which form, for every odd prime \( p \), a \( \mathbb{Q} \)-linearly independent set of numbers, see theorem 2. Our first proofs of these results are analytical, using among others the partial fraction expansions of the \( \text{csc} \) and the \( \text{csc}^2 \), the functional equation for \( L \)-functions and the expressions for \( L(0; \chi) \) and \( L(-1; \chi) \) in terms of the generalized Bernoulli numbers. In the second part we consider the problem from an algebraic point of view, cf. also [6]. This algebraic approach reveals, in our opinion, much more the real nature of the problem. It leads to the general theorem 3, of which the theorems 1 and 2 are special instances. Moreover it contains Chowla's theorem for the cot and Hasse's for the tan.

2. TWO THEOREMS ON THE COSECANT

The analogue for the \( \text{csc} \) of Chowla's theorem on the cot reads as follows:

**Theorem 1.** Let \( p \) denote an odd prime and let \( m = \frac{1}{2}(p-1) \). The \( m \) numbers

\[
\text{csc} \frac{2\pi \ell}{p}, \quad \ell = 1, \ldots, m,
\]

are linearly independent over \( \mathbb{Q} \), if and only if the multiplicative order of \( 2^{\ell} \text{(mod } p) \) is even.

**Proof.** The starting point of the proof is the partial fraction expansion of the \( \text{csc} \), viz.

\[
\text{csc} z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{z^{2n} - 2 \pi^{2n}}, \quad z \neq 0, \pm \pi, \ldots.
\]

Putting \( z = \frac{2\pi \ell}{p}, \quad \ell = 1, \ldots, m, \) one obtains

\[
\frac{\pi}{p} \text{csc} \frac{2\pi \ell}{p} = \frac{1}{2\ell} + \sum_{n=1}^{\infty} \frac{(-1)^n}{np+2\ell} - \frac{(-1)^n}{np-2\ell}.
\]

By means of the well known orthogonality property of characters,
\[
\frac{1}{p-1} \sum_{\chi \in \hat{G}} \frac{\chi(k) \chi(a)}{\chi(a)} = 1, 0,
\]
according to \( k \equiv a \pmod{p} \) and \( k \not\equiv a \pmod{p} \) respectively, where \( \hat{G} \) denotes the group of all Dirichlet characters to the modulus \( p \), we see that

\[
\frac{\pi \csc \frac{2\pi \ell}{p}}{p} = \frac{1}{p-1} \sum_{\chi \in G} \sum_{k=1}^{\infty} \left( \frac{(-1)^{k-1}}{k} \chi(k) \chi(2\ell) \right) \frac{(-1)^{k-1}}{k} \chi(k) \chi(-2\ell) = \]

\[
= \frac{2}{p-1} \sum_{\chi \in \hat{G}'} \chi(2\ell) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \chi(k) , \ \ell = 1, \ldots, m,
\]

where \( \hat{G}' \) denotes the subset of \( \hat{G} \) of the so-called odd characters, that are the characters for which \( \chi(-1) = -1 \). Note that we used that \( k = np \pm 2\ell \) and \( n \) always have the same parity.

Now

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \chi(k) = (\chi(2) - 1)L(1; \chi)
\]

and therefore

\[
\csc \frac{2\pi \ell}{p} = \frac{2p}{\pi(p-1)} \sum_{\chi \in \hat{G}'} \chi(2\ell) (\chi(2) - 1)L(1; \chi) , \ \ell = 1, \ldots, m.
\]

From the functional equation for \( L(s; \chi) \) with \( \chi \in \hat{G}' \), see e.g. [7], p.5, it follows that

\[(2) \quad L(1; \chi) = -\frac{\pi i}{p} \tau(\chi) L(0; \chi) ,
\]

where \( \tau(\chi) \) denotes the ordinary Gauss sum \( \sum_{t=1}^{p-1} \chi(t)e^{2\pi it/p} \). Hence

\[
\csc \frac{2\pi \ell}{p} = -\frac{2i}{p-1} \sum_{\chi \in G} \chi(2\ell) (\chi(2) - 1) \tau(\chi) L(0; \chi) , \ \ell = 1, \ldots, m.
\]

