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Chapter 3

The CM class number one problem for curves of genus 3

ABSTRACT. In this chapter, we give the complete list of CM class number one Galois sextic fields, and under GRH, the complete list of CM class number one non-normal sextic CM fields containing an imaginary quadratic field. We will see in Chapter 4 that the first list is the complete list of CM fields corresponding to CM curves of genus 3 *with rational field of moduli.*

3.1 Introduction

Let K be a sextic CM field and let Φ be a primitive CM type of K. Further, let C be a curve of genus 3 with a simple Jacobian $J(C)$ over $\mathbb{C}$ of type (K,Φ) with CM by $\mathcal{O}_K$. Let (K^r,Φ^r) be the reflex of (K,Φ). Recall that Theorem 1.5.6 implies that if C is defined over the reflex field K^r, then

$$\begin{aligned} I_0(\Phi^r) &:= \{\mathfrak{b} \in I_{K^r} : \mathrm{N}_{\Phi^r}(\mathfrak{b}) = (\alpha) \text{ and } \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b}) = \alpha\overline{\alpha} \text{ for some } \alpha \in K^\times\} \\ &= I_{K^r}. \end{aligned}$$

We say that K is a *CM class number one field* if there exists a primitive CM type Φ such that (K,Φ) has $I_0(\Phi^r) = I_{K^r}$. In this chapter, we will list CM class number one sextic fields by using a similar strategy as in Chapter 2.

We will restrict ourselves to the fields K that contain an imaginary quadratic field. This restriction is not too bad because it covers the most interesting cases:

- CM fields with known explicit CM constructions of genus-3 CM curves:
  - hyperelliptic curves with $K \supset \mathbb{Q}(i)$, Weng [48],
  - Picard curves, Koike and Weng [19],
- all cases where K is Galois over $\mathbb{Q}$ see Section 3.3,
- all CM curves of genus 3 over $\mathbb{Q}$ (see Chapter 4).

Remark 3.1.1. Dodson [12, Section 5.1.1] gives the Galois groups of the remaining sextic CM fields K, i.e., those that do not contain an imaginary quadratic field. For those fields, $K/\mathbb{Q}$ is not normal and the Galois group of a normal closure of K is isomorphic to $(C_2)^3 \rtimes G_0$, where $G_0 \in \{C_3,\ S_3\}$ acts on $(C_2)^3$ by permuting the indices.

Let K be a sextic CM field containing an imaginary quadratic field k and let F be the totally real cubic subfield of K. Since k and F are linearly disjoint over $\mathbb{Q}$, we have $K = Fk$. Totally real cubic fields F are either cyclic or non-normal over $\mathbb{Q}$. If F is non-normal, then the normal closure N_+ of F is a totally real field with Galois group isomorphic to S_3. Let N be a Galois closure of K. Then we have $\mathrm{Gal}(N/\mathbb{Q}) = \mathrm{Gal}(N_+/\mathbb{Q}) \times \mathrm{Gal}(k/\mathbb{Q})$, see Lang [22, Theorem 1.14 in IV]. In particular, we have either $\mathrm{Gal}(N/\mathbb{Q}) = C_3 \times C_2 \cong C_6$ or $\mathrm{Gal}(N/\mathbb{Q}) = S_3 \times C_2 \cong D_6$. Note that N_+ is the maximal totally real subfield of N, whence our notation.

In Section 3.3, we will consider the case that $K/\mathbb{Q}$ is Galois and prove the following.

Theorem 3.1.2. *There exist exactly* 37 *isomorphism classes of cyclic sextic CM fields K such that there exists a primitive CM type Φ satisfying $I_0(\Phi^r) = I_{K^r}$. These fields are exactly the fields listed in Table 3.1.*

In Section 3.4, non-normal sextic CM fields will be considered and the following will be proved.

Theorem 3.1.3. *Assuming GRH, the complete list of the isomorphism classes of CM class number one non-normal sextic CM fields containing an imaginary quadratic field is given in Tables 3.3–3.12.*

3.2 Sextic CM fields containing an imaginary quadratic field

The main result of this section is the following. We will use this result in Sections 3.3 and 3.4 to prove Theorems 3.1.2 and 3.1.3.

Proposition 3.2.1. *Let K be a sextic CM field and let Φ be a CM type of K. Let (K^r, Φ^r) be the reflex of (K, Φ). Suppose that $K^r \cong K$. If K contains a class number one imaginary quadratic field k and $h_K^* = 2^{t_K - 1}$, then*

$$\begin{aligned} I_0(\Phi^r) &:= \{\mathfrak{b} \in I_{K^r} : \mathrm{N}_{\Phi^r}(\mathfrak{b}) = (\alpha) \text{ and } \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b}) = \alpha\overline{\alpha} \text{ for some } \alpha \in K^\times\} \\ &= I_{K^r}. \end{aligned}$$

Lemma 3.2.2. *Let K be a sextic CM field containing an imaginary quadratic field k. Then we have*

$$\mathcal{O}_K^\times = W_K \mathcal{O}_F^\times,$$

where W_K is the group of roots of unity of K.

Proof. This follows from Theorem 5-(i) in Louboutin, Okazaki, Olivier [27]. □

Lemma 3.2.3. *Let K be a sextic CM field containing an imaginary quadratic field k and Φ be a CM type of K. Let F be totally cubic subfield of K. Put $I_K^H = \{\mathfrak{b} \in I_K \mid \overline{\mathfrak{b}} = \mathfrak{b}\}$, where $H := \mathrm{Gal}(K/F)$. Then we have*

$$h_K^* = 2^{t_K - 1}[I_K : I_K^H P_K].$$

Proof. Lemma 2.2.2 tells us that if $\mathcal{O}_K^\times = W_K \mathcal{O}_F^\times$, then $h_K^* = 2^{t_K - 1}[I_K : I_K^H P_K]$. Combine this with Lemma 3.2.2. □

Proof of Proposition 3.2.1. Identify K with K^r via an isomorphism. By Lemma 3.2.3, we have

$$h_K^* = 2^{t_K - 1} \text{ if and only if } I_K = I_K^H P_K.$$

For any $\mathfrak{b} \in \mathrm{I}_F$, we have $\mathrm{N}_{\Phi^r}(\mathfrak{b}) = (\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{b}))$, where $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{b}) \in \mathbb{Z}$. Hence $I_F P_K \subset I_0(\Phi^r)$. We can see from the exact sequence (2.2.1)

$$1 \to I_F \to I_K^H \to \bigoplus_{\mathfrak{p} \text{ prime of } F} \mathbb{Z}/e_{K/F}(\mathfrak{p})\mathbb{Z} \to 1$$

that the elements of I_K^H / I_F are represented by the products of the primes in K that are ramified in K/F. For any such prime $\mathfrak{P}$, let $\mathfrak{p}\mathbb{Z} = \mathfrak{P} \cap F$ and $p = \mathfrak{p} \cap \mathbb{Q}$. Then the following holds

$$\mathrm{N}_{\Phi^r}(\mathfrak{P})^2 = \mathrm{N}_{\Phi^r}(\mathfrak{p}\mathcal{O}_K) = \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})\mathcal{O}_K, \qquad (3.2.1)$$

where $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p}) \in \{p,\ p^2, p^3\}$ depending on the splitting behavior of p in F.

