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In this Chapter we highlight the concepts underlying topological insulating states from a phenomenological point of view and introduce some of the models that will be of use in the subsequent parts. This will entail a bulk and a complementing edge state perspective.

2.1 Bulk perspective

2.1.1 Anomalous quantum Hall and quantum spin Hall models

The conceptual treatment presented in the previous Chapter already indicated the intricate relation between topological band theory and the fermion doubling theorem [33, 34]. This can however be made more explicit by considering the simple lattice regularized Hamiltonian

$$H(k_x, k_y) = A \sin(k_x) \tau_x + A \sin(k_y) \tau_y + [M - 2B(2 - \cos(k_x) - \cos(k_y))] \tau_z,$$  \hspace{1cm} (2.1)

in terms of the Pauli matrices $\tau$, natural units and model parameters $A$, $B$ and $M$ [70]. Furthermore, we have adopted the convention to trivially suppress the creation and annihilation operators for such Hamiltonians in momentum space. A crucial observation regarding the above Hamiltonian then pertains to the fact that the mass term can change sign throughout the Brillouin zone as function of the parameter $M/B$, revealing the defining property that lies at heart of the nonzero Hall conduction of the model. This is particularly easily motivated by expanding Eq. (2.1) around the momenta $(0,0)$, $(\pi,0)$, $(0,\pi)$ and $(\pi,\pi)$ and examining the response to an electromagnetic $U(1)$ gauge potential $A_\mu$. 
Switching to Lagrangian formalism it is evident that such a procedure results in four two-component Dirac or Weyl equations of the type

\[ \mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - \gamma^\mu A_\mu - \hat{M}) \psi, \]  

(2.2)

where \( \bar{\psi} = \psi^\dagger \tau_z \), \( \gamma^0 = \tau_z \), \( \gamma^1 = i \tau_y \), \( \gamma^2 = -i \tau_x \), and the relative signs as well as the mass terms \( \hat{M} \) depend on the specific momentum points. Correspondingly, obtaining the associated polarization tensor \( \Pi_{\mu\nu} \) from the current-current correlation function encompasses a standard result [71, 72] and renders, in addition to the Maxwell part, the well-known topological Chern-Simons term [73]

\[ S_{SC} \sim \mathcal{C}_1 \int dx \, dy \int dt \, \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \]  

(2.3)

The result (2.3) for the Lagrangian (2.2) then finds a natural physical understanding by virtue of the assumed lattice regularization. Indeed, the \((0,0)\) and \((\pi,\pi)\) flavors contribute a Hall conductivity \( \sigma_{xy} = \mathcal{C}_1 = \text{sign}(\hat{M}) \), whereas the other two points contribute \( \sigma_{xy} = -\text{sign}(\hat{M}) \) in units of \( \frac{e^2}{2\pi h} \). Hence, we observe the direct significance of the role of the mass term by simply summing the Hall conductivities. Namely, only when the mass term attains different signs at the different momenta, the total Hall conductivity can acquire a nonzero value and the presence of the Chern-Simons term is assured. Since this term breaks the parity invariance \( \tau_x H(k_x, -k_y) \tau_x = H(k_x, k_y) \) per fermion flavor even in the parity invariant limit \( m \to 0 \), the presence of the Chern-Simons term for the total system is somewhat more commonly referred to as a parity anomaly. The appearance of this anomaly consequently discloses the essential feature underlying the anomalous Hall effect and we note that, apart from subtleties such as the different symmetries of the free graphene model and its number of Dirac flavors, the Haldane model [27] relates to the exact same mechanism.

Accordingly, these considerations can be verified by explicit calculation of the first Chern number (Appendix 1.A) of the associated complex bundle. This procedure reduces for a model of the form

\[ H(k_x, k_y) = d(k) \cdot \tau, \]  

(2.4)

to the particular simple expression

\[ \mathcal{C}_1 = \frac{1}{4\pi} \int dk_x dk_y \varepsilon^{abc} \frac{\partial \hat{d}_a}{\partial k_x} \frac{\partial \hat{d}_b}{\partial k_x} \frac{\partial \hat{d}_c}{\partial k_y}, \]  

