfrak{m}_2 = x\mathfrak{m}_1$ and $t_2(x\overline{x})^{-1} = t_1$. So CM_Φ is bijective to H'_Φ.

Recall $\mathfrak{C}_K := (F_{\gg 0} \times I_K)/\{(x\overline{x}, x\mathcal{O}_K) : x \in K^\times\}$. Define

$$\mathfrak{C}'_K = \{[(b,\mathfrak{c})] \in \mathfrak{C}_K : \mathfrak{c}\overline{\mathfrak{c}} = b\mathcal{O}_F\}.$$

Suppose $\mathfrak{D}_{K/F} \neq \mathcal{O}_K$. Then by Proposition 4.3.5, there exists an element $(t_0,\mathfrak{m}_0)$ in H_Φ. We observe that the map $\mathfrak{C}'_K \to \mathrm{H}'_\Phi : [(b,\mathfrak{c})] \mapsto [(b^{-1}t_0, \mathfrak{c}\mathfrak{m}_0)]$ is a bijection. Hence we have $|\mathfrak{C}'_K| = |\mathrm{H}'_\Phi| = |\mathrm{CM}_\Phi|$.

Moreover, by the assumption $(\mathcal{O}_F^\times)^+ = N_{K/F}(\mathcal{O}_K^\times)$, for every $[(b,\mathfrak{c})] \in \mathfrak{C}'_K$ there is $\epsilon \in \mathcal{O}_K^\times$ such that $b = \epsilon\overline{\epsilon}$. This means that for every $\mathfrak{c} \in I_K$, there is a *unique* class in $\mathfrak{C}'_K$. Hence the group $\mathfrak{C}'_K$ injects into Cl_K via the map $[(b,\mathfrak{c})] \mapsto [\mathfrak{c}]$.

We claim $\mathfrak{C}'_K \cong \ker(\mathrm{Cl}_K \to \mathrm{Cl}_F)$.

By definition, for every $[(b,\mathfrak{c})] \in \mathfrak{C}'_K$, we have $\mathrm{N}_{K/F}(\mathfrak{c}) = b\mathcal{O}_K$ and $b \in F_{\gg 0}$. Therefore, we get $\mathfrak{C}'_K \subset \ker(\mathrm{Cl}_K \to \mathrm{Cl}_F)$ via the injection $[(b,\mathfrak{c})] \mapsto [\mathfrak{c}]$. On the other hand, if $[\mathfrak{a}] \in \ker(\mathrm{Cl}_K \to \mathrm{Cl}_F)$, then we have $\mathrm{N}_{K/F}(\mathfrak{a}) \in P_F$ and $P_F^+ = P_F$ by the assumption $(\mathcal{O}_F^\times)^+ = N_{K/F}(\mathcal{O}_K^\times)$. This proves the claim.

The norm map $\mathrm{N}_{K/F} : \mathrm{Cl}_K \to \mathrm{Cl}_F$ is surjective by Theorem 10.1 in Washington [46] and the fact that the infinite primes ramify in K/F. By the isomorphism theorem we have $\mathrm{Cl}_K/\mathfrak{C}'_K \cong \mathrm{Cl}_F$. Therefore, we get $h_K^* := |\mathrm{Cl}_K|/|\mathrm{Cl}_F| = |\mathfrak{C}'_K| = |\mathrm{CM}_\Phi|$. □

Corollary 4.3.8. *Let K be a cyclic sextic CM field in Table 3.1 on page 56 and let Φ be a primitive CM type of K. Then there are $h_K^* \in \{1,4\}$ isomorphism classes of principally polarized abelian varieties over $\mathbb{C}$ of type (K,Φ) with CM by $\mathcal{O}_K$.*

Proof. The existence of a principally polarized abelian threefold of type (K,Φ) is guaranteed by Corollary 4.3.6. On the other hand, by Lemma 4.3.4, we have $(\mathcal{O}_F^\times)^+ = N_{K/F}(\mathcal{O}_K^\times)$, hence the result follows from Proposition 4.3.7. □

The following proposition is *(ii)* in Theorem 4.3.1.

