# Chapter 1

# Preliminaries

*ABSTRACT. In this chapter, we give the main ingredients that we will use in the later chapters. This chapter contains facts from class field theory, complex multiplication theory and facts related to abelian varieties.*

"All the truths of mathematics are linked to each other, and all means of discovering them are equally admissible."

Adrien-Marie Legendre

## 1.1 Global class field theory

In this section, we follow Neukirch [32].

Let $K$ be a number field and $\mathcal{O}_K$ its ring of integers. We denote places by $\mathfrak{p}$. A *cycle* or *modulus* of $K$ is a formal product

$$\mathfrak{m} = \prod_{\mathfrak{p}} \mathfrak{p}^{n(\mathfrak{p})},$$

where $\mathfrak{p}$ runs over all places of $K$ with $n(\mathfrak{p}) \in \mathbb{Z}_{\geq 0}$ such that $n(\mathfrak{p}) = 0$ for almost all places of $K$. Here $n(\mathfrak{p})$ is 0 or 1 if $\mathfrak{p}$ is real, and 0 if $\mathfrak{p}$ is complex.

## Frobenius automorphisms

Let $L/K$ be a Galois extension of number fields and $\mathfrak{m}$ be a cycle of $K$ divisible by all ramified primes of $L/K$. Given $\mathfrak{P} \nmid \mathfrak{m}$ lying above a finite prime $\mathfrak{p}$ of $K$, there exists a *unique* automorphism $\sigma_{\mathfrak{P}} \in \mathrm{Gal}(L/K)$ satisfying

$$\sigma_{\mathfrak{P}}(x) \equiv x^q \bmod \mathfrak{P} \text{ for all } x \in \mathcal{O}_L,$$

where $\mathcal{O}_L$ is the ring of integers of $L$ and $q = |\mathcal{O}_K/\mathfrak{p}|$. This automorphism is called the *Frobenius automorphism of* $\mathfrak{P}$ and is denoted by

$$\sigma_{\mathfrak{P}} = \left(\frac{L/K}{\mathfrak{P}}\right).$$

## The Artin map for unramified abelian extensions

Let $L/K$ be an unramified *abelian* extension of number fields. Let $\mathfrak{P}$ be a finite prime of $L$ lying above a prime $\mathfrak{p}$ of $K$ and let $\sigma_{\mathfrak{P}} = \left(\frac{L/K}{\mathfrak{P}}\right)$ be its *Frobenius automorphism*. For any element $\tau \in \mathrm{Gal}(L/K)$, we have

$$\left(\frac{L/K}{\tau(\mathfrak{P})}\right) = \tau \left(\frac{L/K}{\mathfrak{P}}\right) \tau^{-1} = \left(\frac{L/K}{\mathfrak{P}}\right).$$

Hence the Frobenius automorphisms $\sigma_{\tau\mathfrak{P}}$ of the primes $\tau\mathfrak{P}$ are the same, hence the Frobenius automorphisms in $\mathrm{Gal}(L/K)$ depends only on $\mathfrak{p} = \mathfrak{P} \cap K$, not on $\mathfrak{P}$ itself. For abelian extensions we use the notation $\left(\frac{L/K}{\mathfrak{p}}\right)$ for $\left(\frac{L/K}{\mathfrak{P}}\right)$ and call this the Frobenius automorphism of $\mathfrak{p}$.

Let $I_K$ be the group of fractional ideals of $\mathcal{O}_K$, which is the free abelian group generated by the prime ideals of $K$.

Then for any $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v(\mathfrak{p})} \in I_K$ with $v(\mathfrak{p}) \in \mathbb{Z}$, we define

$$\left(\frac{L/K}{\mathfrak{a}}\right) = \prod_{\mathfrak{p}} \left(\frac{L/K}{\mathfrak{p}}\right)^{v(\mathfrak{p})}.$$

It is a theorem (Theorem VI.7.1 in Neukirch [32]) that the homomorphism

$$\begin{aligned} r_{K/L} : I_K &\to \mathrm{Gal}(L/K) \\ \mathfrak{a} &\mapsto \left(\frac{L/K}{\mathfrak{a}}\right) \end{aligned} \tag{1.1.1}$$

is surjective. This map is called the *Artin map* for the unramified abelian extension $L/K$.

### Unramified class fields

A number field extension $K \subset L$ is *unramified* if it is unramified at all places of $K$ (including the infinite places). When we say that an infinite place of $L$ is unramified in $L/K$, we mean that it is not a complex place lying over a real place of $K$.

Let $P_K \subset I_K$ be the group of principal ideals of $K$. The *ideal class group* (or *class group*) $\mathrm{Cl}_K$ of $K$ is the quotient group $I_K/P_K$.

**Theorem 1.1.1.** *(Proposition VI.6.9 [32]) Given a number field $K$, there is an unramified finite abelian extension $H_K$ of $K$ such that the* Artin map *(1.1.1) induces an isomorphism*

$$r_K : I_K/P_K \xrightarrow{\sim} \mathrm{Gal}(H_K/K). \qquad \square$$

The field $H_K$ in Theorem 1.1.1 is called the *Hilbert class field* of $K$. It is the maximal abelian extension of $K$ that is unramified at all places of $K$ (see page 399 in Neukirch [32]).

## 1.2 CM fields and CM types

In this section, we mainly follow Lang [20] and Shimura–Taniyama [40].

**Definition 1.2.1.** A *CM field* is a totally imaginary quadratic extension $K$ of a totally real number field $F$. In other words, a CM field is a field $K = F(\sqrt{-\delta})$ for a totally real number field $F$ and a totally positive element $\delta \in F$.

Let $\bar{\cdot} : \mathbb{C} \to \mathbb{C}$ denote the complex conjugation automorphism of $\mathbb{C}$. For every CM field $K$ there exists an automorphism $\rho$ such that for every embedding $\tau : K \to \mathbb{C}$ we have $\bar{\cdot} \circ \tau = \tau \circ \rho$; we call it *complex conjugation* and denote it by $\rho$ or $\bar{\cdot}$. Let $\phi$ be an embedding of a CM field $K$ into any field $N$. Then we denote $\phi \circ \bar{\cdot}$ by $\overline{\phi}$. Note that if $N$ is a CM field or $\mathbb{C}$

then we have $\bar{\cdot} \circ \phi = \overline{\phi}$ because the composite of $\phi$ with any embedding $N \to \mathbb{C}$ is an embedding $K \to \mathbb{C}$.

Let $K$ be a CM field of degree $2g$ and $N'$ be a number field that contains a subfield that is isomorphic over $\mathbb{Q}$ to a normal closure over $\mathbb{Q}$ of $K$.