The prime $\mathfrak{P}$ lies over a rational prime p that is ramified in k, see Lang [21, Proposition 4.8-(ii) in II]. Moreover, the prime p is the unique ramified prime in $k/\mathbb{Q}$. Indeed, by genus theory, if the class number of an imaginary quadratic field k is odd then there is one and only one ramified prime in $k/\mathbb{Q}$. By (3.2.1), we have

$$\mathrm{N}_{\Phi^r}(\mathfrak{P}) = \sqrt{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})\mathcal{O}_K}. \qquad (3.2.2)$$

If $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p}) = p$, then the right hand side of (3.2.2) is generated by $\sqrt{-p}$ if $k \ncong \mathbb{Q}(i)$ and generated by $i+1$ if $k \cong \mathbb{Q}(i)$. Therefore, in both cases we have a generator π in k of $\mathrm{N}_{\Phi^r}(\mathfrak{P})$ such that $\pi\overline{\pi} \in \mathbb{Q}$. Similarly, in cases $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p}) = p^2$ or p^3, there exists a generator π in k of $\mathrm{N}_{\Phi^r}(\mathfrak{P})$ such that $\pi\overline{\pi} \in \mathbb{Q}$.

Hence every element of $I_K^H P_K$, which is I_K, is in $I_0(\Phi^r)$. In particular, we get $I_K = I_0(\Phi^r)$. □

3.3 Cyclic sextic CM fields

In this section, we will prove Theorem 3.1.2. We begin with proving the following proposition which is the main ingredient of the proof of Theorem 3.1.2.

Proposition 3.3.1. *Let K be a cyclic sextic CM field with a primitive CM type Φ. It holds $I_0(\Phi^r) = I_{K^r}$ if and only if $h_K^* = 2^{t_K - 1}$ and $h_k = 1$, where t_K is the number of primes in F that are ramified in K.* □

Let K be a cyclic sextic CM field with $G := \mathrm{Gal}(K/\mathbb{Q}) = \langle y \rangle$. In this notation, complex conjugation $\bar{\cdot}$ is y^3. Then K has a totally real cubic subfield F and an imaginary quadratic subfield k. So $K = kF$.

Proposition 3.3.2. *Given (K, Φ) such that K is a cyclic sextic CM field and Φ is a primitive CM type of K with values in N'. There is an embedding $K \hookrightarrow N'$ such that Φ is $\{\mathrm{id},\ y,\ y^{-1}\}$. Moreover, we then have $K^r = K$ and $\Phi^r = \Phi$.*

Proof. There are 2^3 CM types of K with values in N'. Two of them are induced by the CM types of k and the remaining six are primitive. For simplicity, we consider CM types with values in K. In other words, we identify K with a subfield of N'. The CM type $\{\mathrm{id},\ y,\ y^{-1}\}$ of K is primitive by Corollary 1.2.4 and if we translate this type with the elements of $\mathrm{Gal}(K/\mathbb{Q})$, we get six equivalent primitive CM types. Hence by changing the embedding $K \hookrightarrow N'$, without loss of generality, we have $\Phi = \{\mathrm{id},\ y,\ y^{-1}\}$.

Since K is normal and Φ is primitive, the reflex field K^r is K. Moreover, the reflex type Φ^r is $\Phi^{-1} = \Phi$. □

We now prove the converse of Proposition 3.2.1 for cyclic sextic CM fields.

Proposition 3.3.3. *Let K be a cyclic sextic CM field with a primitive CM type Φ. Suppose $I_0(\Phi^r) = I_{K^r}$. Then we have $h_k = 1$.*

Proof. As K has degree 6, the order μ_K of the group of roots of unity W_K of K is 2, 4, 6, 14, or 18 and it is greater than 2 only if $k = \mathbb{Q}(\sqrt{-d})$

with $d = 1$ 3, 7, or 3 respectively, so in that case we are done as $h_k = 1$ if $d = 1$ 3, or 7. We now suppose $W_K = \{\pm 1\}$.

For any $\mathfrak{a} \in I_k$, we have

$$\mathrm{N}_{\Phi^r}(\mathfrak{a}\mathcal{O}_K) = \mathfrak{a}\overline{\mathfrak{a}}^2\mathcal{O}_K = \overline{\mathfrak{a}}\mathrm{N}_{k/\mathbb{Q}}(\mathfrak{a})\mathcal{O}_K.$$

Then by the assumption $I_0(\Phi^r) = I_K$, we have $\mathfrak{a}\mathrm{N}_{k/\mathbb{Q}}(\mathfrak{a})\mathcal{O}_K = \pi\mathcal{O}_K$ for some $\pi \in K^\times$ such that $\pi\overline{\pi} \in \mathbb{Q}$. Let $\nu = \pi/\mathrm{N}_{k/\mathbb{Q}}(\mathfrak{a})$. Then $\nu\overline{\nu} \in \mathbb{Q}$ and $\nu\mathcal{O}_K = \mathfrak{a}\mathcal{O}_K$. This makes ν unique up to a root of unity hence up to a sign.

The map $\phi : \mathrm{Gal}(K/k) \to \{\pm 1\}$ given by $\phi(\sigma) = \sigma(\nu)/\nu$ is a homomorphism. Since the order of $\mathrm{Gal}(K/k)$ is 3, the map ϕ is trivial. Hence $\sigma(\nu) = \nu$ for every $\sigma \in \mathrm{Gal}(K/k)$. This implies $\nu \in k^\times$ and hence $\mathfrak{a}$ is principal in k. As $\mathfrak{a}$ was arbitrary, we then have $h_k = 1$.

□

Remark 3.3.4. It is known that the imaginary quadratic fields with *class number one* are $k = \mathbb{Q}(\sqrt{-d})$ with $d \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$ (see Baker, Heegner, and Stark [2, 16, 41]).

Proposition 3.3.5. *Let K be a cyclic sextic CM field and let Φ be a primitive CM type of K. Suppose $I_0(\Phi^r) = I_{K^r}$. Then we have $h_K^* = 2^{t_K - 1}$, where t_K is the number of primes in F that are ramified in K.*

Proof. Recall that without loss of generality we have $\Phi = \{\mathrm{id},\ y,\ y^{-1}\}$ and $K^r = K$. By Lemma 3.2.3, we have $h_K^* = 2^{t_K-1}[I_K : I_K^H P_K]$, where $I_K^H = \{\mathfrak{b} \in I_K \mid \overline{\mathfrak{b}} = \mathfrak{b}\}$. So it is enough to show $[I_K : I_K^H P_K] = 1$ under the assumption $I_0(\Phi^r) = I_{K^r}$. For any $\mathfrak{b} \in I_K$, we have the following equality

$$\mathrm{N}_{\Phi^r}(y^{-1}(\mathfrak{b})/y^{-2}(\mathfrak{b})) = \mathfrak{b}\overline{\mathfrak{b}}^{-1}.$$

By the assumption $I_0(\Phi^r) = I_{K^r}$, we get $\mathfrak{b}\overline{\mathfrak{b}}^{-1} = (\beta)$, where $\beta \in K^\times$ and $\beta\overline{\beta} \in \mathbb{Q}$. Here we have $(\beta\overline{\beta}) = (\mathrm{N}_{K^r/\mathbb{Q}}((y^{-1}\mathfrak{b})/(y^{-2}\mathfrak{b}))) = (1)$, by the property of the type norm in Proposition 1.2.6 hence $\beta\overline{\beta} = 1$.