Proposition 4.3.9. *For every cyclic sextic CM field K with $h_K^* = 1$, up to isomorphism, there exists a* unique *principally polarized simple abelian threefold (A,φ) over $\mathbb{C}$ with $\mathrm{End}(A) \cong \mathcal{O}_K$, and such a principally polarized simple abelian threefold has field of moduli $\mathbb{Q}$.*

Proof. Let CM_K be the set of isomorphism classes of principally polarized simple abelian threefold (A,φ) over $\mathbb{C}$ with $\mathrm{End}(A) \cong \mathcal{O}_K$.

We claim that for every primitive CM type Φ of K, there is a bijection between CM_Φ and CM_K. Given any $(A,\varphi) \in \mathrm{CM}_K$, there is an embedding $\theta : \mathcal{O}_K \to \mathrm{End}(A)$ for (A,φ). Let Φ' be the CM type of A. Since A is simple, by Theorem 1.4.1, the CM type Φ' is primitive. Then there is a *unique* $\sigma \in \mathrm{Aut}(K)$ such that $\Phi = \Phi'\sigma$, see the proof of Proposition 3.3.2. So we have $(A,\theta\sigma,\varphi) \in \mathrm{CM}_\Phi$ and this proves the claim.

Since CM_Φ and CM_K are bijective and $|\mathrm{CM}_\Phi| = 1$ by Proposition 4.3.7, we have $|\mathrm{CM}_K| = 1$. This means that for every cyclic sextic CM field K with $h_K^* = 1$, there exists a unique principally polarized simple abelian threefold of type (K,Φ) and every representative (A,φ) of the isomorphism class in CM_K satisfies $(A,\varphi) \cong (\sigma A, \sigma\varphi)$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. Hence $M_{\mathbb{Q}} = \mathbb{Q}$. □

We now suppose that K is a cyclic sextic CM field in Table 3.1 with $h_K^* = 4$ and prove (iii) in Theorem 4.3.1.

Lemma 4.3.10. *Let P be a principally polarized simple abelian threefold over $\mathbb{C}$ that has CM by the maximal order of a CM class number one sextic CM field K. Then we have*

$$({}^{\rho}P : P) = 1,$$

where ρ is complex conjugation.

Proof. Let Φ be a primitive CM type of K and let P be of type $(K,\Phi,t,\mathfrak{m})$, see page 14. By Proposition 3.5.5 in Lang [20], the principally polarized simple abelian threefold ${}^{\rho}P$ is of type $(K,\Phi,t,\overline{\mathfrak{m}})$. Then by (1.5.5), we have $({}^{\rho}P:P)=[(1,\mathfrak{m}/\overline{\mathfrak{m}})]$. So we get $({}^{\rho}P:P)=1$ if and only if $\mathfrak{m}/\overline{\mathfrak{m}}\in P_K$ and is generated by an $\alpha\in K^{\times}$ with $\alpha\overline{\alpha}=1$.

Since all the fields K in Table 3.1 satisfy $I_0(\Phi^r)=I_{K^r}$, by Proposition 3.3.5, we have $I_K=I_K^H P_K$, where $I_K^H=\{\mathfrak{b}\in I_K \mid \overline{\mathfrak{b}}=\mathfrak{b}\}$. So there is $\mathfrak{a}\in I_K^H$ and $(\beta)\in P_K$ such that $\mathfrak{m}=\mathfrak{a}(\beta)$. Then it follows that $\mathfrak{m}/\overline{\mathfrak{m}}=(\beta/\overline{\beta})=:(\alpha)$ and $\alpha\overline{\alpha}=1$ since we have $\mathrm{N}_{K/F}(\mathcal{O}_K^{\times})=(\mathcal{O}_F^{\times})^{+}$ by Lemmata 3.2.2 and 4.3.4. So we get $({}^{\rho}P:P)=1$. □

Recall $\mathfrak{C}_K:=(F_{\gg 0}\times I_K)/\{(x\overline{x},x\mathcal{O}_K):x\in K^{\times}\}$ and $\mathfrak{C}'_K=\{[(b,\mathfrak{c})]\in\mathfrak{C}_K:\mathfrak{c}\overline{\mathfrak{c}}=b\mathcal{O}_F\}$.