The generalized Bernoulli numbers \( B_{n, \chi} \), \( n = 0, 1, \ldots \), \( \chi \) a primitive Dirichlet character to the modulus \( f \), are defined by

\[
\sum_{t=1}^{f-1} \frac{\chi(t)ze^{t/z}}{e^{t/z} - 1} = \sum_{n=0}^{\infty} B_{n, \chi} \frac{z^n}{n!} .
\]
and one has

\[ L(1-n; \chi) = -\frac{1}{n} B_{n, \chi} \]

see [7], §2, theorem 1. This yields our final expression for \( \csc \frac{2\pi \ell}{p} \), viz.

\[ \csc \frac{2\pi \ell}{p} = \frac{2i}{p-1} \sum_{\chi \in \hat{G}} \chi(2\ell)(\overline{\chi}(2)-1)B_{1, \chi} \tau(\overline{\chi}), \ell = 1, \ldots, m. \]

Now the proof is finished in the same way as AYOUB's proof in [1] of Chowla's theorem. Suppose that the m rational numbers \( c_\ell, \ell = 1, \ldots, m, \) are such that

\[ \sum_{\ell=1}^{m} c_\ell \csc \frac{2\pi \ell}{p} = 0. \]

In view of (4) this implies

\[ \sum_{\chi \in \hat{G}} \left[ (\overline{\chi}(2)-1)B_{1, \chi} \sum_{\ell=1}^{m} c_\ell \chi(\ell) \right] \tau(\overline{\chi}) = 0. \]

From the definition of the numbers \( B_{n, \chi} \) it follows that with \( \chi \) a non-principal character to the modulus \( p, \)

\[ B_{n, \chi} \in \mathbb{Q}(\mathbb{Z}^{p-1}). \]

In fact, for the numbers \( B_{1, \chi} \), we have

\[ B_{1, \chi} = \frac{1}{p-1} \sum_{t=1}^{p-1} \chi(t)t, \chi \text{ non-principal}, \]

which follows from IWASAWA [7], p.10, last formula. Hence the whole expression between square brackets in (5) belongs to the field \( \mathbb{Q}(\mathbb{Z}^{p-1}) \). But, as AYOUB showed in [1], the numbers

\[ \tau(\chi), \chi \in \hat{G}, \]
are linearly independent over this field. Thus

\[ (\tilde{\chi}(2)-1)B_{1,\chi} \sum_{\ell=1}^{m} c_{\ell}\chi(\ell) = 0, \quad \chi \in \hat{G}'. \]

From (2), (3) and from \( L(1;\chi) \neq 0 \) we see that

\[ (7) \quad B_{1,\chi} \neq 0, \quad \chi \in \hat{G'} \]

and hence that

\[ (\tilde{\chi}(2)-1) \sum_{\ell=1}^{m} c_{\ell}\chi(\ell) = 0, \quad \chi \in \hat{G}'. \]

Let \( p \) be a prime for which the multiplicative order \( k \) of 2 \((\text{mod} \ p)\) is even, say \( k = 2\kappa \). Then \( 2^k \equiv -1 \ (\text{mod} \ p) \) and therefore \( (\chi(2))^k = \chi(-1) = -1, \quad \chi \in \hat{G}'. \) Hence for those \( p \) we always have \( \chi(2) \neq 1, \quad \chi \in \hat{G}' \) and thus

\[ \sum_{\ell=1}^{m} c_{\ell}\chi(\ell) = 0, \quad \chi \in \hat{G}'. \]

Since the matrix

\[ (\chi(\ell)), \quad \ell = 1, \ldots, m, \quad \chi \in \hat{G}', \]

is non-singular we must have

\[ c_{\ell} = 0, \quad \ell = 1, \ldots, m, \]

which proves the if-part of theorem 1.