Then the proof continues as in the proof of Lemma 2.2.3: there is a $\gamma \in K^\times$ such that $\beta = \overline{\gamma}\gamma^{-1}$ by Hilbert's Theorem 90. Thus we have $\mathfrak{b} = \overline{\gamma\mathfrak{b}} \cdot (\frac{1}{\gamma}) \in I_K^H P_K$ and therefore $I_K = I_K^H P_K$. □

Proof of Proposition 3.3.1. It follows from Propositions 3.2.1, 3.3.2, 3.3.3, and 3.3.5. □

Theorem 3.1.2. *There exist exactly* 37 *isomorphism classes of cyclic sextic CM fields* K *such that there exists a primitive CM type* Φ *satisfying* $I_0(\Phi^r) = I_{K^r}$*. These fields are exactly the fields listed in Table 3.1.*

Proof. By Proposition 3.3.1, we have

$$I_0(\Phi^r) = I_{K^r} \text{ if and only if } h_K^* = 2^{t_K - 1} \text{ and } h_k = 1.$$

If $h_k = 1$, then there is only one ramified prime in k, say p. Under the assumption $I_0(\Phi^r) = I_{K^r}$, we have $h_k = 1$ and hence all ramified primes in K/F lie over p (see Proposition 4.8-(ii) in II of Lang [21]). So if $I_0(\Phi^r) = I_{K^r}$, then we have $t_K \leq 3$ and therefore, we get $h_K^* = 2^{t_K - 1} \leq 2^2$. Thanks to Park and Kwon [34], we have the list of cyclic sextic CM fields with $h_K^* \leq 4$. And of all the sextic cyclic CM fields listed in Table 3 in Kwon and Park [34], those that satisfy $h_K^* = 2^{t_K - 1}$ and $h_k = 1$ are listed in Table 3.1. □

In Table 3.1, K is a cyclic sextic CM field that contains an imaginary quadratic field k, and F is the totally real cubic subfield that is defined as being the splitting field of an irreducible monic polynomial $p(X)$. Furthermore, d_k is the absolute value of the discriminant of k and h_F is the class number of F. In column C some CM fields have $*$, this indicates that a rational model of the corresponding CM curve is known, see Section 4.4.

Table 3.1: All CM class number one cyclic sextic CM fields

$h_K^* = 1$							
d_k	$p(X)$	h_F	C	$\lvert d_k \rvert$	$p(X)$	h_F	C
3	X^3+X^2-4X+1	1	*	7	X^3-3X-1	1	
3	X^3+X^2-2X-1	1	*	7	$X^3+8X^2-51X+27$	3	
3	X^3-3X-1	1	*	7	X^3+6X^2-9X+1	3	
3	$X^3+X^2-10X-8$	1	*	7	$X^3+X^2-30X+27$	3	
3	$X^3+X^2-14X+8$	1	*	7	$X^3+4X^2-39X+27$	3	
3	$X^3+3X^2-18X+8$	3		8	X^3+X^2-4X+1	1	
3	X^3+6X^2-9X+1	3		8	X^3+X^2-2X-1	1	
3	$X^3+3X^2-36X-64$	3		11	X^3+X^2-2X-1	1	
4	X^3+2X^2-5X+1	1	*	19	X^3+2X^2-5X+1	1	
4	X^3-3X-1	1	*	19	$X^3+9X^2-30X+8$	3	
4	X^3+X^2-2X-1	1	*	19	$X^3+7X^2-66X-216$	3	
7	X^3+X^2-4X+1	1		43	$X^3+X^2-14X+8$	1	
7	X^3+X^2-2X-1	1	*	67	$X^3+2X^2-21X-27$	1	

$h_K^* = 4$							
d_k	$p(X)$	h_F	C	d_k	$p(X)$	h_F	C
3	$X^3+4X^2-15X-27$	1		7	X^3+2X^2-5X+1	1	
3	$X^3+2X^2-21X-27$	1		8	$X^3+X^2-10X-8$	1	
4	$X^3+X^2-14X+8$	1		11	$X^3+X^2-14X+8$	1	
4	$X^3+X^2-10X-8$	1	*	11	X^3+2X^2-5X+1	1	
4	$X^3+3X^2-18X+8$	3		19	X^3-3X-1	1	
7	$X^3+X^2-24X-27$	1					

3.4 Non-normal sextic CM fields

In this section, we will prove Theorem 3.1.3. As in Section 3.3, we begin with proving the following proposition, which is the analogue of Proposition 3.3.1 in the case of non-normal sextic CM fields.

Proposition 3.4.1. *Let K be a non-normal sextic CM field containing an imaginary quadratic field k. Let Φ be a primitive CM type of K. Let F be the totally real cubic subfield of K. Then $I_0(\Phi^r) = I_{K^r}$ holds if and only if $h_K^* = 2^{t_K-1}$ and $h_k = 1$, where t_K is the number of primes in F that are ramified in K.*

Let K be a non-normal sextic CM field containing $\mathbb{Q}(\sqrt{-d})$, where $d \in \mathbb{Z}_{>0}$. The normal closure N of K is a dihedral CM field of degree 12 with Galois group $G := \mathrm{Gal}(N/\mathbb{Q}) = \langle x,\ y : y^6 = x^2 = 1,\ xyxy = 1\rangle$, where $\langle x\rangle$ fixes K. The complex conjugation $\bar{\cdot}$ is y^3 in this notation. The normal closure of F is the maximal totally real subfield N_+ of N, which is fixed by $\langle y^3\rangle$. We have the following diagram of some of the subfields of N.

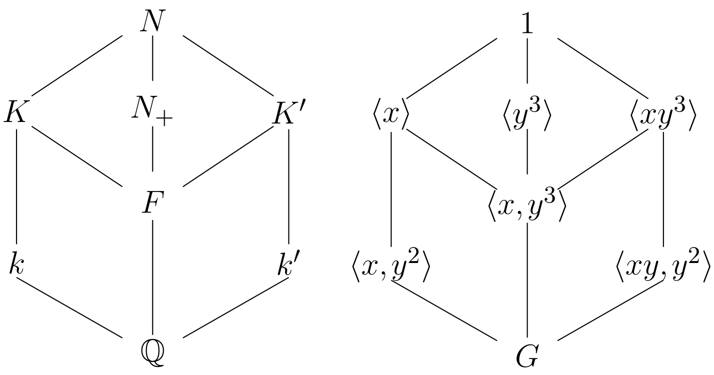

Figure 3.1: Some subfields and subgroups

The extension N/F is biquadratic, and the extensions N/k, N/k' and $N_+/\mathbb{Q}$ are dihedral Galois of degree 6.

Let N' be a number field that contains a subfield isomorphic to N.

Proposition 3.4.2. *Given* (K, Φ) *such that* K *is a non-normal sextic CM field that contains an imaginary quadratic subfield and* Φ *is a primitive CM type of* K *with values in* N'*. There is an embedding* $N \hookrightarrow N'$ *such that* Φ *is* $\{\mathrm{id},\ y|_K,\ y^{-1}|_K\}$*. Moreover, the reflex field* K^r *is* K *and the reflex type* Φ^r *is* Φ*.*

Proof. There are 2^3 CM types of K with values in N'. Two of them are induced by the CM types of k. The remaining six are primitive. For simplicity, we consider CM types with values in N. In other words, we identify N with a subfield of N'. The CM type $\{\mathrm{id},\ y|_K,\ y^{-1}|_K\}$ of K is primitive by Proposition 1.2.3 and if we translate this type with the elements of the unique cyclic subgroup of order 6, we get six equivalent primitive CM types. Hence by changing the embedding $N \hookrightarrow N'$ by an appropriate power of y, without loss of generality, we have $\Phi = \{\mathrm{id},\ y|_K,\ y^{-1}|_K\}$.