Lemma 4.3.11. *Let K be a cyclic sextic CM field with $h_K^*=4$ and let Φ be a primitive CM type of K. Let F be the totally real cubic subfield of K. Let $p\mathcal{O}_F=\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ and $\mathfrak{p}_i\mathcal{O}_K=\mathfrak{P}_i^2$. Suppose $I_0(\Phi^r)=I_{K^r}$.*

Then there is $t_i\in F_{\gg 0}$ such that $\mathfrak{p}_i^{h_F}=t_i\mathcal{O}_F$ for each $i\in\{1,2,3\}$ and we have

$$\mathfrak{C}'_K=\{[(1,\mathcal{O}_K)]\}\cup\{[(t_i,\mathfrak{P}_i^{h_F})]:i=1,2,3\}$$

of order 4.

Proof. The Galois group $\mathrm{Gal}(K/\mathbb{Q})$ acts on $\{\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3\}$ transitively, hence they all have the same order in Cl_K.

By the assumption $I_0(\Phi^r)=I_{K^r}$, Proposition 3.3.5 implies $I_K=I_K^H P_K$. By Table 3.1, the class number h_F is odd, so the group Cl_F injects into Cl_K. Therefore, we have

$$\mathrm{Cl}_K=I_K^H P_K/P_K=\langle \mathrm{Cl}_F,[\mathfrak{P}_1],[\mathfrak{P}_2],[\mathfrak{P}_3]\rangle.$$

Since $h_K^*=4$, the order of $\langle[\mathfrak{P}_1],[\mathfrak{P}_2],[\mathfrak{P}_3]\rangle$ is divisible by 4. Hence $[\mathfrak{P}_i]$ has even order in Cl_K.

By Lemma 4.3.4, there is $t_i\in F_{\gg 0}$ such that $\mathfrak{p}_i^{h_F}=t_i\mathcal{O}_F$. Hence we have $[(t_i,\mathfrak{P}_i^{h_F})]\in\mathfrak{C}'_K$ as $\mathrm{N}_{K/F}(\mathfrak{P}_i^{h_F})=\mathfrak{p}_i^{h_F}=t_i\mathcal{O}_F$. By Proposition 3.3.1, under the assumption $I_0(\Phi^r)=I_{K^r}$, the imaginary quadratic field $k\subset K$

has class number one. Hence there is only one ramified prime in k, say p. Then we have $\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3 = \sqrt{-p}\mathcal{O}_K$ if $k = \mathbb{Q}(\sqrt{-d})$ with $d \neq 1$ or $\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3 = (1+i)\mathcal{O}_K$ otherwise. Since $[\mathfrak{P}_i]$ has even order, no two of $[\mathfrak{P}_i^{h_F}]$ for $i \in \{1,2,3\}$ are the same class in Cl_K and none are the trivial class. Moreover, in the proof of Proposition 4.3.7, we proved $|\mathfrak{C}'_K| = h^*_K$. Therefore we get $\mathfrak{C}'_K = \{[(1,\mathcal{O}_K)]\} \cup \{[(t_i,\mathfrak{P}_i^{h_F})] : i = 1,2,3\}$ of order 4.

□

The following proves *(iii)* in Theorem 4.3.1.

Proposition 4.3.12. *Let K be a cyclic sextic CM field with $h^*_K = 4$ and let Φ be a primitive CM type of K. Let F be the totally real cubic subfield of K. Suppose $I_0(\Phi^r) = I_{K^r}$. Then there are four isomorphism classes of principally polarized simple abelian threefolds over $\mathbb{C}$.* Exactly *one of these classes has $\mathbb{Q}$ as the field of moduli and the other three have field of moduli F.*

Proof. Let CM_Φ be the isomorphism classes of principally polarized *simple* abelian threefolds over $\mathbb{C}$ of type (K,Φ) with CM by $\mathcal{O}_K$. The set CM_Φ is not empty by Corollary 4.3.6 and, indeed, by Proposition 4.3.7 it holds that $|\mathrm{CM}_\Phi| = 4$.