Suppose now that the order \( k \) of 2 \((\text{mod} \ p)\) is odd; hence \( k \leq m \). For every complex number \( \zeta \) with \( \zeta \neq 0, \pm 1, \pm i \), one has

\[ (8) \quad \zeta(\zeta^{-1} - 1)^{-1} = (\zeta^2 - \zeta^{-2})^{-1} + \zeta^2 (\zeta^2 - \zeta^{-2})^{-1}. \]

A repeated application of (8) on its own last term yields, with \( \zeta \) a primitive \( p \)-th root of unity,
\[ \zeta(\zeta - \zeta^{-1})^{-1} = \sum_{j=1}^{k} \left( \zeta^{2^j} - \zeta^{-2^j} \right)^{-1} + \zeta^2 \left( \zeta^{2^k} - \zeta^{-2^k} \right)^{-1} \]
or
\[ \sum_{j=0}^{k-1} \left( \zeta^{2^j} - \zeta^{-2^j} \right)^{-1} = 0. \]

Observing that \(-1\) is not contained in the multiplicative group generated by 2, modulo \(p\), and that the cosecant is an odd function, we see that (9) is nothing else than a relation
\[ \prod_{\ell=1}^{m} c_\ell \csc \frac{2\pi \ell}{p} = 0, \ c_\ell = 0, \ \pm 1, \ \text{not all} \ c_\ell = 0, \ \ell = 1, \ldots, m. \]

In their studies on the representation of \(-1\) as a sum of squares, FEIN, GORDON & SMITH [5] and CONNELL [4], characterized the primes \(p\) for which the condition of our theorem 1 is fulfilled. It is easy to see that for \(p \equiv 3, 5 \pmod{8}\), the order of 2 (mod \(p\)) is always even and that for \(p \equiv 7 \pmod{8}\) this order is always odd. In [4] and [5] one finds a calculation of the asymptotic density of the primes \(p\) for which the order of 2 is even, among all odd primes. This density is \(17/24\).

For every odd prime \(p\) a set of \(m\) \(\mathbb{Q}\)-linearly independent numbers, in terms of values of the cosecant, is given by the following

**THEOREM 2.** Let \(p\) denote an odd prime and let \(m = \frac{1}{2}(p-1)\). Then the \(m\) numbers
\[ \csc^2 \frac{2\pi \ell}{p}, \ \ell = 1, \ldots, m, \]
are linearly independent over \(\mathbb{Q}\).

**PROOF.** The proof is quite similar to that of theorem 1. Starting with
\[ \csc^2 z = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n\pi)^2}, \ z \neq 0, \ \pm \pi, \ldots, \]
Clearly,
\[ C_\chi \in \mathbb{Q}(e^{\frac{2\pi i}{p}-1}), \quad C_\chi \neq 0, \quad \chi \in \hat{G}^n. \]

For future use we note the following analogue to (6), which follows from [7], p.10:

\[ \frac{\chi}{\tau} = \frac{1}{p} \sum_{t=1}^{p-1} \chi(t)t^2, \quad \chi \neq \chi_0. \]

So we have found the following expression for \( \csc^2 \frac{2\pi \ell}{p} \):

\[ \csc^2 \frac{2\pi \ell}{p} = \frac{2}{\tau} \sum_{\chi \in \hat{G}^n} \chi(2\ell) C_\chi \tau(\chi), \quad \ell = 1, \ldots, m. \]

From this, every relation

\[ \sum_{\ell=1}^{m} c_\ell \csc^2 \frac{2\pi \ell}{p} = 0, \quad c_\ell \in \mathbb{Q}, \quad \ell = 1, \ldots, m, \]

leads to

\[ \sum_{\ell=1}^{m} c_\ell \chi(\ell) = 0, \quad \chi \in \hat{G}^n \]

and since the matrix

\[ (\chi(\ell)), \quad \ell = 1, \ldots, m, \quad \chi \in \hat{G}^n \]

is non-singular, this is only possible when \( c_\ell = 0, \quad \ell = 1, \ldots, m. \)