The CM type $\Phi_N = \{\mathrm{id}, y, y^{-1}, x, xy, xy^{-1}\}$ of N is induced by the CM type Φ. By definition (see page 5), the reflex field K^r is the fixed field of $\{\gamma : \gamma \in \mathrm{Gal}(N/\mathbb{Q}),\ \Phi_N^{-1}\gamma = \Phi_N^{-1}\} = \langle x\rangle$. Hence the reflex field K^r is K and the reflex type Φ^r is Φ. □

Proposition 3.4.3. *Let K be a non-normal sextic CM field containing an imaginary quadratic field and let Φ be a primitive CM type of K. Suppose $I_0(\Phi^r) = I_{K^r}$. Then $h_K^* = 2^{t_K-1}$, where t_K is the number of primes in F that are ramified in K.*

Proof. The idea is similar to the proof of Proposition 3.3.5.

Put $I_K^H = \{\mathfrak{b} \in I_K \mid \overline{\mathfrak{b}} = \mathfrak{b}\}$ and $P_K^H = P_K \cap I_K^H$, where $H := \mathrm{Gal}(K/F)$. Then by Lemma 3.2.3 we have

$$h_K^* = 2^{t_K-1}[I_K : I_K^H P_K].$$

On the other hand, Lemma 2.2.3 tells us the following. If for every $\mathfrak{b} \in I_K$, we have

$$\mathrm{N}_{\Phi^r}\mathrm{N}_{\Phi}(\mathfrak{b}) = (\beta)\mathfrak{b}\overline{\mathfrak{b}}^{-1} \text{ and } \beta\overline{\beta} \in \mathbb{Q} \tag{3.4.1}$$

with $\beta \in K^\times$, then $[I_K : I_K^H P_K] \leq [I_{K^r} : I_0(\Phi^r)]$.

Without loss of generality, we can take $\Phi = \{\mathrm{id},\ y|_K,\ y^{-1}|_K\}$. Then for any $\mathfrak{a} \in I_K$, we have the following equality

$$\mathrm{N}_{\Phi^r}\mathrm{N}_{\Phi}(\mathfrak{a}) = \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})\mathrm{N}_{\Phi^r}(\mathfrak{a})\mathfrak{a}\overline{\mathfrak{a}}^{-1}. \tag{3.4.2}$$

By the assumption $I_0(\Phi^r) = I_{K^r}$, there exists $\alpha \in K^\times$ such that $\mathrm{N}_{\Phi^r}(\mathfrak{a}) = (\alpha)$ and $\alpha\overline{\alpha} \in \mathbb{Q}$. Moreover, the assumption $I_0(\Phi^r) = I_{K^r}$ also implies that there is a $\beta \in K^\times$ such that $\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})\mathfrak{a}\overline{\mathfrak{a}}^{-1}(\alpha) = (\beta)$ and $\beta\overline{\beta} \in \mathbb{Q}$. So the CM type (K, Φ) satisfies (3.4.1), by the assumption $I_0(\Phi^r) = I_{K^r}$. Therefore, Lemma 2.2.3 implies $[I_K : I_K^H P_K] \leq [I_{K^r} : I_0(\Phi^r)] = 1$. Hence the result follows. □

Proposition 3.4.4. *Let K be a non-normal sextic CM field and let k be an imaginary quadratic field such that K contains k. Let Φ be a primitive CM type of K. Suppose $I_0(\Phi^r) = I_{K^r}$. Then we have either*

(i) $k = \mathbb{Q}(\sqrt{-d})$ with $d \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$ or

(ii) $k = \mathbb{Q}(\sqrt{-d})$ *with* $d \in \{23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907\}$ *and for every* $[\mathfrak{a}] \in \mathrm{Cl}_k$ *of order* 3*, we have* $K = k(\sqrt[3]{\alpha})$ *for all* $\alpha \in k$ *such that* $\mathfrak{a}^3 = (\alpha)$*.*

Proof. By Baker, Heegner, Stark [2, 16, 41], the list *(i)* consists of all imaginary quadratic fields whose class number is 1 and by Arno [1], the list *(ii)* consists of all imaginary quadratic fields whose class number is 3.

Let N, K, F, k, and k' be as in Figure 3.1. Denote the Galois group of $N/\mathbb{Q}$ by G. Without loss of generality, we can take $\Phi = \{\mathrm{id},\ y|_K,\ y^{-1}|_K\}$ and $K^r = K$. Given $\mathfrak{a} \in I_k$, we have

$$\begin{aligned} \mathrm{N}_{\Phi^r}(\overline{\mathfrak{a}}\mathcal{O}_{K^r})\mathcal{O}_N &= (\overline{\mathfrak{a}}\mathcal{O}_N)y(\overline{\mathfrak{a}}\mathcal{O}_N)y^{-1}(\overline{\mathfrak{a}}\mathcal{O}_N) \\ &= \mathfrak{a}^2\overline{\mathfrak{a}}\mathcal{O}_N = \mathfrak{a}\mathrm{N}_{k/\mathbb{Q}}(\mathfrak{a})\mathcal{O}_N. \end{aligned}$$

Then by the assumption $I_0(\Phi^r) = I_K$, we have $\mathfrak{a}\mathrm{N}_{k/\mathbb{Q}}(\mathfrak{a})\mathcal{O}_N = \pi\mathcal{O}_N$ for some $\pi \in K^\times$ such that $\pi\overline{\pi} \in \mathbb{Q}$.

Let $\nu = \pi/\mathrm{N}_{k/\mathbb{Q}}(\mathfrak{a})$. Then $\nu\overline{\nu} \in \mathbb{Q}$ and $\nu\mathcal{O}_N = \mathfrak{a}\mathcal{O}_N$. We would like to imitate the proof of Proposition 3.3.3 and show $\nu \in k^\times$ in order to conclude $h_k = 1$. Since K is the field fixed by $\langle x\rangle$, to show that ν is fixed by $\mathrm{Gal}(N/k) = \langle x,\ y^2\rangle$ it would be enough to prove $y^2\nu = \nu$.

However, we cannot prove this. Instead we will show that if $h_k \neq 1$, then $h_k = 3$ and $K = k(\sqrt[3]{\alpha})$, where $(\alpha) = \mathfrak{a}^3$ for every ideal class $[\mathfrak{a}] \in \mathrm{Cl}_k$ with order 3.

Suppose that $h_k \neq 1$.

Claim 1. The subgroup $\langle y^2\rangle \subset G$ fixes all elements of the group W_N of roots of unity.

Proof. Suppose that $\mu_N = m$. Then $\mathbb{Q}(\zeta_m)$ is contained in the maximal abelian subfield M of N. Since the Galois group of M is $G/[G,G] \cong \langle x,y\rangle/\langle y^2\rangle$, every element in W_N is fixed by $\langle y^2\rangle$. This proves the claim.

The map $\phi : \langle y^2\rangle \to W_N$ given by $\phi(y^2) = (y^2\nu)/\nu$ is a 1-cocyle. But y^2 acts trivially as $y^2|_M$ is the identity map, hence ϕ is a homomorphism. It depends only on $\mathfrak{a}$ because y^2 fixes the elements of W_N. So we denote this homomorphism ϕ by $\phi_{\mathfrak{a}}$.

Claim 2. The map

$$\begin{aligned} \psi : I_k &\to \mathrm{Hom}(\langle y^2\rangle, W_N) \\ \mathfrak{a} &\mapsto \phi_{\mathfrak{a}} \end{aligned}$$

is a homomorphism and induces an injective homomorphism from the class group Cl_k to $\mathrm{Hom}(\langle y^2\rangle, W_N)$.