**Definition 1.2.2.** Let $K$ and $N'$ be as above. A *CM type* of $K$ with values in $N'$ is a set $\Phi$ of embeddings $\phi : K \to N'$ such that exactly one embedding of each of the $g$ complex conjugate pairs $\phi$, $\overline{\phi} : K \to N'$ is in $\Phi$. We say that $(K, \Phi)$ is a *CM pair* or *CM type*.

Let $K_0$ be a proper CM subfield of $K$. Let $\Phi_{K_0}$ be a CM type of $K_0$ with values in $N'$. Then the CM type of $K$ *induced* by $\Phi_{K_0}$ is $\{\phi \in \mathrm{Hom}(K, N') : \ \phi|_{K_0} \in \Phi_{K_0}\}$.

We say that a CM type $\Phi$ of a CM field is *primitive* if it is not induced from a CM type of a proper CM subfield. The following proposition is a criterion for the primitiveness of a CM type.

If $\gamma$ is an automorphism of $K$, then we define CM type $\Phi\gamma$ as the set of embeddings $\phi \circ \gamma$ for $\phi \in \Phi$, and if $\gamma$ is an automorphism of $N'$, then we define CM type $\gamma\Phi$ as the set of embeddings $\gamma \circ \phi$ for $\phi \in \Phi$.

**Proposition 1.2.3.** *(Shimura–Taniyama [40, Propostion 26]) Let $K$ be a CM field and let $N$ be a normal closure over $\mathbb{Q}$ of $K$. Then $N$ is a CM field. Let $N'$ be as above.*

*Let $\Phi$ be a CM type of $K$ with values in $N'$ and let $\Phi_N$ be the CM type of $N$ with values in $N'$ induced from $\Phi$. Then $(K, \Phi)$ is primitive if and only if*

$$\mathrm{Gal}(N/K) = \{\gamma \in \mathrm{Gal}(N/\mathbb{Q}) : \Phi_N\gamma = \Phi_N\}. \qquad \square$$

**Corollary 1.2.4.** *With the notation in Proposition 1.2.3, suppose that $K$ is normal over $\mathbb{Q}$. Then $(K, \Phi)$ is primitive if and only if there is no non-trivial element $\gamma \in \mathrm{Gal}(K/\mathbb{Q})$ satisfying $\Phi\gamma = \Phi$.* $\square$

We say that CM types $\Phi_1$ and $\Phi_2$ of $K$ are *equivalent* if there is an automorphism $\sigma$ of $K$ such that $\Phi_1 = \Phi_2\sigma$ holds.

Let $K$ be a CM field of degree $2g$ and $N'$ be a number field that contains a subfield that is isomorphic over $\mathbb{Q}$ to a normal closure over $\mathbb{Q}$

of $K$. We make $N'$ smaller and from now on we assume $N \cong N'$. Let $\Phi$ be a CM type of $K$ with values in $N'$ and let $\Phi_N$ be the CM type of $N$ with values in $N'$ induced from $\Phi$. Here $\Phi_N$ is a set of isomorphisms $\phi : N \to N'$, so we can take the inverses $\phi^{-1} : N' \to N$. Let $\Phi_N^{-1} = \{\phi^{-1} : \phi \in \Phi_N\}$. Then the subfield $K^r$ of $N'$ corresponding to the subgroup $\{\gamma : \gamma \in \mathrm{Gal}(N'/\mathbb{Q}),\ \Phi_N^{-1}\gamma = \Phi_N^{-1}\}$ is a CM field with the *primitive* CM type $\Phi^r = \Phi_N^{-1}|_{K^r}$. Moreover, we have

$$K^r = \mathbb{Q}(\{\sum_{\phi\in\Phi} \phi(x) \mid x \in K\}) \subset N'.$$

For details, see Shimura–Taniyama [40, Proposition 28].

The field $K^r$ is called the *reflex field* of $(K, \Phi)$ and $\Phi^r$ is called the *reflex type* of $(K, \Phi)$. The pair $(K^r, \Phi^r)$ is called the *reflex* of $(K, \Phi)$.

**Lemma 1.2.5.** *Let $K$ be a CM field and let $\Phi$ be a CM type of $K$. Then the reflex field $K^{rr}$ of $(K^r, \Phi^r)$ is a subfield of $K$ with the primitive CM type $\Phi^{rr}$. If $\Phi$ is primitive, then $K^{rr} = K$ and $\Phi^{rr} = \Phi$.*

*Proof.* This follows from the definition of the *reflex field* and Proposition 1.2.3. □

The *type norm* of a CM pair $(K, \Phi)$ is the multiplicative map

$$\begin{aligned} \mathrm{N}_\Phi : K &\to K^r, \\ x &\mapsto \prod_{\phi\in\Phi} \phi(x). \end{aligned}$$

**Proposition 1.2.6.** *(Shimura–Taniyama [40, Proposition 29]) Let $K$ be a CM field and let $\Phi$ be a CM type of $K$ with values in $N'$. Let $\mathfrak{a} \in I_K$ and $x \in K$. Then there is an ideal $\mathrm{N}_\Phi(\mathfrak{a})$ of $K^r$ such that $\mathrm{N}_\Phi(\mathfrak{a})\mathcal{O}_{N'} = \prod_{\phi\in\Phi} \phi(\mathfrak{a})\mathcal{O}_{N'}$ and we have*

$$\begin{aligned} \mathrm{N}_\Phi(\mathfrak{a})\overline{\mathrm{N}_\Phi(\mathfrak{a})} &= \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})\mathcal{O}_{K^r}, \\ \mathrm{N}_\Phi(x)\overline{\mathrm{N}_\Phi(x)} &= \mathrm{N}_{K/\mathbb{Q}}(x) \in \mathbb{Q}. \end{aligned}$$ □

## 1.3 Abelian varieties

In this chapter, we refer to Lang [20] and Shimura–Taniyama [40].

Let $k$ be a field. In this thesis, we will use the following definitions. By a *variety* over $k$, we mean a geometrically integral, separated scheme of finite type over $\mathrm{Spec}(k)$. Curves, respectively surfaces, respectively threefolds are varieties are of dimension 1, respectively 2, respectively 3. We will always assume that curves, surfaces and threefolds are projective, smooth over $k$.

By an *abelian variety* over $k$, we mean a complete irreducible group variety over $k$. It is known that abelian varieties are smooth, projective, and commutative. Let $A$ and $B$ be abelian varieties over $k$. A *morphism* $\lambda$ of $A$ to $B$ is a morphism of varieties that respects the group structure. If $A$ and $B$ are of the same dimension and $\lambda$ is surjective, then it is called an *isogeny*. If an isogeny $\lambda : A \to B$ exists, then $A$ and $B$ are called *isogenous*. A non-zero abelian variety is said to be *simple* if it is not isogenous to a product of abelian varieties of lower dimensions.