(2.5)
in terms of the unit vector \( \hat{d} \). We notice that in this case the quantization is rather manifest; the integrand is nothing but the Jacobian of a mapping of the map \( \hat{d}(k) : T^2 \rightarrow S^2 \). Hence, integration results in the image of the Brillouin zone on the two sphere, which is a topological (Pontryagin) winding number of quantized value \( 4\pi \nu \), \( \nu \in \mathbb{Z} \). Explicit evaluation of the Hamiltonian (2.1) then shows that indeed

\[
C^1 = \begin{cases}
0, & 0 > \frac{m}{B} \text{ or } \frac{m}{B} > 8 \\
1, & 0 < \frac{m}{B} < 4 \\
-1, & 4 < \frac{m}{B} < 8
\end{cases}, \tag{2.6}
\]

conforming the above continuum results.

![Diagram](image)

Figure 2.1: A particular illustrative way to visualize the topological non-trivial phases of the Hamiltonian (2.1) is by plotting the vector field \( \hat{d}(k) \) over the Brillouin zone. This reveals a skyrmion centered around \( k = 0 \) for \( 0 < M/B < 4 \), whereas the skyrmion is centered around \( k = (\pi, \pi) \) for \( 4 < M/B < 8 \).

The relation between the anomalous quantum Hall effect and the quantum spin Hall effect, as illustrated in the previous Chapter with the Haldane and Kane-Mele model [31, 32], can subsequently be exploited to define an analogous model exhibiting a \( \mathbb{Z}_2 \) topological phase. Combining two time reversed copies of Eq.(2.1), the Hamiltonian attains the form

\[
\mathcal{H}(k_x, k_y) = \begin{pmatrix}
H(k) & 0 \\
0 & H^*(-k)
\end{pmatrix} = A \sin(k_x) \gamma_x + A \sin(k_y) \gamma_y + M(k) \gamma_0, \tag{2.7}
\]

with \( M(k) = M - 2B(2 - \cos(k_x) - \cos(k_y)) \), \( \gamma^0 = \tau_z \otimes \sigma_0 \), \( \gamma^1 = \tau_x \otimes \sigma_z \), \( \gamma^2 = \tau_y \otimes \sigma_0 \) and the Pauli spin matrices \( \sigma \) acting in the spin space. As the spin Hall conductivity \( \sigma_{xy} = (\sigma_{xy}^+ - \sigma_{xy}^-) \), a \( \mathbb{Z}_2 \) topological phase is expected for \( 0 < M/B < 4 \) and \( 4 < M/B < 8 \). This simple argumentation can be shown to be profound. Specifically, the \( \mathbb{Z}_2 \) invariant
\( v \) is explicitly given by \((-1)^v = \prod_{\Gamma_i} \text{sign}(M(\Gamma_i))\), in terms of the time reversal invariant (TRI) momenta \( \Gamma_i \) comprising \((0,0), (\pi,0), (0,\pi)\) and \((\pi,\pi)\) [38], verifying the \( \mathbb{Z}_2 \) topological entity of the above parameter regimes also in the presence of \( \langle S_z \rangle \) breaking perturbations that do not close the gap. As a matter of fact, the above simple model pertains to the effective description introduced by Bernevig, Hughes and Zhang [40] used to describe the actual experimental viable realization of the quantum spin Hall effect in CdTe/HgTe/CdTe quantum wells [41] and finds its origin in the same regularization considerations as above.

### 2.1.2 Three dimensional \( \mathbb{Z}_2 \) topological band insulators

The generic form of the \( 2 + 1 \) dimensional Hamiltonian (2.7) can readily be generalized to three spatial dimensions, resulting in

\[
\mathcal{H}(k_x, k_y, k_z) = A_1 \sin(k_x)\gamma_x + A_1 \sin(k_y)\gamma_y + A_2 \sin(k_z)\gamma_z + M(k)\gamma_0. \tag{2.8}
\]

Where in the above, it is assumed that the above Dirac matrices \( \gamma \) satisfy the Clifford algebra and are odd (even) under time reversal for the odd (even) momentum terms. As in the two dimensional case, the topological properties are determined by the momentum depended mass term. Taking \( M(k) = M - 2B(3 - \cos(k_x) - \cos(k_y) - \cos(k_z)) \), for example, calculation of the \( \mathbb{Z}_2 \) invariant still amounts to multiplying the signs of the mass terms at the TRI points, which in this case constitute the vertices of the first octant of the cubic Brillouin zone, and confirms the existence of the non-trivial phases for the gapped parameter regimes \( 0 < M/B < 4, 4 < M/B < 8 \) and \( 4 < M/B < 12 \). Strikingly, the simple model (2.8) also meets its lower dimensional counterpart in terms of application to physical systems, as its form encompasses the lattice regularized version of the effective continuum description of the topological phases in the second generation Bi\(_2\)Se\(_3\) family of materials [43, 44, 45].