Proof. For $i = 1, 2$ let $\mathfrak{a}_i$ be a fractional ideal of I_k and ν_i be an element in $K^\times$ such that $\mathfrak{a}_i\mathcal{O}_N = \nu_i\mathcal{O}_N$ and $\nu_i\overline{\nu}_i \in \mathbb{Q}$. We have

$$\begin{aligned}\psi(\mathfrak{a}_1\mathfrak{a}_2)(y^2) = (y^2(\nu_1\nu_2))/\nu_1\nu_2 &= ((y^2\nu_1)/\nu_1)((y^2\nu_2)/\nu_2)\\ &= (\psi(\mathfrak{a}_1)\psi(\mathfrak{a}_2))(y^2),\end{aligned}$$

hence ψ is a homomorphism. Moreover, we clearly have $P_k \subset \ker\psi$. On the other hand, if $\mathfrak{a} \in \ker\psi$, then $\psi(\mathfrak{a})$ is the identity homomorphism and so ν is fixed by y^2. This implies that ν is fixed by $\mathrm{Gal}(N/k)$ as $\nu \in K$ is also fixed by x. So we have $\nu \in k$, and hence $\mathfrak{a} \in P_k$. Then the isomorphism theorem proves the claim.

Since the order of $\langle y^2\rangle$ is 3, the order of the image of ψ is divisible by 3. So h_k divides 3 and since we assumed $h_k \neq 1$, we get $h_k = 3$. For every generator $[\mathfrak{a}]$ of Cl_k, there is $\nu \in K^\times$ and $\alpha \in k$ such that $\mathfrak{a}^3 = \alpha\mathcal{O}_k$, $\nu = \mathfrak{a}\mathcal{O}_K$ and $\nu_i\overline{\nu}_i \in \mathbb{Q}$. Hence $\nu^3 = \alpha$ up to a root of unity in K, so up to ± 1. So $\nu = \sqrt[3]{\alpha} \in K$ and hence we have $K = k(\sqrt[3]{\alpha})$.

Therefore, we proved that the imaginary quadratic field in a sextic non-normal CM field satisfying $I_0(\Phi^r) = I_{K^r}$ is one of the fields in the proposition. □

Proposition 3.4.5. *None of the fields K in (ii) in Proposition 3.4.4 are sextic CM fields with CM class number one.*

Proof. Let k be any of the imaginary quadratic fields in *(ii)* in Proposition 3.4.4. For a generator $[\mathfrak{a}]$ of Cl_k such that $\mathfrak{a}^3 = \alpha\mathcal{O}_k$, we let $K = k(\sqrt[3]{\alpha})$. Let F be the maximal totally real subfield of the CM field K. A direct computation gives $h_K^* \neq 2^{t_K-1}$, therefore, by Proposition 3.4.3, the CM field K is not a CM class number one field. □

Proof of Proposition 3.4.1. If $I_0(\Phi^r) = I_{K^r}$, then by Proposition 3.4.3 we have $h_K^* = 2^{t_K-1}$, and by Propositions 3.4.4 and 3.4.5 we have that the imaginary quadratic fields is as required.

Conversely, if k is one of the imaginary quadratic fields in the proposition and $h_K^* = 2^{t_K-1}$, then by Propositions 3.2.1 and 3.4.2 we have $I_0(\Phi^r) = I_{K^r}$. □

By combining the following two theorems, under GRH, we give a lower bound for the relative class number h_K^* of a non-normal sextic CM field K containing an imaginary quadratic field. Then using this result, under GRH, we give an upper bound on the discriminant of the totally real cubic subfield F of a *CM class number one* non-normal sextic CM field K, see Proposition 3.4.9.

Theorem 3.4.6. *(Louboutin [25, Theorem 2]) If F is a totally real cubic number field, then*

$$\mathrm{Res}_{s=1}(\zeta_F) \leq \frac{1}{8}\log^2 d_F,$$

where ζ_F is the Dedekind zeta function and d_F is the discriminant. □

Theorem 3.4.7. *(J. Oesterlé) For any number field K different from $\mathbb{Q}$ for which the Riemann Hypothesis for the Dedekind zeta function ζ_K holds, we have*

$$\mathrm{Res}_{s=1}(\zeta_K) \geq \frac{e^{-3/2}}{\sqrt{\log|d_K|}} \exp\left(\frac{-1}{\sqrt{\log|d_K|}}\right).$$

Proof. See Theorem 14 in Bessassi [5]. □

Combining these theorems with the analytic class number formula (see page 35), we get the following.

Theorem 3.4.8. *Let K be a non-normal sextic CM field containing an imaginary quadratic field and let F be the totally real cubic subfield of K. Then, under the Riemann Hypothesis (RH) for $\zeta_K(s)$, we have*

$$h_K^* \geq \frac{\mu_K}{e^{3/2}\pi^3} \frac{\sqrt{|d_K|/d_F}}{(\log d_F)^2 \sqrt{\log(|d_K|)}} \exp\left(\frac{-1}{\sqrt{\log|d_K|}}\right), \tag{3.4.3}$$

where μ_K is the order of the group of roots of unity W_K of K.

Proof. Recall $Q_K := [\mathcal{O}_K^\times : W_K \mathcal{O}_F^\times]$. We have (see Washington [46, Chapter 4]):

$$h_K^* = \frac{Q_K \mu_K}{8\pi^3} \sqrt{\frac{|d_K|}{d_F}} \frac{\mathrm{Res}_{s=1}(\zeta_K)}{\mathrm{Res}_{s=1}(\zeta_F)}.$$

If we assume the Riemann Hypothesis for ζ_K, then combining Theorem 3.4.6 and Theorem 3.4.7, we obtain the lower bound (3.4.3). □

Proposition 3.4.9. *Let K be a* CM class number one *non-normal sextic CM field containing an imaginary quadratic field k. Let F be the totally real cubic subfield of K. Then we have $k = \mathbb{Q}(\sqrt{-d})$ with $d \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$ and an upper bound on d_F is given in Table 3.2.*

$\lvert d_k \rvert$	$d_F \leq$
3	$6 \cdot 10^9$
4	$3 \cdot 10^{10}$
7	$1.4 \cdot 10^{10}$
8	$7 \cdot 10^{10}$
11	$8 \cdot 10^9$
19	$4 \cdot 10^9$
43	$1.4 \cdot 10^9$
67	$8 \cdot 10^8$
163	$3 \cdot 10^8$

Table 3.2: Upper bounds on d_F for *CM class number one* non-normal sextic CM fields K containing an imaginary quadratic field k.

Proof. By Proposition 3.4.1, the CM field K contains one of the imaginary quadratic fields in the proposition and $h_K^* = 2^{t_K - 1}$. Since $h_k = 1$ we have $t_k \leq 3$. Moreover, the factor $\exp(-1/\sqrt{\log |d_K|})$ increases monotonically with $|d_K|$ and $|d_K|/\log|d_K|$ increases monotonically with $|d_K| > 2$. Hence for every fixed d_F, the right hand side of (3.4.3) increases monotonically with $|d_K| > e$.