We denote by $\mathrm{End}(A)$ the ring of homomorphisms of $A$ to itself over $\overline{k}$ and we put $\mathrm{End}_0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$.

### 1.3.1 Polarizations and the dual variety

This section basically follows Lang [20, 3.4].

By a divisor on a variety, we always mean a *Cartier* divisor. We say that two divisors $X_1$ and $X_2$ on an abelian variety $A$ over a field $k$ are *algebraically equivalent* ($X_1 \sim X_2$) if there is a connected algebraic set $T$, two points $t_1,\ t_2 \in T$ and a divisor $Z$ on $A \times T$ such that $Z|_{t_i} = X_i$ for $i = 1,\ 2$. The divisors $X_1$ and $X_2$ are *linearly equivalent* if there is a rational function $f \in k(A)^\times$ such that $X_1 = X_2 + (f)$. For details, see Hartshorne [15].

Let $\mathcal{D}_a(A)$ and $\mathcal{D}_l(A)$ respectively be the group of divisors on $A$ over $\overline{k}$ that are algebraically equivalent to 0 and the group of divisors on $A$ over $\overline{k}$ that are linearly equivalent to 0. There exists an abelian variety $A^*$, that is called the *dual variety* of $A$, whose group of $\overline{k}$-points is canonically isomorphic to $\mathrm{Pic}^0(A) := \mathcal{D}_a(A)/\mathcal{D}_l(A)$. Let $X$ be an ample divisor over $\overline{k}$ on an abelian variety $A$ and let $[X]$ denote the linear equivalence class

of $X$. Let $X_a$ be the translation of $X$ by an element $a \in A$. Then the map

$$\begin{aligned} \varphi_X : A &\to \mathrm{Pic}^0(A) \\ a &\mapsto [X_a - X], \end{aligned} \tag{1.3.1}$$

induces an isogeny $\varphi_X : A \to A^*$.

**Proposition 1.3.1.** *(Serre [37]) Two divisors $X_1$ and $X_2$ are algebraically equivalent if and only if $\varphi_{X_1} = \varphi_{X_2}$.* □

**Definition 1.3.2.** An isogeny $\varphi : A \to A^*$ induced by (1.3.1) is called a *polarization* of $A$. It is said to be a *principal polarization* if $\varphi$ is an isomorphism.

We understand by a *polarized abelian variety* a pair $(A, \varphi)$ formed by an abelian variety $A$ and a polarization $\varphi$ of $A$. We say that a polarized abelian variety $(A, \varphi)$ is defined over a field $k$ if $A$ and $\varphi$ are defined over $k$.

Every polarization $\varphi$ on $A$ induces an involution as follows. Each endomorphism $\lambda \in \mathrm{End}_0(A)$ has a dual

$$\lambda^* : A^* \to A^* : \quad [Y] \mapsto [\lambda^{-1}(Y)] \tag{1.3.2}$$

and for every $\lambda \in \mathrm{End}_0(A)$, we define

$$\lambda' = \varphi^{-1} \lambda^* \varphi \in \mathrm{End}_0(A).$$

The map sending $\lambda$ to $\lambda'$ is an involution of $A$ and called the *Rosati involution determined by* $\varphi$.

Let $(A_1, \varphi_1)$ and $(A_2, \varphi_2)$ be two polarized abelian varieties of the same dimension. A homomorphism $\lambda$ of $A_1$ to $A_2$ is called a *homomorphism* of $(A_1, \varphi_1)$ to $(A_2, \varphi_2)$ if the following diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\lambda} & A_2 \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\ A_1^* & \longrightarrow & A_2^* \end{array}$$

commutes.

**Proposition 1.3.3.** *(Shimura–Taniyama [40, Theorem 2 and Proposition 14]) Let $(A, \varphi)$ be a polarized abelian variety over a characteristic 0 field $k$. Then there exists a field $k_0 \subset k$ with the following property: For all $\sigma : k \to \overline{k}$, it holds that $(A, \varphi)$ and $(\sigma A, \sigma\varphi)$ are isomorphic over $\overline{k}$ if and only if $\sigma$ is the identity map on $k_0$.* □

**Definition 1.3.4.** The field $k_0$ in Proposition 1.3.3 is called the *the field of moduli* of $(A, \varphi)$.

## Jacobian of curves

The *Jacobian* $J(C)$ of a curve $C/k$ of genus $g$ is a certain *principally polarized* abelian variety of dimension $g$ such that we have $J(C)(\overline{k}) = \mathrm{Pic}^0(C_{\overline{k}})$; for details we refer to [29].

**Theorem 1.3.5.** *(Torelli) Two algebraic curves over $\mathbb{C}$ are isomorphic if and only if their Jacobians are isomorphic as polarized abelian varieties.*

*Proof.* This is Theorem 11.1.7 of Birkenhake–Lange [6]. □

**Theorem 1.3.6.**

*(i) (Weil) Every principally polarized abelian surface over $\mathbb{C}$ is either a product of elliptic curves with the product polarization or the Jacobian of a smooth projective curve of genus 2.*

*(ii) (Matsusaka–Ran) Every principally polarized abelian threefold over $\mathbb{C}$ is either the Jacobian of a smooth curve of genus 3 or a principally polarized product of a principally polarized abelian surface with an elliptic curve or of three elliptic curves.*

*Proof.* The assertion (i) is Satz 2 of Weil [47]. The assertions (i) and (ii) are consequences of the Matsusaka–Ran criterion in [28, 35], also see Corollary 11.8.2 in Birkenhake–Lange [6]. □

### 1.3.2 Complex abelian varieties

A *lattice* in $\mathbb{C}^g$ is a discrete subgroup of maximal rank in $\mathbb{C}^g$. It is a free abelian group of rank $2g$. The quotient $\mathbb{C}^g/\Lambda$ is called a *complex torus*. A *Riemann form* on $\mathbb{C}^g$ is a skew symmetric $\mathbb{R}$-bilinear map $E$ such that the form $\mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C} : (x, y) \mapsto E(x, iy)$ is positive definite symmetric.

If $A$ is a $g$-dimensional abelian variety over $\mathbb{C}$, then there is a complex torus $\mathbb{C}^g/\Lambda$ that is isomorphic (as a complex Lie group) to $A$ via an analytic isomorphism $\iota : \mathbb{C}^g/\Lambda \to A$. Following Lang [20, 3.4] we describe the notion of *polarization* in the complex analytic setting as follows. Let $X$ be an *ample* divisor on an abelian variety $A$ over $\mathbb{C}$ and let $\varphi_X$ be a polarization on $A$. Then $\iota^{-1}(X)$ is an analytic divisor of $\mathbb{C}^g/\Lambda$, and its pull back to $\mathbb{C}^g$ is defined by a *theta function* $f_X$. There is a Riemann form $E_X$ associated to $f_X$ . It is obtained from the functional equation of $f_X$, see Lang [20, page 68]. We say that $E_X$ is associated to $X$ via $\iota$. Two divisors are algebraically equivalent if and only if the associated Riemann forms are the same.