### 2.1.3 Field theories of \( \mathbb{Z}_2 \) topological band insulators

Although the outlined mechanism can directly be employed to construct the above models of \( \mathbb{Z}_2 \) topological band insulators, an elegant reminiscent procedure provides for a route of also addressing the effective field theory [69]. This description is rooted in the generalization of the \( 2 + 1 \) dimensional integer quantum Hall effect (IQHE) to a \( 4 + 1 \) dimensional time reversal invariant analogue [74, 75], the origin of which can actually
be understood from a geometrical argument [76] that is closely related to the Hopf maps [77]. Specifically, the $2+1$ dimensional IQHE can be apprehended by mapping the problem to the two-sphere $S^2$, with a Dirac monopole at its center [78]. Topological non-trivial configurations then arise as the $U(1)$ gauge potentials extending from the North and South pole are matched non-trivially at the equator, which is in fact possible due to the natural isomorphism between the circle $S^1$ and the gauge group $U(1)$. This geometrical coincidence occurs similarly in the case of the four-sphere $S^4$, where the equator $S^3$ is isomorphic to the gauge group $SU(2)$ and, on a heuristic level, the IQHE in four spatial dimensions can effectively be regarded as the $SU(2)$ variant of the familiar $2+1$ dimensional quantum Hall effect.

Correspondingly, the above analysis concerning the $2+1$ dimensional IQHE can directly be translated. In particular, Eq. (2.3) finds its generalization in

$$S_{cs} \sim \mathcal{C}^2 \int d^4 x \int dt \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau \quad \mu, \nu, \rho, \sigma, \tau \in \{0, 1, 2, 3, 4\}. \quad (2.9)$$

Here, the second Chern Chern number $\mathcal{C}^2$, associated with the non-Abelian Berry’s phase field, plays the exact same role as the the first Chern character (Appendix 1.A), arising similarly in the context of the electromagnetic response of models of the type as in Equations (2.7) and (2.8). More importantly, the $4+1$ dimensional Chern-term can subsequently be subjected to a fundamental procedure known as dimensional reduction, resulting in the effective response theory governing $(2+1)$ and $(3+1)$ dimensional $\mathbb{Z}_2$ topological band insulators. Basically, retaining momentum in one direction results in a parameter family of dimensionally reduced Hamiltonians, which are connected due to the von Neumann-Wigner theorem [79]. In this specific gauge, the reduced action then exactly reproduces the Laughlin argument. Indeed, the momentum gets shifted by the gauge field and defines a periodic interpolation parameter $\theta(x,t)$, which is directly related to the charge polarization $P[\theta(x,t)]$ during such a cycle [20, 21]. Resultantly, the effective action can be written as

$$S_{3D} = \frac{1}{4\pi} \int d^3 x \int dt \varepsilon^{\mu\nu\sigma\tau} P[\theta(x,t)] \partial_\mu A_\nu \partial_\sigma A_\tau \quad \mu, \nu, \sigma, \tau \in \{0, 1, 2, 3\}. \quad (2.10)$$

Time reversal symmetry then dictates that the difference in second Chern character of any two continuous interpolations (Laughlin cycles) associated with such descended Hamiltonians is even and hence defines two different classes, characterized by the parity of the second Chern number [69]. This is in direct accordance with the underlying $\mathbb{Z}_2$ classification. In fact, time reversal invariance actually constrains the value of $P$.
to a constant, characterizing the topological and trivial insulating classes by $P = 0, \frac{1}{2}$, respectively. Finally, its presence also allows for a similar topological distinction for the reminiscent $2 + 1$ dimensional response theory, as can be obtained by utilizing the extra possible reduction step.