3. AN ALGEBRAIC APPROACH

We consider a more general problem. Let \( K \) be a field, \( G \) a finite abelian group of order \( n \), with \( n \) prime to the characteristic of \( K \), and \( M \) a module over the group ring \( K[G] \). For \( \alpha \in M \) we are interested in calculating the \( K \)-dimension of \( K[G] \cdot \alpha = \{ r \cdot \alpha \mid r \in K[G] \} \) and, more generally, in finding all \( K \)-linear relations between the elements \( \alpha_\ell, \ldots, \alpha_m \).
Define the map $K[G] \rightarrow M$ by sending $r$ to $ra$, for $r \in K[G]$. The kernel $I$ of this map is called the annihilator of $\alpha$. It is an ideal of $K[G]$ which can be viewed as the space of linear relations between the elements $\sigma \alpha, \sigma \in G$. Obviously, $\dim_K K[G].\alpha = n - \dim_K I$; so the question is how to determine $I$.

First we consider the case that $K$ contains all $e$-th roots of unity, where $e$ is the exponent of $G$. Then the group of characters

$$\hat{G} = \{ \chi : G \rightarrow K^* \mid \chi \text{ is a group homomorphism} \}$$

has order $n$. If we put

$$e_\chi = \frac{1}{n} \sum_{\sigma \in G} \chi^{-1}(\sigma) \sigma \in K[G]$$

then the set \( \{ e_\chi \mid \chi \in \hat{G} \} \) is a $K$-basis for $K[G]$. More precisely, an element

$$r = \sum_{\sigma \in G} k_\sigma \sigma$$

of $K[G]$, with $k_\sigma \in K$, has the following representation on the basis $\{ e_\chi \mid \chi \in \hat{G} \}$:

$$r = \sum_{\chi \in \hat{G}} \left( \sum_{\sigma \in G} k_\sigma \chi(\sigma) \right) e_\chi \quad (13)$$

It is well known and easily proved that multiplication in $K[G]$ is performed componentwise on the basis $\{ e_\chi \mid \chi \in \hat{G} \}$:

$$\left( \sum_{\chi \in \hat{G}} k_\chi e_\chi \right) \cdot \left( \sum_{\chi \in \hat{G}} k'_\chi e_\chi \right) = \sum_{\chi \in \hat{G}} k_\chi k'_\chi e_\chi$$

for $k_\chi, k'_\chi \in K$. Thus we see that the ring $K[G]$ is isomorphic to the product of $n$ copies of $K$, with componentwise ring operations. It follows that every ideal $J$ of $K[G]$ has the form.
for a subset $S$ of $\hat{G}$. So there are precisely $2^n$ ideals of $K[G]$. We say that $J$ corresponds to $S$ if (14) holds; clearly $\dim_K(J) = \#S$.

The annihilator $I$ of $e$ now corresponds to

$$\{\chi \in \hat{G} \mid e \chi \in I\} = \{\chi \in \hat{G} \mid \sum_{\sigma \in G} \chi^{-1}(\sigma)\sigma e = 0\}.$$  

We conclude that the space of linear relations between the $\sigma e$, $\sigma \in G$, is completely determined by the set of characters $\chi$ for which

$$\sum_{\sigma \in G} \chi^{-1}(\sigma)\sigma e = 0 \quad (\epsilon M).$$

In particular we have

$$\dim_K K[G].e = \#\{\chi \in \hat{G} \mid \sum_{\sigma \in G} \chi^{-1}(\sigma)\sigma e \neq 0\}.$$  

In order to deal with general $K$, we choose a field extension $K < K'$ such that $K'$ contains all $e$-th roots of unity, and apply the above results to the $K'[G]$-module $M = K' \otimes_K M$. Then the annihilators $I$ and $I'$ of $e - 1 \otimes e$ in $K[G]$ and $K'[G]$, respectively, determine each other by

$$I' = K' \otimes I \subset K' \otimes K[G] = K'[G],$$

$$I = I' \cap K[G] \text{ (inside } K'[G]).$$

Further

$$\dim_K I = \dim_K I',$$

and $I'$ corresponds to

$$\{\chi \in \hat{G} \mid \sum_{\sigma \in G} \chi^{-1}(\sigma) \otimes \sigma e = 0\}$$

where $\hat{G}$ is the group of characters $G \to K'^*$.  