Let $X$ be an ample divisor on an abelian variety $A$ and $E$ be the Riemann form associated to $X$ via $\iota : \mathbb{C}^g/\Lambda \to A$. We can consider $\mathbb{C}^g$ as the dual vector space of itself over $\mathbb{R}$ with respect to the Riemann form $E$ that is, we identify $y \in \mathbb{C}^g$ with $E(\cdot, y)$. Let us denote by $\Lambda^*$ the set of all vectors of $x \in \mathbb{C}^g$ such that $E(x, y) \in \mathbb{Z}$ for every $y \in \Lambda$. Then $\Lambda^*$ is a discrete group in $\mathbb{C}^g$ and $\mathbb{C}^g/\Lambda^*$ is a complex torus, which we call the *dual* of the complex torus $\mathbb{C}^g/\Lambda$. Then there is an analytic isomorphism $\iota^* : \mathbb{C}^g/\Lambda^* \to A^*$ making the following diagram commutative

$$\begin{array}{ccc} \mathbb{C}^g/\Lambda & \xrightarrow{\ \iota\ } & A \\ {\scriptstyle \lambda : x \mapsto x}\downarrow & & \downarrow{\scriptstyle \varphi_X} \\ \mathbb{C}^g/\Lambda^* & \xrightarrow{\ \iota^*\ } & A^* \end{array} .$$

## 1.4 Abelian varieties with complex multiplication

This section is a summary of Birkenhake–Lange [6, 13.3] and Lang [20, 1.4].

We say that an abelian variety $A$ over a field $k$ of dimension $g$ has *complex multiplication* (CM) by a CM field $K$ if $K$ has degree $2g$ and there is an embedding $\theta : K \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$. We say that $A$ has CM by an order $\mathcal{O} \subset K$ if there is an embedding $\theta : K \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$ such that $\theta^{-1}(\mathrm{End}(A)) = \mathcal{O}$.

The *tangent space* $\mathrm{Tgt}_0(A)$ of $A$ at the unit point 0 of $A$ is a vector space over $k$ of dimension $g$.

Let $A$ be an abelian variety over $\mathbb{C}$ with CM by $K$ via the embedding $\theta : K \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$. Then there exists a unique set $\Phi$ of embeddings $K \to \mathbb{C}$ such that the representation of $K$ on $\mathrm{End}_{\mathbb{C}}(\mathrm{Tgt}_0(A))$ via $\theta$ is equivalent to $\oplus_{\phi \in \Phi} \phi$, see Shimura–Taniyama [40, §5.2]. We call $\Phi$ the *CM type* of $K$. The CM type $\Phi$ is uniquely determined by $(A, \theta)$. We say that $(A, \theta)$ is an *abelian variety of type* $(K, \Phi)$. Furthermore, if $\theta(\mathcal{O}_K) \subset \mathrm{End}(A)$ holds, then we say that $(A, \theta)$ is an *abelian variety of type* $(K, \Phi)$ *with CM by* $\mathcal{O}_K$.

We say that $(A, \theta)$ is defined over a field $k$ if $A$ is defined over $k$ and every element of $\theta(\mathcal{O}_K) \subset \mathrm{End}(A)$ is defined over $k$.

**Theorem 1.4.1** (Shimura, §8.2)**.** *Let $K$ be a CM field and let $\Phi$ be a CM type of $K$. An abelian variety $(A, \theta)$ of type $(K, \Phi)$ is simple if and only if $\Phi$ is primitive.* □

### 1.4.1 Construction of abelian varieties with CM

By a *lattice* in an algebraic number field $K$ of finite degree over $\mathbb{Q}$, we mean a finitely generated $\mathbb{Z}$-submodule of $K$ that spans $K$ over $\mathbb{Q}$.

Let $K$ be a CM field of degree $2g$. To every CM type $\Phi$ of $K$ and $\mathbb{Z}$-lattice $\mathfrak{m}$ of $K$, we associate an abelian variety $A_{\Phi,\mathfrak{m}}$ as follows. The tensor product $K \otimes_{\mathbb{Q}} \mathbb{R}$ is an $\mathbb{R}$-vector space of dimension $2g$. The CM type $\Phi = \{\phi_1, \ldots, \phi_g\}$ induces a $\mathbb{C}$-algebra structure on $K \otimes_{\mathbb{Q}} \mathbb{R}$ via the $\mathbb{R}$-algebra isomorphism

$$\begin{aligned}\tilde{\Phi} : K \otimes_{\mathbb{Q}} \mathbb{R} &\to \mathbb{C}^g \\ \alpha \otimes a &\mapsto {}^t(a\phi_1(\alpha), \ldots, a\phi_g(\alpha)).\end{aligned}$$

By $\tilde{\Phi}(\mathfrak{m})$, we mean the group of all elements $\tilde{\Phi}(\alpha)$ with $\alpha \in \mathfrak{m}$. Then $\tilde{\Phi}(\mathfrak{m})$ is a lattice in $\mathbb{C}^g$ and $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m})$ is a *complex torus*. Hence the quotient $A_{\Phi,\mathfrak{m}} := (K \otimes_{\mathbb{Q}} \mathbb{R})/\mathfrak{m}$ is isomorphic to the complex torus $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m})$.

**Proposition 1.4.2.** *(Birkenhake–Lange [6, Proposition 13.3.1] and Lang [20, Theorem 1.4.1-(iii)]) With the notation above, the complex torus $A_{\Phi,\mathfrak{m}}$ is an abelian variety and has a natural CM structure given by the action of $\mathcal{O}_K$ on $\mathfrak{m}$.* □

In this thesis, we use the complex torus $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m})$ instead of $A_{\Phi,\mathfrak{m}}$ as a realization of an abelian variety over $\mathbb{C}$ conforming to the notation of Lang [20] and Shimura–Taniyama [40].

For each $\alpha \in K$, we let $\mathrm{S}_\Phi(\alpha)$ be the matrix $\mathrm{diag}(\phi_1(\alpha), \ldots, \phi_g(\alpha))$.