The action (2.10) finds a direct physical motive, as it provides for the condensed matter realization of the axion Lagrangian or $\theta$-vacuum, familiar in the context of quantum chromodynamics [80, 81], with $\theta$ playing the role of the axion field. Essentially, the surface between the topological vacuum and the trivial vacuum provides for an axion domain wall structure, as detailed in the next section. Moreover, as the the $\theta$-term couples the magnetic and electric field, a magnetic field then induces a charge polarization, resulting in the so-called topological magneto-electric effect (TME). As a result, $\mathbb{Z}_2$ topological insulators thus also provide for a physical setting in which these exotic field theories might find an application [82], possibly culminating in striking features such as for example dyons and the Witten effect [69, 83].

2.2 Edge state perspective

We conclude this introductory part by accentuating the above notions from the perspective of the edge states, being the very signature of topological band structures. Specifically, this perspective motivates the manifestation of the underlying concept of anomalies in Dirac field theories and their relation to nontrivial states of free fermion matter directly in the context of domain walls. This basic mechanism finds its conceptual origin in the rather surprising context of the $1 + 1$ dimensional effective field theory of polyacetylene. Hence, we will first step back a dimension to revisit the well-known Su-Schrieffer-Heeger model.

2.2.1 Jackiw-Rebbi solitons in the Su-Schrieffer-Heeger model

The Su-Schrieffer-Heeger (SSH) model [84, 85, 86], introduced as an effective description of the conducting polymer polyacetylene, turns out out to host all the necessary ingredients for the rich concepts of fermion number fractionalization and domain wall states [87, 88]. Consider a ring of (trans)polyacetylene consisting of $N$ monomers of mass $M$ (Figure 2.2). Assuming that the relevant degree of freedom is the displacement $u$ of the monomers, a Peierls tight-binding model including electron-phonon coupling to
2.2 Edge state perspective

Figure 2.2: Chemical notation of (trans) polyacetylene. The conducting polymer undergoes at half-filling a Peierls instability into a dimerized state. Nonetheless, there are two topologically distinct vacua, which resultantly allow for interpolating domain wall configurations.

the lowest order is readily obtained [86]. A calculation, resembling the mean-field theory of conventional superconductivity and its quasiparticle spectrum [89], then shows that at half-filling the system subsequently has two separate dimerized ground state configurations, associated with spontaneous symmetry breaking (Figure 2.2). Moreover, similar to 1 + 1 dimensional sine-Gordon theory or $\phi^4$ theory, the system subsequently allows for soliton configurations interpolating between this so-called A phase and B phase; introducing an order parameter

$$\phi_j = (-1)^j u_j,$$

(2.11)

taking values $\langle \phi_j \rangle = \pm \eta$ in the A/B phase, numerical calculations [84, 85] show that there is a phonon field configuration $\phi_j \approx \eta \tanh((j - j_0)/\xi)$ approaching the A (B) phase when $j \to \pm \infty$.

Of topical interests are the consequences for the electronic states, the relevant physics of which can be addressed by expanding the bands around the Fermi energy. In particular, such an expansion results in two spin-degenerate linear dispersing branches ($\pm$), which are coupled by virtue of the dimerization wave number $2k_f$ (being a consequence of the doubling of the unit cell), and therefore results in the real space Hamiltonian

$$H(x) = \Psi(x)^\top (\tau_z(-i\partial_x) + \phi(x)\tau_x)\Psi(x),$$

(2.12)

with $\Psi(x) = (\psi(x)_+, \psi(x)_-)^\top$. Focussing on the soliton back ground configuration, it is easy to see that Eq. (2.12) admits a single normalizable "Jackiw-Rebbi" zero-
energy solution of the form $\psi_0(x) \sim \phi_+ e^{-\int_0^x dx' m(x')} [90, 91]$, where $\tau_y \phi_\pm = \pm \phi_\pm$ and it is assumed that the mass profile exhibits a sign change from negative to positive at $x = 0$. The appearance of this zero energy state is in fact the epitome of the profound interplay of topology and Dirac fermionic states as formalized by the Atiyah-Singer index theorem [92, 93], which directly relates the number of zero-energy modes to the topological index of the elliptic differential operator (2.12).