**Conclusion:** Let $K$ be a field and $G$ a finite abelian group of order prime to the characteristic of $K$. Let $K'$ be an extension field of $K$.
containing the $e$-th roots of unity, with $e = \exp(G)$, and let $\hat{G}$ be the
group of characters $\chi: G \to K^\times$. Then for every $K[G]$-module $M$ and every
$\alpha \in M$ we have

$$\dim_K K[G].\alpha = \#\{\chi \in \hat{G} \mid \sum_{\sigma \in G} \chi^{-1}(\sigma) \otimes \sigma \alpha \neq 0 \text{ in } K' \otimes_K M\}.$$ 

Further, the space of linear relations between $\{\sigma \alpha \mid \sigma \in G\}$ is com-
pletely determined by the set of $\chi \in \hat{G}$ for which $\sum_{\sigma \in G} \chi^{-1}(\sigma) \otimes \sigma \alpha = 0$.

We apply this to the situation $K = \mathbb{Q}$, $M = \mathbb{Q}(\zeta)$ with $p$ prime, and
$G = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$; here $\zeta_p$ denotes a primitive $p$-th root of unity and
$M$ is a $K[G]$-module in an obvious way. We take $K' = \mathbb{Q}$.

For $t \in \mathbb{Z}$, $p \nmid t$, let $\sigma_t$ be the element of $G$ mapping $\zeta$ to $\zeta_p^t$.
Then $G = \{\sigma_t \mid 1 \leq t \leq p-1\}$, and writing $\chi(t)$ for $\chi(\sigma_t)$ we see that $G$
can be identified with the set of Dirichlet characters with conductor
dividing $p$.

The condition $\sum_{\sigma \in G} \chi^{-1}(\sigma) \otimes \sigma \alpha = 0$ can be expressed conveniently
in terms of the coefficients of a representation

$$\alpha = \sum_{t=1}^{p-1} a_t \zeta_p^t \quad (a_t \in \mathbb{Q}).$$

Notice that such a representation exists, since $\{\zeta_p^t \mid 1 \leq t \leq p-1\}$ is
a $\mathbb{Q}$-basis for $\mathbb{Q}(\zeta_p)$. A short computation shows

$$\sum_{\sigma \in G} \chi^{-1}(\sigma) \otimes \sigma \alpha = (\sum_{t=1}^{p-1} \chi(t)a_t)(\sum_{u=1}^{p-1} \chi^{-1}(u) \otimes \zeta_p^u).$$

The second factor on the right (essentially a Gauss sum) is a nonzero
element of $M'$, by the linear independence of $\{\zeta_p^u \mid 1 \leq u \leq p-1\}$ over
$\mathbb{Q}$; so

$$\sum_{\sigma \in G} \chi^{-1}(\sigma) \otimes \sigma \alpha = 0 \iff \sum_{t=1}^{p-1} \chi(t)a_t = 0.$$

We have proved:

**Theorem 3.** Let $p$ be a prime number, and let $\alpha$ be an algebraic number
of the form
\[ \alpha = \sum_{t=1}^{p-1} a_t \zeta_p^t \]

with \( a_t \in \mathbb{Q}, \, 1 \leq t \leq p-1 \), where \( \zeta_p \) denotes a primitive \( p \)-th root of unity. Then the dimension of the \( \mathbb{Q} \)-vector space generated by the conjugates of \( \alpha \) is equal to the number of Dirichlet characters \( \chi \) to the modulus \( p \) for which \( \sum_{t=1}^{p-1} \chi(t)a_t \neq 0 \). Also, the set of these \( \chi \) completely determines the set of all linear relations between the conjugates of \( \alpha \).