We first prove that there is *at least* one isomorphism class in CM_Φ with field of moduli $\mathbb{Q}$. Then we prove there is *only* one such isomorphism class in CM_Φ.

Let $\mathrm{Aut}(\mathbb{C})$ act on CM_Φ with $(\sigma,[P]) \mapsto [{}^\sigma P]$. Under the assumption $I_0(\Phi^r) = I_{K^r}$, Theorem 1.5.6 implies that for each $[P] \in \mathrm{CM}_\Phi$, we have $M_\mathbb{Q} \subset K^r \cong K$. Identify K with K^r. Then Theorem 4.2.1-(i) tells us $M_K = K$. Moreover, Lemma 4.3.10 says that ρ acts trivially on CM_Φ and hence $G' = \mathrm{Gal}(K/\mathbb{Q})/\langle\rho\rangle$ acts on CM_Φ. Since $|G'| = 3$, by the orbit-stabilizer theorem (see Lang [22, Proposition 5.5.1 in I]) the size of each orbit is 1 or 3. This means that the action is either trivial or has one orbit of length 1 and one orbit of length 3. This implies that there is *at least* one isomorphism class in CM_Φ with field of moduli $\mathbb{Q}$. We will now show that the action of G' on CM_Φ is not trivial, in other words, we will prove that there is *only* one isomorphism class in CM_Φ with field of moduli $\mathbb{Q}$.

Suppose that G' acts on CM_Φ trivially. This implies that the field of moduli $M_\mathbb{Q}$ of each $[P] \in \mathrm{CM}_\Phi$ is $\mathbb{Q}$ and P is fixed by $\mathrm{Gal}(M_K/\mathbb{Q})$. Hence for every $[P] \in \mathrm{CM}_\Phi$, we have $({}^\sigma P : P) = 1$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. Let $[P']$ be an element of CM_Φ such that $P \ncong P'$. Then by Proposition 1.5.2 and (1.5.6), there is a non-trivial $[(b, \mathfrak{c})] \in \mathfrak{C}'_K$ such that $(P' : P) = [(b, \mathfrak{c})]$.

Let $\mathrm{Gal}(K/\mathbb{Q}) = \langle y \rangle$ and let $\sigma \in \mathrm{Aut}(\mathbb{C})$ be such that $\sigma|_K = y$. Then by Proposition 4.2.5-(iii), we have

$$({}^\sigma P' : P') = [((yb)b^{-1}, (y\mathfrak{c})\mathfrak{c}^{-1})] \in \mathfrak{C}'_K.$$

Therefore, we have ${}^\sigma P' \cong P'$ if and only if $(y\mathfrak{c})/\mathfrak{c}$ is a principal ideal.

Let $p\mathcal{O}_F = \mathfrak{p}_1^2\mathfrak{p}_2^2\mathfrak{p}_3^2$ and $\mathfrak{p}_i\mathcal{O}_K = \mathfrak{P}_i^2$. Then by Lemma 4.3.11, we have

$$\mathfrak{C}'_K = \{[(1, \mathcal{O}_K)]\} \cup \{[(t_i, \mathfrak{P}_i^{h_F})] : i = 1, 2, 3\}$$

Since $\mathrm{Gal}(K/\mathbb{Q}) = \langle y \rangle$ acts on $\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3\}$ transitively, without loss of generality we have

$$y\mathfrak{P}_1 = \mathfrak{P}_2, y\mathfrak{P}_2 = \mathfrak{P}_3, \text{and}\, y\mathfrak{P}_3 = \mathfrak{P}_1.$$

So by Lemma 4.3.11, we have $[(t_2t_1^{-1}, (\mathfrak{P}_2\mathfrak{P}_1^{-1})^{h_F})] \neq 0$. Hence ${}^\sigma P'$ is not isomorphic to P' for some $\sigma \in \mathrm{Aut}(\mathbb{C})$. This proves that G' does not act on CM_Φ trivially.