The single domain wall mode subsequently leads to the notion of fractionalized quantum numbers as a result of the crucial presence of the apparent charge conjugation symmetry, generated by the anti-commuting $\tau_y$ matrix. Introducing a full Dirac sea, to assure that the spectrum is bounded from below, it is straightforward to calculate the density $\rho_s(E)$ of single-particle states in the presence of the soliton relative to the vacuum $\rho(E)$. Exploiting the completeness relation, the change in the charge of the ground state is found to be

$$Q = -\frac{1}{2} \int_0^\infty dE (\rho_s(E) - \rho_s(-E)), \quad (2.13)$$

revealing that due to the spectral symmetry only the zero modes contributes half a charge. More generally, one can show that for a traceless Hermitian operator $Q$, representing a constant of motion, the ground state average $\langle Q \rangle$ is given by

$$\langle Q \rangle = \frac{1}{2} \left( \sum_{\text{occupied}} \Psi_E^\top Q \Psi_E - \sum_{\text{unoccupied}} \Psi_E^\top Q \Psi_E \right), \quad (2.14)$$

where the sum has the interpretation of an integral when the spectrum is continuous and $\{\Psi_E\}$ is the complete set of eigenstates of the Hamiltonian with energy $E$. Taking into account the spin degeneracy and assuming that $S_z$ is a good quantum number, this form immediately exposes the spin-charge separation. Namely, populating one of the modes leads to two types of spinons, having spin $S_z = \pm \frac{1}{2}$ but no net relative charge $Q = 0$, whereas unoccupied or doubly occupied configurations result in chargons or holons, characterized by $S_z = 0$ and $Q = \pm 1$. Physically, in the case of the polyacetylene ring, a soliton and anti-soliton are obviously created in pairs. The valence band experiences a depletion of one state for each pair and thus, by charge conjugation symmetry, the conduction band then also loses a total of one state. These states form two pairs located at mid gap when the solitons are separated by large distances as compared to the characteristic length.

Although it is evident that the SSH model exemplifies a $1 + 1$ dimensional topological band insulator, where the role of protecting symmetry is now played by the charge
2.2 Edge state perspective

conjugation symmetry, we finally note that the connection to the above notions can be made even more explicit by obtaining the quantum numbers via an alternative approach. Basically, coupling the Dirac fermions to a slowly varying complex field that represents the soliton, the response is readily calculated using the Golstone-Wilczek formula [94]. This procedure then shows that response theory is of the same form as Eq. (2.10). Accordingly, the polarization can similarly be obtained from dimensionally reducing the 2 + 1 dimensional quantum Hall effect, while imposing charge conjugation symmetry [69].

2.2.2 Two dimensional topological band insulators

The Jackiw-Rebbi solitons in the Su-Schrieffer-Heeger model provide for a particularly accessible illustration of the conceptual mechanism of the formation of zero-modes bound to domain walls. However, such constructions have been a subject of extensive study in field theory and find straightforward generalizations in higher dimensions in the form of the so-called Callan-Harvey effect [95]. This line of reasoning then directly provides for the effective description of the surface states, characterizing nontrivial band structures. In particular, it is straightforward to see that a Dirac Hamiltonian with a mass profile that changes sign form positive to negative at \( x = 0 \)

\[
H(x, k_y) = -i\tau_y \partial_x + k_y \tau_y + m(x) \tau_z
\]  

(2.15)

has a normalizable domain wall solution of the form \( \psi_0(x) \sim \phi_\pm e^{ik_y} e^{-\int_0^x dx' m(x')} \), where \( \tau_y \phi_\pm = \pm \phi_\pm \). This directly signifies the correspondence between the existence of spatially localized edges states and the topological characterization of the vacua on either sides of the interface \( x = 0 \), as signaled by the sign change of the mass. Moreover, these surface states have a definite chirality. Namely, as an actual system will have two edges, the non-normalizable solution, with an overall sign in the exponent, is also part of the spectrum and represents the normalizable solution at the opposite surface.