Furthermore, we have $|d_K| = d_F^2 d_r$, where $d_r := |\mathrm{N}_{F/\mathbb{Q}}(\Delta_{K/F})|$. If we replace $|d_K|$ with $d_F^2 d_r$ and divide both sides of (3.4.3) by the constant $\mu_K/(e^{3/2}\pi^3)$, then we get

$$\frac{h_K^* e^{3/2}\pi^3}{\mu_K} \geq \sqrt{\frac{d_r d_F}{(\log d_F)^4 \log(d_r d_F^2)}} \exp\left(\frac{-1}{\sqrt{\log(d_r d_F^2)}}\right). \qquad (3.4.4)$$

For every fixed d_r, each of the two factors of the right hand side of (3.4.4) increases monotonically with $d_F > 103$. Hence the right hand side of (3.4.4) increases monotonically with d_F for every fixed d_r.

Claim. Let p be the prime that ramifies in k. Then we have

(i) $d_r \geq |d_k|$ if $t_K = 1$,

(ii) $d_r \geq \max\{p^2, |d_k|\}$ if $t_K = 2$,

(iii) $d_r \geq |d_k|^3$ if $t_K = 3$.

Proof. By Lemma 20 in Louboutin, Okazaki, Olivier [27], we have that $|d_k|$ divides d_r.

Every prime $\mathfrak{p}$ of F that ramifies in K/F divides $\Delta_{K/F}$. So $\Delta_{K/F}$ has at least t_K prime factors. Hence $\mathrm{N}_{F/\mathbb{Q}}(\Delta_{K/F})$ has at least t_K prime factors with multiplicity. And these factors all must be equal to p because it is the unique prime that ramifies in $k/\mathbb{Q}$. So this proves (ii).

If $t_K = 3$, then p splits completely in F and all the primes above p in F ramify in K. So $\gcd(d_k, d_F) = 1$. Since $|d_k|^3$ divides $|d_k|^3|\mathrm{N}_{k/\mathbb{Q}}(\Delta_{K/k})|$ $= |d_K| = d_F^2 d_r$ and $\gcd(d_k, d_F) = 1$, we have that $|d_k|^3$ divides d_r. This proves the last assertion. Hence we proved the claim.

For every d_k in Table 3.2 and every $t_K \in \{1,2,3\}$, if we take d_F from the right hand side of the table, use $h_K^* = 2^{t_K-1}$, $\mu_K = \mu_k$ and use the lower bound on d_r from (i)–(iii), then we get that the right hand side of (3.4.4) is larger than the left hand side. By monotonicity of the right hand side of (3.4.4) in terms of d_r and d_F, this gives a contradiction with (3.4.4) for every d_F larger than the bound in Table 3.2. □

By Proposition 3.3.3, we know that every CM class number one sextic CM field K contains a *class number one* imaginary quadratic field k. Thus there is only one prime that ramifies in k. On the other hand, the ramified primes in K/F are lying above the prime that ramifies in k (see Proposition 4.8-(ii) in II of Lang [21]). Hence the relative class number h_K^* is at most 4, and so by Proposition 3.4.9, we get that if $I_0(\Phi^r) = I_{K^r}$ and the *RH* holds for ζ_K, then the bound for d_F is given in Table 3.2.

We will list all fields up to that bound. Then the following lemma will help us eliminate sextic CM fields that do not have trivial CM class group.

Lemma 3.4.10. *Let K be a non-normal sextic CM field containing an imaginary quadratic field k. Assuming $I_0(\Phi^r) = I_{K^r}$, if a rational prime l splits completely in $K/\mathbb{Q}$, then $l \geq \sqrt{d_F/(6|d_k|)}$.*

Proof. Let l be a rational prime that splits completely in K and $\mathfrak{l}$ be a prime in K lying above l. By the assumption $I_0(\Phi^r) = I_{K^r}$, there exists $\pi \in K^\times$ such that $\mathrm{N}_{\Phi^r}(\mathfrak{l}) = \pi\mathcal{O}_K$ and $\pi\overline{\pi} = l$.

We claim that $K = \mathbb{Q}(\pi)$. Let N be a normal closure of K with $\mathrm{Gal}(N/\mathbb{Q}) = \langle x,\ y : y^6 = x^2 = 1,\ xyxy = 1\rangle$, where K is fixed by $\langle x\rangle$.

If $\sigma\pi$ and π have the same ideal factorization in N, then σ satisfies

$$(\sigma\mathrm{N}_{\Phi^r}(\mathfrak{l}))\mathcal{O}_N = (\mathrm{N}_{\Phi^r}(\mathfrak{l}))\mathcal{O}_N. \tag{3.4.5}$$

Since N is a normal closure of K, the rational prime l splits completely in N. Then the equality (3.4.5) holds only if $\sigma\Phi_N^r = \Phi_N^r$, equivalently $\Phi_N\sigma = \Phi_N$. Since Φ is primitive, this implies that $\sigma \in \langle x\rangle$. So if $\sigma\pi = \pi$, then $\sigma \in \mathrm{Gal}(N/K)$, hence $\mathbb{Q}(\pi) = K$.

We claim that $\{1, \pi, \overline{\pi}\}$ is linearly independent over k. Assume that they are linearly dependent over k. Then there exist $a, b \in k$ such that $\overline{\pi} = a + b\pi$. Then we have $b\pi^2 + a\pi - l = 0$, hence $\{1, \pi, \pi^2\}$ are linearly dependent over k. But this is a contradiction since the degree of K/k is 3.

Therefore, we have

$$\Delta_{K/k} \leq |\mathrm{disc}(1, \pi, \overline{\pi})| = \begin{vmatrix} 1 & \pi & \overline{\pi} \\ \pi & \pi^2 & l \\ \overline{\pi} & l & \overline{\pi}^2 \end{vmatrix} \leq 3! \cdot l^2 \tag{3.4.6}$$

as $|\pi| = \sqrt{l}$ for every $K \hookrightarrow \mathbb{C}$. Hence we get $|\mathrm{N}_{k/\mathbb{Q}}(\Delta_{K/k})| \leq (6l^2)^2$.

By Lemma 20 in [27], we have $|d_K| \geq |d_k| d_F^2$. So we get

$$|d_k| d_F^2 \leq |d_K| = |\mathrm{N}_{k/\mathbb{Q}}(\Delta_{K/k})||d_k|^3 \leq (6l^2)^2 |d_k|^3.$$

Hence we get $\sqrt{d_F/(6|d_k|)} \leq l$. □

Algorithm 3.4.11. ***Output:*** *Assuming GRH, the output is the complete list of non-normal sextic CM fields with a primitive CM type Φ satisfying $I_0(\Phi^r) = I_{K^r}$ and containing an imaginary quadratic field k.*

Step 1. Enumerate all totally real non-normal cubic number fields F up to $d_F \leq 7 \cdot 10^{10}$, using the algorithm in Belabas [4].

Step 2. *For each F construct $K = F(\sqrt{-d})$, where $d \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$.*

Step 3. *Eliminate fields K that have totally split primes under the bound $\sqrt{d_F/(6|d_k|)}$.*

Step 4. *For each sextic CM field K, compute the class numbers of K and its totally real subfield F under the GRH. Then test whether $h_K^* = 2^{t_K-1}$, where t_K is the number of primes in F that are ramified in K.*

Proof. Note that Step 3 of the algorithm does not affect the validity of the algorithm by Lemma 3.4.10, only speeds up the computation.