In order to derive theorem 1 from theorem 3 we can take
\[ \alpha = 2p(\zeta_p - \zeta_p^{-1})^{-1}, \]
since the set of conjugates of \( \alpha \) equals
\[ \{ \pm \text{ip.csc}(2\pi t/p) \mid 1 \leq t \leq m \}. \]

An elementary computation shows
\[ \alpha = \sum_{t \text{ odd}} (t-p) \zeta_p^t + \sum_{t \text{ even}} t \zeta_p^t \]

where \( t \) ranges over the odd integers in the set \( \{1, 2, \ldots, p-1\} \) and over the even ones, respectively. So we must determine for which \( \chi \) the sum
\[ \sum_{t \text{ odd}} \chi(t)(t-p) + \sum_{t \text{ even}} \chi(t)t \]
vanishes. We have
\[ 2(1-\chi(2)) \sum_{t=1}^{p-1} \chi(t)t = 2 \sum_{t=1}^{p-1} \chi(t)t - 2 \sum_{t=1}^{p-1} \chi(2t)t \]
\[ = 2 \sum_{t \text{ even}} \chi(t)(t-2t) + 2 \sum_{t \text{ odd}} \chi(t)(t-\frac{1}{2}(t+p)) \]
\[ = \sum_{t \text{ odd}} \chi(t)(t-p) + \sum_{t \text{ even}} \chi(t)t. \]

Therefore the sum (15) vanishes if and only if
\[ \chi(2) = 1 \text{ or } \sum_{t=1}^{p-1} \chi(t)t = 0. \]
which by (6), (7) and $B_1, \chi = 0$ for $\chi \in \hat{G}^n$, $\chi \neq \chi_0$, see [7], p. 10, is the same as

$$\chi(2) = 1 \text{ or } \chi(-1) = 1.$$ 

We conclude that the dimension of the $\mathbb{Q}$-vector space generated by the conjugates of $\alpha$ is equal to the number of odd characters $\chi$ for which $\chi(2) \neq 1$. So the dimension is $m$ if and only if $\chi(2) \neq 1$ for every odd character, which happens if and only if the multiplicative order of 2 mod $p$ is even.

This proves theorem 1. Theorem 2 is derived by analogous computations, using the non-vanishing of the sum

$$\sum_{t=1}^{p-1} \chi(t)t^2, \chi \in \hat{G}^n,$$

cf. (12).

Finally, we determine all linear relations between the conjugates of $\alpha = 2p(\zeta_p^{-1})^{-1}$, for an odd prime $p$. If $I' \subset \mathbb{C}[G]$ is the annihilator of $\alpha(= 1 \otimes \alpha)$ then by the above proof $I'$ corresponds to

(16) \quad \{ \chi \mid \chi(2) = 1 \text{ or } \chi(-1) = 1 \}.

Let $J$ be the ideal of $\mathbb{C}[G]$ generated by $1 + \sigma_{-1}$ and $1 + \sigma_2 + \sigma_2^2 + \ldots + \sigma_2^{k-1}$, where $k$ is the multiplicative order of 2 mod $p$. We claim that $I' = J$, and to prove this it suffices to show that $J$ also corresponds to (16).

By (13) we have $e \chi \in J$ if and only if some $r = \sum_{\sigma \in G} k(\sigma) \sigma \in J$ satisfies $\sum_{\sigma \in G} k(\sigma) \chi(\sigma) \neq 0$; since $J$ is generated by $1 + \sigma_{-1}$ and $1 + \sigma_2 + \ldots + \sigma_2^{k-1}$, this happens if and only if $1 + \chi(-1) \neq 0$ or $1 + \chi(2) + \ldots + \chi(2)^{k-1} \neq 0$, which in turn is equivalent to $\chi(-1) = 1$ or $\chi(2) = 1$. So indeed $J$ corresponds to the set (16).

It follows that the annihilator $I$ of $\alpha$ in $\mathbb{Q}[G]$ is generated by $1 + \sigma_{-1}$ and $1 + \sigma_2 + \ldots + \sigma_2^{k-1}$. That means...
(17) \[ \alpha + \sigma_{-1}(\alpha) = 0 \]

(18) \[ \alpha + \sigma_2(\alpha) + \ldots + \sigma_{k-1}(\alpha) = 0, \]

and all \( \varphi \)-linear relations between the conjugates of \( \alpha \) can be derived from these two by conjugation and linearity. (Further (18) follows from (17) if \( k \) is even).

Alternatively, one can prove this by verifying (17) and (18) directly, cf. (9); dimension considerations then show that there cannot be "more" relations.

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