This general mechanism finds immediate application in the concrete setting of the anomalous Hall model (2.1). Breaking the translational invariance in Eq. (2.1) in one direction and solving the spectrum as function of the projected momentum \( k \) readily reveals the edge states. For \( 0 < M/B < 4 \), the spectrum features two low-energy states that transverse the band gap forming a Dirac cone at \( k = 0 \), whereas for \( 4 < M/B < 8 \) this cone is located at \( k = \pi \) (Figure 2.3). These states can explicitly be shown to
be of the anticipated chiral nature in accordance with the Chern numbers (2.6). That is, for $0 < M/B < 4$ the surface states propagate clockwise along the perimeter of the system, while for $4 < M/B < 8$ the direction is reversed. More interestingly, the solitonic configuration of the mass for the specific value of $k$ also exposes the manifestation of the anomaly at the edge. Focussing on one edge, the coefficient of response term (2.3) is set by the sum of the topological charges, being $\pm 1/2$ depending on the mass of each flavor. Across the interface the mass term changes sign inducing a similar anomalous term that cancels the bulk anomaly. This anomaly-cancellation argument lies at heart of the effective understanding of the integer as well as fractional quantum Hall effect [96], and in essence entails the Callan-Harvey effect [95].

Figure 2.3: Spectra of Eq. (2.1) on a cylinder. Specifically, periodic boundary conditions were imposed in the $\hat{y}$-direction, whereas the $\hat{x}$-direction featured open boundary conditions. Due to the $C_4$ symmetry the result with the boundary conditions interchanged is equivalent. The edge states traverse the gap forming a Dirac cone. As the two states are localized to opposite surfaces, the total system features chiral edge states.

The translation to the spin Hall effect subsequently amounts to a routine procedure, exploiting the physical perspective. Specifically, in case of $S_z$ conservation, each spin copy of the model (2.7) is characterized by one surface state culminating in a total system that features a pair of counter propagating helical or spin filtered edge states. Generally speaking, the usual Luttinger liquid theory [97, 98] describing the chiral edge states of a quantum Hall system is hence modified to

$$H(k) = \int \frac{dk}{2\pi} (\psi_{k,\uparrow} v k \psi_{k,\uparrow} - \psi_{k,\downarrow} v k \psi_{k,\downarrow}) + H_{\text{pert}},$$

(2.16)

where $v$ is the effective velocity. The edge state perspective directly exposes the crucial role of time reversal symmetry (TRS), as it prohibits backscattering terms opening a gap.
in Eq. (2.16) for each individual edge. This then conveys the persistence of the edge states in more general scenarios that include spin-conservation breaking perturbation terms and signifies the bulk-boundary correspondence in a rather concrete manner.

2.2.3 Three dimensional $\mathbb{Z}_2$ topological band insulators

The protection mechanism based on the spatial separation of the surfaces and the resultant role of time reversal symmetry finds an immediate generalization to $3 + 1$ dimensions, providing for the edge state perspective on the existence of the three dimensional $\mathbb{Z}_2$ topological band insulator. Let us for concreteness derive the anticipated edge states explicitly in case of the above model (2.8) in a continuum setting. Expanding the model around $k = 0$, the Hamiltonian attains the form

$$H_{\text{eff}}(k_x, k_y, k_z) = A_2 k_x \gamma^1 + A_2 k_y \gamma^2 + A_1 k_z \gamma^3 + M(k) \gamma^0,$$

(2.17)

where in analogy to [45] it is assumed that $M(k) = M - B(k_x^2 + k_y^2 + k_z^2)$ and the $\Gamma$ matrices are specified by $\gamma^0 = \tau_z \otimes \sigma_0$, $\gamma^1 = \tau_x \otimes \sigma_x$, $\gamma^2 = \tau_y \otimes \sigma_y$, $\gamma^3 = \tau_z \otimes \sigma_z$. Considering the Hamiltonian in a half-plane geometry for $z > 0$, it is readily verified that the defining equation for the anticipated $k_x, k_y = 0$ edge states

$$\mathcal{H}(0, 0, -i \partial_z) \Psi(z) = [-i A_1 \gamma^3 \partial_z + \gamma^0 (M + B \partial_z^2)] \Psi(z) = 0,$$

(2.18)

admits solutions of the form

$$\Psi_\downarrow(z) = \begin{pmatrix} 0 \\ \Psi(z) \end{pmatrix}, \quad \Psi_\uparrow(z) = \begin{pmatrix} \Psi(z) \\ 0 \end{pmatrix}.$$  

(2.19)