Suppose that a non-normal sextic CM field $K = F(\sqrt{-d})$ with $d \in \mathbb{Z}_{>0}$ satisfies $I_0(\Phi^r) = I_{K^r}$. Then by Proposition 3.4.1, we have $h_K^* = 2^{t_K-1}$, and without loss of generality $d \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$. Therefore, there are at most 3 ramified primes in K/F. This implies $t_K \leq 3$ and hence $h_K^* = 2^{t_K-1} \leq 4$. Then by Proposition 3.4.9, under GRH, for each k we get the upper bound for d_F as in Table 3.2. Hence $d_F < 7 \cdot 10^{10}$. Therefore, the CM field K is listed by the algorithm. Conversely, all the listed fields satisfy $I_0(\Phi^r) = I_{K^r}$ by Propositions 3.2.1 and 3.4.2. □

We implemented the algorithm in SageMath [36] using Belabas' *cubic* software [3] for Step 1 and using Pari [33]'s bnfinit function with flag $= 0$ for computing the class numbers under the GRH without computing the generators of the unit group. The fields that we obtained are in Tables 3.3–3.12. The implementation is online at [18]. This computation takes few weeks on a computer.

So we proved Theorem 3.1.3.

In Tables 3.3–3.12, the notation is as follows: K is a non-normal sextic CM field that contains $\mathbb{Q}(\sqrt{-d})$ for some $d \in \mathbb{Z}_{>0}$; F is the totally real cubic subfield of K and F is defined by a monic irreducible polynomial $p(X)$; h_K^* is the relative class number h_K/h_F, where h_K and h_F are the class numbers of K and F, respectively.

Table 3.3: Under GRH, the complete list of $p(X) \in \mathbb{Q}[X]$ such that $K = \mathbb{Q}(\sqrt{-3})[X]/p(X)$ is a CM class number one non-normal sextic CM field.

$h_K^* = 1$			
$p(X)$	h_F	$p(X)$	h_F
$X^3 - 6X - 2$	1	$X^3 + 3X^2 - 24X + 8$	1
$X^3 - 6X - 1$	1	$X^3 + 4X^2 - X - 2$	1
$X^3 - 9X - 2$	1	$X^3 + 4X^2 - 3X - 3$	1
$X^3 - 18X - 12$	1	$X^3 + 4X^2 - 5X - 3$	1
$X^3 + X^2 - 3X - 1$	1	$X^3 + 4X^2 - 7X - 4$	1
$X^3 + X^2 - 4X - 1$	1	$X^3 + 4X^2 - 12X - 12$	1
$X^3 + X^2 - 7X - 1$	1	$X^3 + 4X^2 - 16X - 4$	1
$X^3 + X^2 - 9X - 3$	1	$X^3 + 4X^2 - 18X - 12$	1
$X^3 + X^2 - 16X + 8$	1	$X^3 + 4X^2 - 22X + 8$	1
$X^3 + X^2 - 20X - 12$	1	$X^3 + 5X^2 - X - 2$	1
$X^3 + X^2 - 22X - 16$	1	$X^3 + 5X^2 - 6X - 6$	1
$X^3 + X^2 - 30X - 18$	1	$X^3 + 5X^2 - 10X - 2$	1
$X^3 + 2X^2 - 3X - 2$	1	$X^3 + 5X^2 - 12X - 6$	1
$X^3 + 2X^2 - 4X - 2$	1	$X^3 + 5X^2 - 18X - 24$	1
$X^3 + 2X^2 - 5X - 3$	1	$X^3 + 5X^2 - 22X - 8$	1
$X^3 + 2X^2 - 14X - 12$	1	$X^3 + 6X^2 - 2$	1
$X^3 + 2X^2 - 15X - 6$	1	$X^3 + 6X^2 - 3X - 2$	1
$X^3 + 2X^2 - 22X + 4$	1	$X^3 + 6X^2 - 3X - 6$	1
$X^3 + 3X^2 - 3X - 2$	1	$X^3 + 6X^2 - 6X - 12$	1
$X^3 + 3X^2 - 4X - 2$	1	$X^3 + 6X^2 - 9X - 4$	1
$X^3 + 3X^2 - 6X - 2$	1	$X^3 + 6X^2 - 21X - 36$	1
$X^3 + 3X^2 - 6X - 3$	1	$X^3 + 7X^2 - 3$	1
$X^3 + 3X^2 - 9X - 5$	1	$X^3 + 7X^2 + X - 3$	1
$X^3 + 3X^2 - 12X - 8$	1	$X^3 + 7X^2 - 8X - 12$	1
$X^3 + 3X^2 - 18X - 8$	1	$X^3 + 7X^2 - 14X - 24$	1
$X^3 + 3X^2 - 18X - 16$	1	$X^3 + 9X^2 - 6X - 24$	1

Table 3.4: Under GRH, the complete list of $p(X) \in \mathbb{Q}[X]$ such that $K = \mathbb{Q}(\sqrt{-3})[X]/p(X)$ is a CM class number one non-normal sextic CM field.

$h_K^* = 2$			
$p(X)$	h_F	$p(X)$	h_F
$X^3 - 5X - 1$	1	$X^3 + 2X^2 - 5X - 1$	1
$X^3 - 8X - 2$	1	$X^3 + 2X^2 - 14X - 4$	1
$X^3 - 14X - 4$	1	$X^3 + 3X^2 - 8X - 8$	1
$X^3 + X^2 - 5X + 1$	1	$X^3 + 3X^2 - 14X - 8$	1
$X^3 + X^2 - 6X - 2$	1	$X^3 + 4X^2 - 2X - 2$	1
$X^3 + X^2 - 8X - 2$	1	$X^3 + 4X^2 - 5X - 2$	1
$X^3 + X^2 - 9X + 1$	1	$X^3 + 5X^2 - X - 3$	1
$X^3 + X^2 - 12X - 8$	1	$X^3 + 6X^2 + X - 2$	1
$X^3 + 2X^2 - 3X - 1$	1		

$h_K^* = 4$			
$p(X)$	h_F	$p(X)$	h_F
$X^3 - 22X - 12$	1	$X^3 + 3X^2 - 16X - 12$	1
$X^3 + 3X^2 - 4X - 3$	1	$X^3 + 6X^2 - X - 3$	1
$X^3 + 3X^2 - 10X - 6$	1	$X^3 + 6X^2 - 4X - 12$	1
$X^3 + 3X^2 - 7X - 3$	1		

Table 3.5: Under GRH, the complete list of $p(X) \in \mathbb{Q}[X]$ such that $K = \mathbb{Q}(\sqrt{-4})[X]/p(X)$ is a CM class number one non-normal sextic CM field.

$h_K^* = 1$			
$p(X)$	h_F	$p(X)$	h_F
$X^3 - 6X - 2$	1	$X^3 + 3X^2 - 8X - 2$	1
$X^3 - 14X - 4$	1	$X^3 + 3X^2 - 24X - 18$	1
$X^3 - 22X - 12$	1	$X^3 + 4X^2 - 3X - 4$	1
$X^3 + X^2 - 3X - 1$	1	$X^3 + 4X^2 - 22X - 4$	1
$X^3 + X^2 - 4X - 1$	1	$X^3 + 4X^2 - 27X - 36$	1
$X^3 + X^2 - 5X + 1$	1	$X^3 + 5X^2 - 4X - 6$	1
$X^3 + X^2 - 7X - 1$	1	$X^3 + 5X^2 - 8X - 6$	3
$X^3 + X^2 - 12X - 8$	1	$X^3 + 5X^2 - 12X - 8$	1
$X^3 + X^2 - 36X + 18$	1	$X^3 + 6X^2 - 3X - 2$	1
$X^3 + 2X^2 - 3X - 2$	1	$X^3 + 6X^2 - 4X - 12$	1
$X^3 + 2X^2 - 4X - 2$	1	$X^3 + 6X^2 - 7X - 6$	1
$X^3 + 2X^2 - 11X - 6$	1	$X^3 + 8X^2 + X - 4$	1
$X^3 + 2X^2 - 18X - 8$	1	$X^3 + 8X^2 - 15X - 36$	1
$X^3 + 2X^2 - 18X + 8$	1	$X^3 + 13X^2 + 12X - 12$	1
$X^3 + 3X^2 - 4X - 2$	1		