**Theorem 1.4.3.** *(Lang [20, Theorem 1.4.1-(ii)]) Let $(K,\Phi)$ be a CM pair and let $(A,\theta)$ be an abelian variety of type $(K,\Phi)$ with CM by $\mathcal{O}_K$. Then there is a fractional ideal $\mathfrak{m} \in I_K$ and an analytic isomorphism $\iota : \mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}) \to A(\mathbb{C})$ such that the diagram*

$$\begin{array}{ccc} \mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}) & \longrightarrow & A(\mathbb{C}) \\ {\scriptstyle S_\Phi(\alpha)}\downarrow & & \downarrow{\scriptstyle \theta(\alpha)} \\ \mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}) & \longrightarrow & A(\mathbb{C}) \end{array}$$

*commutes for all $\alpha \in \mathcal{O}_K$.* □

**Definition 1.4.4.** We say that an *abelian variety* $(A,\theta)$ *of type* $(K,\Phi)$ *is of type* $(K,\Phi,\mathfrak{m})$ if there is a fractional ideal $\mathfrak{m} \in I_K$ and an analytic isomorphism $\iota : \mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}) \to A(\mathbb{C})$.

**Definition 1.4.5.** Let $(A,\theta)$ be an abelian variety of type $(K,\Phi)$ and let $X$ be an ample divisor on $A$. We say that $(A,\theta)$ is $\Phi$-*admissible* with respect to the polarization $\varphi_X$ if $\theta(K)$ is stable under the *Rosati* involution.

**Theorem 1.4.6.** *(Lang [20, Theorem 1.4.5-(iii)] If an abelian variety* $(A,\theta)$ *of type* $(K,\Phi,\mathfrak{m})$ *is* simple*, then it is* $\Phi$-admissible *with respect to every polarization.* □

**Definition 1.4.7.** Let $(A_1,\theta_1)$ and $(A_2,\theta_2)$ be abelian varieties of type $(K,\Phi)$ with CM by $\mathcal{O}_K$. A *homomorphism* $\lambda$ from $A_1$ to $A_2$ is called a homomorphism from $(A_1,\theta_1)$ to $(A_2,\theta_2)$ if it satisfies

$$\lambda\theta_1(\alpha) = \theta_2(\alpha)\lambda$$

for every $\alpha \in \mathcal{O}_K$.

**Proposition 1.4.8.** *(Shimura–Taniyama [40, Proposition 1 in §14]) Let* $(K,\Phi)$ *be a primitive CM type. Let* $(A_1,\theta_1)$ *and* $(A_2,\theta_2)$ *be abelian varieties over* $k \subset \mathbb{C}$ *of type* $(K,\Phi)$*. Then every homomorphism from* $A_1$ *into* $A_2$ *over* $k$ *is a homomorphism from* $(A_1,\theta_1)$ *to* $(A_2,\theta_2)$ *over* $k$*.* □

Let $(A,\theta)$ be a $g$-dimensional abelian variety of CM type $(K,\Phi)$. Let $\mathfrak{a}$ be a $\mathbb{Z}$-lattice in $K$. Let $(\alpha_1,\dots,\alpha_{2g})$ be a basis of $\mathfrak{a}$ over $\mathbb{Z}$. We obtain a homomorphism

$$\lambda_{\mathfrak{a}} : A \to A^{2g}$$

such that $x \mapsto (\alpha_1 x,\dots,\alpha_{2g}x)$ for all $x \in A$. If $\mathfrak{a} \neq 0$, then $\lambda_{\mathfrak{a}}$ is an isogeny to its image $\lambda_{\mathfrak{a}}(A)$ (see page 56 in Lang [20]).

A homomorphism $\lambda$ of $(A_1,\theta_1)$ onto $(A_2,\theta_2)$ is called an $\mathfrak{a}$-*multiplication* if there is a commutative diagram

$$\begin{array}{ccc} (A_1,\theta_1) & \xrightarrow{\quad\lambda_{\mathfrak{a}}\quad} & (\lambda_{\mathfrak{a}}(A_1),\lambda_{\mathfrak{a}}\theta_1) \\ & {\scriptstyle\lambda}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & (A_2,\theta_2) & \end{array}$$

of homomorphisms as in Definition 1.4.7.

An $\mathfrak{a}$-multiplication is uniquely determined up to an isomorphism, for the details, see Lang [20, 3.2] and Shimura–Taniyama [40, 7.1].

Let $(A_1,\theta_1)$ and $(A_2,\theta_2)$ be $g$-dimensional abelian varieties over $\mathbb{C}$ of a primitive CM type $(K,\Phi)$, analytically represented by $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}_1)$ and

$\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}_2)$ respectively. If a non-zero $\gamma \in K$ is such that $\gamma\mathfrak{m}_1 \subset \mathfrak{m}_2$, then there exists a homomorphism $\gamma_{\theta_1,\theta_2}$ such that the following diagram

$$\begin{array}{ccc} \mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}_1) & \longrightarrow & A_1 \\ {\scriptstyle \mathrm{S}_\Phi(\gamma)}\downarrow & & \downarrow{\scriptstyle \gamma_{\theta_1,\theta_2}} \\ \mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}_2) & \longrightarrow & A_2 \end{array}$$

is commutative with $S_\Phi(\gamma)$ as on page 11. Observe that $\gamma_{\theta_1,\theta_2}$ gives an isogeny of $(A_1,\theta_1)$ onto $(A_2,\theta_2)$.

**Proposition 1.4.9.** *(Lang [20, Proposition 3.2.6], Shimura–Taniyama [40, Proposition 15 in 7.4]) Let $K$ be a CM field and let $\Phi$ be a primitive CM type of $K$. Let $(A_1,\theta_1)$ and $(A_2,\theta_2)$ be abelian varieties over $\mathbb{C}$ of types $(K,\Phi,\mathfrak{m}_1)$ and $(K,\Phi,\mathfrak{m}_2)$ respectively (see Definition 1.4.4). If $\gamma \neq 0$ is an element of $\mathfrak{m}_1^{-1}\mathfrak{m}_2$, then $\gamma_{\theta_1,\theta_2}$ is a $\gamma\mathfrak{m}_2^{-1}\mathfrak{m}_1$-multiplication of $(A_1,\theta_1)$ onto $(A_2,\theta_2)$. Every isogeny of $(A_1,\theta_1)$ onto $(A_2,\theta_2)$ is equal to $\gamma_{\theta_1,\theta_2}$ for some such $\gamma$.* □

**Corollary 1.4.10.** *Any two abelian varieties of the same primitive CM type $(K,\Phi)$ are isogenous to each other.* □

**Proposition 1.4.11.** *Let $(K,\Phi)$ be a primitive CM pair and let $(A_1,\theta_1)$ and $(A_2,\theta_2)$ be $g$-dimensional abelian varieties over $\mathbb{C}$ of types $(K,\Phi,\mathfrak{m}_1)$ and $(K,\Phi,\mathfrak{m}_2)$ respectively (see Definition 1.4.4). Let $[\mathfrak{m}_i]$ denote the class of $\mathfrak{m}_i$ in the class group $\mathrm{Cl}_K$. Then $(A_1,\theta_1)$ and $(A_2,\theta_2)$ are isomorphic if and only if $[\mathfrak{m}_1] = [\mathfrak{m}_2]$.*