This proves that for exactly one isomorphism class in CM_Φ, we have $M_\mathbb{Q} = \mathbb{Q}$. Moreover, since ρ acts trivially on CM_Φ, the other three isomorphism classes in CM_Φ have field of moduli $F^r = F$. □

4.4 Genus-3 CM curve examples over $\mathbb{Q}$

In this section we give some examples.

Example 4.4.1. *The curve*

$$C : y^2 = x^7 + 1$$

has CM by $\mathbb{Z}[\zeta_7]$ via $\zeta_7(x, y) = (\zeta_7 x, y)$ of type $\Phi = \{1, \overline{3}, \overline{3}^2\} \subset (\mathbb{Z}/7\mathbb{Z})^\times$. It is defined over $\mathbb{Q}$. See (II) on page 76 in Shimura [38]

Example 4.4.2. *The curve*

$$C : y^3 = x^4 - x$$

is a Picard curve which has CM by $\mathbb{Z}[\zeta_9]$ *via* $\zeta_9(x,y) = (\zeta_9^3 x, \zeta_9 y)$ *of type* $\Phi = \{1, \overline{2}, \overline{2}^2\} \subset (\mathbb{Z}/9\mathbb{Z})^\times$. *It is defined over* $\mathbb{Q}$. *See Lemma 5.1-(a) in [8].*

Example 4.4.3. *The curve*

$$C : y^2 = (x^3 - x^2 - 2x + 1)^2 x - 2x$$

is a hyperelliptic curve over $\mathbb{Q}$ *with CM by* $\mathbb{Z}[\zeta_7 + \zeta_7^{-1}, i]$ *and a primitive CM type* $\Phi = \{(\overline{0}, \overline{1}), (\overline{1}, \overline{2}), (\overline{0}, \overline{3})\} \subset \mathrm{Gal}(K/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/7\mathbb{Z})^\times / \langle \pm 1 \rangle$. *See Proposition 4 in Tautz, Top and Verberkmoes [44].*

Example 4.4.4.

(i) For hyperelliptic curves corresponding to the fields $K = F(i)$, *where* $F \cong \mathbb{Q}[X]/(p(X))$ *and*

$$p(X) \in \{X^3 - 3X - 1,\ X^3 + 2X^2 - 5X + 1,\ X^3 + X^2 - 2X - 1,\ X^3 + X^2 - 10X - 8\},$$

models have been computed that are correct up to some precision over $\mathbb{C}$. *These models are defined over* $\mathbb{Q}$, *see Weng [48].*

Note that the sextic CM field $K = F(i)$ *with* $F \cong \mathbb{Q}[X]/(p(X))$, *where* $p(X) = X^3 + X^2 - 2X - 1$ *corresponds to the CM curve in Example 4.4.3.*

(ii) For Picard curves corresponding to the fields $K = F(\zeta_3)$, *where* $F \cong \mathbb{Q}[X]/(p(X))$ *and*

$$p(X) \in \{X^3 + X^2 - 4X + 1,\ X^3 + X^2 - 2X - 1,\ X^3 + X^2 - 10X - 8,\ X^3 - 3X - 1,\ X^3 + X^2 - 14X + 8\}$$

models have been computed that are correct up to some precision over $\mathbb{C}$. *These models are defined over* $\mathbb{Q}$, *see Koike–Weng [19].*

Note that the sextic CM field $K = F(\zeta_3)$ with $F \cong \mathbb{Q}[X]/(p(X))$, where $p(X) = X^3 - 3X - 1$ corresponds to the CM curve in Example 4.4.2.

Example 4.4.5. *Let C be a Picard curve defined over a field k_0 with* $\mathrm{char}(k_0) \neq 2$ *and* 3. *Without loss of generality, we may assume that C is given by*

$$C : y^3 = x^4 + g_2 x^2 + g_3 x + g_4, \text{ where } g_i \in k_0.$$

If $g_2 g_3 \neq 0$, then C is defined over the field of moduli, see Koike–Weng [19, page 504].