$h_K^* = 2$			
$p(X)$	h_F	$p(X)$	h_F
$X^3 - 4X - 1$	1	$X^3 + 3X^2 - 6X - 2$	1
$X^3 + X^2 - 6X - 2$	1	$X^3 + 3X^2 - 8X - 8$	1
$X^3 + X^2 - 10X + 2$	1	$X^3 + 3X^2 - 10X - 6$	1
$X^3 + X^2 - 16X - 4$	1	$X^3 + 3X^2 - 16X - 12$	1
$X^3 + X^2 - 16X + 8$	1	$X^3 + 4X^2 - X - 2$	1
$X^3 + 2X^2 - 4X - 1$	1	$X^3 + 4X^2 - 5X - 2$	1
$X^3 + 2X^2 - 6X - 3$	1	$X^3 + 5X^2 - X - 2$	1
$X^3 + 2X^2 - 14X - 12$	1	$X^3 + 5X^2 - 2X - 2$	1
$X^3 + 2X^2 - 18X - 12$	1	$X^3 + 6X^2 - 6X - 12$	1
$X^3 + 3X^2 - 3X - 2$	1	$X^3 + 7X^2 + 2X - 2$	1

$h_K^* = 4$	
$p(X)$	h_F
$X^3 + 3X^2 - 10X - 8$	1

Table 3.6: Under GRH, the complete list of $p(X) \in \mathbb{Q}[X]$ such that $K = \mathbb{Q}(\sqrt{-7})[X]/p(X)$ is a CM class number one non-normal sextic CM field.

$h_K^* = 1$			
$p(X)$	h_F	$p(X)$	h_F
$X^3 - 6X - 2$	1	$X^3 + 3X^2 - 30X - 9$	1
$X^3 + X^2 - 16X - 1$	3	$X^3 + 4X^2 - 7X - 3$	1
$X^3 + X^2 - 8X - 1$	1	$X^3 + 4X^2 - 7X - 7$	1
$X^3 + X^2 - 36X + 27$	1	$X^3 + 4X^2 - 9X - 1$	1
$X^3 + X^2 - 42X - 45$	1	$X^3 + 5X^2 - 6X - 3$	1
$X^3 + 2X^2 - 4X - 1$	1	$X^3 + 5X^2 - 18X - 27$	1
$X^3 + 2X^2 - 7X - 3$	1	$X^3 + 6X^2 - 5X - 7$	1
$X^3 + 2X^2 - 39X + 27$	1	$X^3 + 6X^2 - 5X - 3$	1
$X^3 + 3X^2 - 10X - 3$	1	$X^3 + 8X^2 - X - 5$	1
$X^3 + 3X^2 - 10X - 7$	1	$X^3 + 9X^2 - 3$	1
$X^3 + 3X^2 - 16X - 9$	1		

$h_K^* = 2$	
$p(X)$	h_F
$X^3 + X^2 - 4X - 1$	1
$X^3 + 2X^2 - 5X - 3$	1
$X^3 + 2X^2 - 3X - 1$	1

$h_K^* = 4$	
$p(X)$	h_F
$X^3 - 7X - 1$	1
$X^3 + 5X^2 - 2X - 3$	1
$X^3 + 3X^2 - 6X - 1$	1

Table 3.7: Under GRH, the complete list of $p(X) \in \mathbb{Q}[X]$ such that $K = \mathbb{Q}(\sqrt{-8})[X]/p(X)$ is a CM class number one non-normal sextic CM field.

$h_K^* = 1$			
$p(X)$	h_F	$p(X)$	h_F
X^3+X^2-3X-1	1	$X^3+X^2-24X-4$	1
$X^3+2X^2-13X-4$	1	$X^3+X^2-16X-8$	3
X^3+3X^2-6X-2	1	X^3+3X^2-8X-8	1
X^3+4X^2-X-2	1	$X^3+4X^2-18X-16$	1
X^3+5X^2-6X-2	1	$X^3+6X^2-10X-20$	1
X^3+8X^2-9X-2	1	$X^3+6X^2-22X-20$	1
$h_K^* = 2$			
$p(X)$	h_F	$p(X)$	h_F
X^3-4X-1	1	X^3+5X^2-2X-2	1
$X^3-13X-2$	1	X^3+6X^2-X-4	1
$X^3+X^2-10X-2$	1	X^3+8X^2+3X-2	1
X^3+2X^2-3X-2	1	$X^3+3X^2-16X-8$	1
X^3+3X^2-4X-2	1	$X^3+4X^2-14X-16$	1
X^3+4X^2-3X-4	1	$X^3+4X^2-10X-8$	1
X^3+5X^2-2	1		

Table 3.8: Under GRH, the complete list of $p(X) \in \mathbb{Q}[X]$ such that $K = \mathbb{Q}(\sqrt{-11})[X]/p(X)$ is a CM class number one non-normal sextic CM field.

$h_K^* = 1$			
$p(X)$	h_F	$p(X)$	h_F
X^3-5X-1	1	$X^3+2X^2-18X-8$	1
X^3+X^2-7X-2	1	X^3+4X^2-5X-2	1
X^3+X^2-8X-1	1	X^3+4X^2-6X-1	1
$X^3+X^2-16X-2$	1	$X^3+5X^2-26X-8$	1
$X^3+X^2-16X-8$	3	$X^3+5X^2-26X-32$	1
$X^3+2X^2-18X+4$	1	$X^3+7X^2-14X-16$	1

$h_K^* = 2$	
$p(X)$	h_F
X^3+2X^2-3X-2	1
$h_K^* = 4$	
$p(X)$	h_F
X^3+5X^2-2X-2	1

Table 3.9: Under GRH, the complete list of $p(X) \in \mathbb{Q}[X]$ such that $K = \mathbb{Q}(\sqrt{-19})[X]/p(X)$ is a CM class number one non-normal sextic CM field.

$h_K^* = 1$			
$p(X)$	h_F	$p(X)$	h_F
X^3+2X^2-8X-3	1	$X^3+6X^2-6X-12$	1
X^3+4X^2-3X-3	1	$X^3+7X^2-30X-54$	1
$X^3+4X^2-14X-8$	1	X^3+8X^2+X-3	1
X^3+6X^2-X-4	1		

Table 3.10: Under GRH, the complete list of $p(X) \in \mathbb{Q}[X]$ such that $K = \mathbb{Q}(\sqrt{-43})[X]/p(X)$ is a CM class number one non-normal sextic CM field.

$h_K^* = 1$	
$p(X)$	h_F
$X^3+X^2-22X-8$	1
$X^3+2X^2-18X-12$	1
X^3-5X-1	1

Table 3.11: Under GRH, the complete list of $p(X) \in \mathbb{Q}[X]$ such that $K = \mathbb{Q}(\sqrt{-67})[X]/p(X)$ is a CM class number one non-normal sextic CM field.

$h_K^* = 1$	
$p(X)$	h_F
X^3+2X^2-4X-1	1

Table 3.12: Under GRH, the complete list of $p(X) \in \mathbb{Q}[X]$ such that $K = \mathbb{Q}(\sqrt{-163})[X]/p(X)$ is a CM class number one non-normal sextic CM field.

$h_K^* = 1$	
$p(X)$	h_F
X^3+3X^2-8X-8	1