*Proof.* Suppose that $\lambda$ is an isomorphism of $(A_1,\theta_1)$ onto $(A_2,\theta_2)$. By Proposition 1.4.9 there is a non-zero $\gamma \in \mathfrak{m}_1^{-1}\mathfrak{m}_2$ such that $S(\gamma)$ gives an isomorphism between $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}_1)$ and $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}_2)$. Therefore we get $S(\gamma)\tilde{\Phi}(\mathfrak{m}_1) = \tilde{\Phi}(\mathfrak{m}_2)$ and hence $\gamma\mathfrak{m}_1 = \mathfrak{m}_2$, so we have $[\mathfrak{m}_1] = [\mathfrak{m}_2]$.

Conversely, if $[\mathfrak{m}_1] = [\mathfrak{m}_2]$ then there is a non-zero $\alpha \in \mathcal{O}_K$ such that $\alpha\mathfrak{m}_1 = \mathfrak{m}_2$. Therefore, the map $S(\alpha)$ gives an isomorphism between $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}_1)$ and $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}_2)$. Hence by Proposition 1.4.8, the map $S(\alpha)$ induces an isomorphism between $(A_1,\theta_1)$ and $(A_2,\theta_2)$. □

**Proposition 1.4.12.** *(Shimura–Taniyama [40, Proposition 30 in 8.5]) Let $A$ be a simple abelian variety over a field $k \subset \mathbb{C}$ with CM by $K$ via $\theta : K \hookrightarrow \mathrm{End}_0(A)$ of CM type $\Phi$. Then $\theta$ is over $k$ if and only if the reflex field $K^r$ of $(K, \Phi)$ is contained in $k$.* □

**Proposition 1.4.13.** *(Shimura [39, (5.5.17) and Proposition 5.14]) Let $(K, \Phi)$ be a primitive CM pair and let $(A, \theta)$ be an abelian variety over $\mathbb{C}$ and of type $(K, \Phi)$. If an automorphism $\sigma$ of $\mathbb{C}$ is the identity map on the reflex field $K^r$, then $(\sigma A, \sigma\theta)$ is of type $(K, \Phi)$.* □

## 1.5 Polarized simple abelian varieties with complex multiplication

Let $K$ be a CM field and let $\Phi$ be a primitive CM type of $K$. Let $(A, \theta)$ be an abelian variety of type $(K, \Phi, \mathfrak{m})$ with CM by $\mathcal{O}_K$. Let $X$ be an ample divisor on $A$ and $E(u, w)$ be the Riemann form on $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m})$ associated to $X$. Then there is an element $t \in K^\times$ (see Shimura–Taniyama [40, Theorem 4 in §6.2] and use Theorem 1.4.6) such that

$$E(\tilde{\Phi}(x), \tilde{\Phi}(y)) = \mathrm{tr}_{K/\mathbb{Q}}(tx\overline{y}) \tag{1.5.1}$$

for every $(x, y) \in K \times K$, and the element $t$ satisfies

$$\overline{t} = -t, \ \mathrm{Im}(\phi(t)) > 0 \quad \text{for all } \phi \in \Phi. \tag{1.5.2}$$

Since we obtained $t$ from an ample divisor, by (1.3) in Shimura [38], we have

$$\mathrm{tr}_{K/\mathbb{Q}}(t\mathfrak{m}\overline{\mathfrak{m}}) = \mathbb{Z}. \tag{1.5.3}$$

Let $\varphi$ be the polarization corresponding to $X$. Then we say that the polarized abelian variety $P := (A, \theta, \varphi)$ is *of type* $(K, \Phi, t, \mathfrak{m})$.

Let $A^*$ be the dual variety of $A$ (recall the definition of $A^*$ from Section 1.3.1). For every $\alpha \in K$, put

$$\theta^*(\alpha) = \theta(\overline{\alpha})^*,$$

where $\theta(\alpha)^*$ is the transpose homomorphism of $\theta(\alpha)$. In [40, 3.3 & 6.3], Shimura shows that $\theta^*$ is an isomorphism of $K$ into $\mathrm{End}_0(A^*)$ and $(A^*, \theta^*)$

is of type $(K,\Phi)$, and $(A^*,\theta^*)$ is analytically represented by the complex torus $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m}^*)$, where

$$\mathfrak{m}^* = \{\beta \in K \mid \mathrm{tr}_{K/\mathbb{Q}}(\beta\overline{\mathfrak{m}}) \subset \mathbb{Z}\}. \tag{1.5.4}$$

**Proposition 1.5.1.** *Let $(A_1,\theta_1)$ and $(A_2,\theta_2)$ be abelian varieties of primitive CM type $(K,\Phi)$. If an isogeny $\lambda$ from $(A_1,\theta_1)$ onto $(A_2,\theta_2)$ is an $\mathfrak{a}$-multiplication, then $\lambda^*$ from $(A_1^*,\theta_1^*)$ onto $(A_2^*,\theta_2^*)$ is an $\overline{\mathfrak{a}}$-multiplication.*

*Proof.* It is Proposition 6 in Shimura–Taniyama [40, 14.4]. □

Let $\mathfrak{D}_{K/\mathbb{Q}}$ be the *different* (the inverse of the dual of $\mathcal{O}_K$ relative to the trace form on $K/\mathbb{Q}$) of $K$.

**Proposition 1.5.2.** *Let $P=(A,\theta,\varphi)$ be a polarized abelian variety of type $(K,\Phi,t,\mathfrak{m})$. Then the isogeny $\varphi: A \to A^*$ is a $t\mathfrak{D}_{K/\mathbb{Q}}\mathfrak{m}\overline{\mathfrak{m}}$-multiplication.*

*Moreover, if $P$ is principally polarized, then $t\mathfrak{D}_{K/\mathbb{Q}}\mathfrak{m}\overline{\mathfrak{m}} = \mathcal{O}_K$.*

*Proof.* By the definition of $\mathfrak{m}^*$ (1.5.4), we have

$$\mathfrak{m}^* = (\mathfrak{D}_{K/\mathbb{Q}}\overline{\mathfrak{m}})^{-1}.$$

Then by (1.3.1), the isogeny $\varphi: A \to A^*$ is represented by the matrix $\mathrm{S}_\Phi(t)$. Hence by Proposition 1.4.9, the polarization $\varphi$ is a $t\mathfrak{D}_{K/\mathbb{Q}}\mathfrak{m}\overline{\mathfrak{m}}$-multiplication from $(A,\theta)$ onto $(A^*,\theta^*)$ (also see Shimura–Taniyama [40, 14.3]).