Specifically, the full solution for $\psi(z)$ is given by

$$\psi_0(z) = (\alpha e^{\lambda_+ z} + \beta e^{\lambda_- z}) \phi_\uparrow + (\zeta e^{\lambda_+ z} + \eta e^{\lambda_- z}) \phi_\downarrow, \quad \alpha, \beta, \zeta, \eta \in \mathbb{C},$$

(2.20)

where $\lambda_\pm = -A_1 \pm \sqrt{4MB + A_1^2}$ and $\tau_y \phi_\pm = \pm \phi_\pm$. Crucially, the normalization condition $\text{Re}[\lambda_\pm] < 0$ (taking $\eta, \zeta = 0$) or $\text{Re}[\lambda_\pm] > 0$ (taking $\alpha, \beta = 0$) can only be satisfied when the band inversion condition $MB > 0$ is met. This criterium is again in direct correspondence with the nontrivial characterization of Eq. (2.17). That is, for $MB > 0$ the continuum description is in the inverted regime and associated with the topological phase, whereas for $MB < 0$ the system is in the trivial regime [45].
Finally, the obtained surface states can also be exploited to project the Hamiltonian onto the subspace spanned by $|\Psi_\uparrow(z)\rangle$ and $|\Psi_\downarrow(z)\rangle$, resulting in a surface Hamiltonian that yet another time illustrates the general principles. Concretely, the result in leading order in $k$ is

$$H_{\text{surface}}(k_x,k_y) = \begin{pmatrix} 0 & A_2(k_x - ik_y) \\ A_2(k_x + ik_y) & 0 \end{pmatrix} \quad (2.21)$$

and provides for an adequate description of the experimentally observed edge states in the second generation materials, such as for example Bi$_2$Se$_3$ [43, 44]. Coupling the system to a magnetic field $B = B_x e_z$ then reveals the manifestation of the Callan-Harvey mechanism. Taking the specific gauge $A_z = 0$, the Hamiltonian (2.21) becomes

$$H(k_x,k_y) = \sigma_x(k_x + A_x) + \sigma_y(k_y + A_y) - gB\sigma_z, \quad (2.22)$$

where the Zeeman term now acts as the effective mass. Consequently, the relativistic Landau levels come in the standard form

$$\varepsilon_{n,\sigma_3} = \pm [(2n + 1)B - \sigma_3 B + (gB)^2]^{\frac{1}{2}} \quad n = 0,1,2,3,..., \quad \sigma_3 = \pm 1. \quad (2.23)$$

The symmetry of the spectrum in conjunction with the fact that $\varepsilon_{0,1}$ only carries half of the degeneracy of the other levels explicitly indicates the relation with the above solitons, showing that the surface similarly accumulates charge and spin. More interestingly, as $gB$ acts the effective mass a familiar consideration shows that, in case of two opposite surfaces, the two edges of the system feature an opposite Hall conductivity $\sigma_{xy} = \frac{i}{2} \text{sign}(gB)$, in units of $e^2/h$. Hence, the total response term including both surfaces can be written as

$$S_{\text{eff}} = \frac{1}{4\pi} \left[ \int_{S_1} dxdy \int dt \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \int_{S_2} dxdy \int dt \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right], \quad (2.24)$$

where we have reverted to natural units ($\hbar = e = c = 1$), $\mu, \nu, \rho \in \{0,1,2\}$ and $S_1, S_2$ denote the two relevant surfaces. Expressing Eq. (2.24) as a volume integral then explicitly demonstrates the bulk-boundary correspondence. Specifically, the $\theta$-term (2.10)

$$S_{3D} = \frac{1}{4\pi} \int d^3x \int dt \varepsilon^{\mu\nu\sigma\tau} \theta \partial_\mu A_\nu \partial_\sigma A_\tau \quad (2.25)$$

with $\theta = \frac{1}{2}$ is clearly a total derivative and integrates to the above boundary theory, reconfirming the intricate relation between the topological bulk characterization and its manifestation on the edge. That is, on the closed manifold the response is governed
by the action (2.10) and the polarization $P$ only takes the distinguishable time reversal invariant values 0 or $\frac{1}{2}$, as the additional integer part is associated with gauge transformations and does not represent a physical quantity [69]. Correspondingly, in the case of a system with open boundaries, the domain wall structure of the polarization is in direct accordance with this topological term and resultantly physically signifies the topological distinction of the two vacua on either side of the boundary.