Moreover, the kernel of $S_\Phi(t)$ is $\tilde{\Phi}(t^{-1}\mathfrak{m}^*)/\tilde{\Phi}(\mathfrak{m})$, which is isomorphic to $t^{-1}\mathfrak{m}^*/\mathfrak{m}$. Hence we have $\ker(\varphi) = (t\mathfrak{D}_{K/\mathbb{Q}}\overline{\mathfrak{m}})^{-1}/\mathfrak{m}$. This implies that if $P$ is *principally* polarized, that is $\ker(\varphi)=1$, then we have $t\mathfrak{D}_{K/\mathbb{Q}}\mathfrak{m}\overline{\mathfrak{m}} = \mathcal{O}_K$. □

Put $\mathfrak{f} := t\mathfrak{D}_{K/\mathbb{Q}}\mathfrak{m}\overline{\mathfrak{m}}$. We now show that there is an $\mathcal{O}_F$-ideal $\mathfrak{f}_0$ such that $\mathfrak{f} = \mathfrak{f}_0\mathcal{O}_K$. By definition, we have $\mathfrak{D}_{K/\mathbb{Q}} = \mathfrak{D}_{K/F}\mathfrak{D}_{F/\mathbb{Q}}$ and moreover the ideal $\mathfrak{D}_{K/F}$ is generated by the elements $(\alpha-\overline{\alpha})$ for $\alpha \in \mathcal{O}_K$. Since $\overline{t} = -t$, we have $t(\alpha-\overline{\alpha}) \in F$ for every $\alpha \in \mathcal{O}_K$. Hence there is an $\mathcal{O}_F$-ideal $\mathfrak{f}_1$ such that $t\mathfrak{D}_{K/F} = \mathfrak{f}_1\mathcal{O}_K$. On the other hand, the ideal $\mathfrak{D}_{F/\mathbb{Q}}\mathfrak{m}\overline{\mathfrak{m}}$ is an $\mathcal{O}_F$-ideal. So if we put $\mathfrak{f}_0 = \mathfrak{f}_1\mathfrak{D}_{F/\mathbb{Q}}\mathfrak{m}\overline{\mathfrak{m}}$, then we get $\mathfrak{f} = \mathfrak{f}_0\mathcal{O}_F$.

Remark that by definition, the ideal $\mathfrak{f}$ is determined by $P=(A,\theta,\varphi)$. Set $\mathfrak{f}(P) := \mathfrak{f}$. We say that $P$ is *of type* $(K,\Phi,\mathfrak{f}(P))$.

**Proposition 1.5.3.** *(Shimura–Taniyama [40, Proposition 2 in 14.2]) Let $(A,\theta)$ be an abelian variety over $\mathbb{C}$ of type $(K,\Phi,\mathfrak{m})$. Let $F$ be the maximal totally real subfield of $K$. Let $X_1$ and $X_2$ be two ample divisors on $A$, and let $E_1$, $E_2$ be the Riemann forms on $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m})$ associated to $X_1$, $X_2$ respectively. Let $t_i$ be the element of $K^\times$ satisfying* (1.5.2) *for the form $E_i$. Then $t_1^{-1}t_2$ is a totally positive element in $F$ and we have*

$$\varphi_{X_1}^{-1}\varphi_{X_2} = \theta(t_1^{-1}t_2) \in \mathrm{End}(A)\otimes\mathbb{Q}. \qquad \square$$

**Proposition 1.5.4.** *(Shimura–Taniyama [40, Proposition 3 in 14.2]) Let the notation be as in Proposition 1.5.3. The polarized abelian varieties $(A,\varphi_{X_1})$ and $(A,\varphi_{X_2})$ are isomorphic if and only if there exist $\epsilon \in \mathcal{O}_K^\times$ such that $t_1^{-1}t_2 = \epsilon\bar{\epsilon}$.* $\square$

### 1.5.1 Classes of polarized simple abelian varieties with CM

For a given *polarized simple abelian variety* $P=(A,\theta,\varphi)$ over $\mathbb{C}$ of primitive CM type $(K,\Phi)$ we can find a $\mathbb{Z}$-lattice $\mathfrak{m}$ in $K$ such that $A$ is isomorphic to $\mathbb{C}^g/\tilde{\Phi}(\mathfrak{m})$. There exists $t\in K^\times$ satisfying (1.5.1) and (1.5.2) such that $P=(A,\theta,\varphi)$ is *of type* $(K,\Phi,t,\mathfrak{m})$. We say that $P$ is *of type* $(t,\mathfrak{m})$ if $(K,\Phi)$ is fixed. Put $\mathfrak{f}(P) := t\mathfrak{D}_{K/\mathbb{Q}}\mathfrak{m}\overline{\mathfrak{m}}$. We call $(t,\mathfrak{m})$ and $(t',\mathfrak{m}')$ equivalent if the following holds

$$t = b\bar{b}t' \text{ and } \mathfrak{m}b = \mathfrak{m}' \text{ with } b\in K^\times.$$

For given $(t,\mathfrak{m})$ satisfying (1.5.2) and (1.5.3), there is a polarized simple abelian variety $P=(A,\theta,\varphi)$ of type $(K,\Phi,t,\mathfrak{m})$, which is unique up to isomorphism (see page 67 in Shimura [38]).

We denote the group of totally positive elements in $F$ by $F_{\gg 0}$. Set

$$\mathfrak{C}_K := (F_{\gg 0}\times I_K)/\{(x\overline{x}, x\mathcal{O}_K) : x\in K^\times\}.$$

We define the multiplication of two classes $[(\xi_1,\mathfrak{c}_1)]$ and $[(\xi_2,\mathfrak{c}_2)]$ in $\mathfrak{C}_K$ by

$$[(\xi_1,\mathfrak{c}_1)][(\xi_2,\mathfrak{c}_2)] = [(\xi_1\xi_2,\mathfrak{c}_1\mathfrak{c}_2)].$$

This set becomes a group with the identity element $[(1, \mathcal{O}_K)]$. It is clear that the group $\mathfrak{C}_K$ is abelian.

Let $P_i = (A_i, \theta_i, \varphi_i)$ be of type $(K, \Phi, t_i, \mathfrak{m}_i)$ for $i \in \{1,\ 2\}$. By (1.5.2), we have $t_1^{-1}t_2 \in F_{\gg 0}$, hence we get an element $[(t_1^{-1}t_2, \mathfrak{m}_1\mathfrak{m}_2^{-1})]$ in $\mathfrak{C}_K$. We define

$$(P_2 : P_1) := [(t_1^{-1}t_2, \mathfrak{m}_1\mathfrak{m}_2^{-1})] = [(\xi, \mathfrak{c})] \in \mathfrak{C}_K. \tag{1.5.5}$$

Then by the definition of $\mathfrak{f}(P_i)$, we have

$$\mathfrak{f}(P_1)\mathfrak{f}(P_2)^{-1} = \xi^{-1}\mathrm{N}_{K/F}(\mathfrak{c}). \tag{1.5.6}$$

**Proposition 1.5.5.** *(Shimura–Taniyama [40, Proposition 10 in 14.7])* *We have $(P_2 : P_1) = [(1, \mathcal{O}_K)]$ if and only if $P_2$ is isomorphic to $P_1$.* □

## 1.5.2 The first main theorem of CM

**Theorem 1.5.6** (The first main theorem of complex multiplication, [40, Main Theorem 1])**.** *Let $(K, \Phi)$ be a primitive CM type with $[K : \mathbb{Q}] = 2g$ and let $(K^r, \Phi^r)$ be its reflex. Let $P = (A, \theta, \varphi)$ be a polarized simple abelian variety of type $(K, \Phi)$ with CM by $\mathcal{O}_K$. Let $M$ be the field of moduli of $(A, \varphi)$. Then $K^r \cdot M$ is the unramified class field over $K^r$ corresponding to the ideal group*

$$I_0(\Phi^r) := \{\mathfrak{b} \in I_{K^r} :\ \mathrm{N}_{\Phi^r}(\mathfrak{b}) = (\alpha),\ \alpha\overline{\alpha} \in \mathbb{Q} \text{ for some } \alpha \in K^\times\}.$$

We give a part of the proof because we will use ideas from this in Chapter 4.

*Proof.* Let $P = (A, \theta, \varphi)$ be of type $(K, \Phi, t, \mathfrak{m})$ with CM by $\mathcal{O}_K$. Put $\mathfrak{f}(P) = t\mathfrak{D}_{K/\mathbb{Q}}\mathfrak{m}\overline{\mathfrak{m}}$. Let $(A, \theta, \varphi)$ be defined over an algebraic number field of finite degree $k$ such that $k$ is normal over $K^r$. Since there are only finitely many $\sigma A$ up to $\mathbb{C}$-isomorphism of $A$ over $K^r$, such a field $k$ exists and the field of moduli $M$ of $(A, \varphi)$ is contained in $k$.

Let $\sigma \in \mathrm{Gal}(k/K^r)$. Then by Proposition 1.4.13 the abelian variety $(\sigma A, \sigma\theta)$ is of type $(K, \Phi)$ and hence by Corollary 1.4.10, the abelian variety $(A, \theta)$ is isogenous to $(\sigma A, \sigma\theta)$. Let $X$ be the ample divisor satisfying $\varphi = \varphi_X$. By Proposition 1.5.2, the isogeny $\varphi_X$ is an $\mathfrak{f}(P)$-multiplication.

Let $\mathfrak{m}' \in I_K$ and $t' \in K$ such that $\sigma P$ is of type $(K, \Phi, t', \mathfrak{m}')$. The dual of $(\sigma A, \sigma\theta)$ is $(\sigma A^*, \sigma\theta^*)$ and $\varphi_{(\sigma X)} = \sigma\varphi_X$ is an $\mathfrak{f}(P)$-multiplication, see page 124 Shimura–Taniyama [40]. So we have $\mathfrak{f}(\sigma P) = \mathfrak{f}(P)$ hence by (1.5.6), we get

$$\mathrm{N}_{K/F}(\mathfrak{m}'^{-1}\mathfrak{m}) = t^{-1}t'.$$

This concludes $(\sigma P : P) = (\mathrm{N}_{K/F}(\mathfrak{m}'^{-1}\mathfrak{m}), \mathfrak{m}'^{-1}\mathfrak{m}) \in \mathfrak{C}_K$.

On the other hand, for every $\sigma_1, \sigma_2 \in \mathrm{Gal}(k/K^r)$, we have

$$(\sigma_2\sigma_1 P : P) = (\sigma_1 P : P)(\sigma_2 P : P)$$

and the map

$$\begin{aligned} \psi : \mathrm{Gal}(k/K^r) &\to \mathfrak{C}_K \\ \sigma \quad &\mapsto (\sigma P : P), \end{aligned}$$

gives a surjective homomorphism see §15.2 in Shimura–Taniyama [40].

By Proposition 1.5.5, we have $\sigma \in \ker(\psi)$ if and only if $\sigma P$ and $P$ are isomorphic. Moreover, by the definition of $M$ and by Proposition 1.4.8, it holds that $\sigma P$ and $P$ are isomorphic if and only if $\sigma$ fixes the field $M$. Therefore, we have $\ker(\psi) = \mathrm{Gal}(k/MK^r)$ and hence the image of $\psi$ in $\mathfrak{C}_K$ is isomorphic to $\mathrm{Gal}(MK^r/K^r)$. Since $\mathfrak{C}_K$ is abelian, the image of $\psi$ in $\mathfrak{C}_K$ is abelian and so the extension $MK^r/K^r$ is abelian.

It remains to show that $MK^r$ is unramified over $K^r$ corresponding to the subgroup $I_0(\Phi^r)$. For this, we refer to page 127 in Shimura–Taniyama [40, §15]. □

We say that a curve $C$ has CM by an order of a CM field $K$ if the endomorphism ring of its Jacobian $J(C)$ is an order in $K$. Moreover, we say that a curve $C$ is *of type* $(K, \Phi)$, if its Jacobian $J(C)$ *is of type* $(K, \Phi)$.

**Corollary 1.5.7.** *If a curve $C$ is of primitive type $(K, \Phi)$ with CM by $\mathcal{O}_K$ and defined over $K^r$, then the CM class group $I_{K^r}/I_0(\Phi^r)$ is trivial.* □

**Definition 1.5.8.** Let the notation be as in the preceding theorem. The quotient $I_{K^r}/I_0(\Phi^r)$ is called the *CM class group* of $(K, \Phi)$. We say that the CM field $K$ has *CM class number one* if there exists a primitive CM type $\Phi$ such that $(K, \Phi)$ satisfies $I_0(\Phi^r) = I_{K^r}$.

**Definition 1.5.9.**

- The *CM class number one problem for CM fields of degree* $2g$ is the problem of finding all CM class number one pairs $(K, \Phi)$ of degree $2g$.
- The *CM class number one problem for curves of genus* $g$ is the problem of finding all curves of genus $g$ that have a simple Jacobian with CM by the maximal order of a CM class number one field of degree $2g$.

We skip the second and the third main theorems of complex multiplication as we do not need them.