Dynamics of a charged spinning particle in a Reissner-Nordström geometry

THESIS
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We study the dynamics of a charged spinning particle in a Reissner-Nordström geometry using the hamiltonian formalism. We use covariant instead of canonical momentum and a worldline along which the spin tensor is covariantly constant. We find the equations of motion as well as the constants of motion and give a full characterization of the circular orbits for a minimal hamiltonian. We study the spin and charge dependence of the innermost stable circular orbit.

In the last part we introduce a non-minimal hamiltonian, including spin-spin interaction and an interaction between the spin tensor and the electromagnetic field. We show that the conserved quantities that we found with the minimal hamiltonian are still constants of motion.
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Introduction

1.1 Compact objects

This thesis presents a framework for the computation of the motion of charged compact spinning objects in the geometry of a charged black hole. There are three main types of stellar compact objects: white dwarfs, neutron stars and black holes. These objects differ from normal stars in the sense that the inward gravitational pull is not balanced by thermal pressure, but, in the case of white dwarfs and neutron stars, by degeneracy pressure [1]. Black holes are even more exotic and are the result of unstoppable gravitational collapse (an introduction of black holes will be given below). Compact objects typically have a small radius and a very high density: White dwarfs have radii of $10^{-2} R_\odot$ and typical densities of $10^6 \text{ g/cm}^3$ (for comparison, the density of the sun is $1.4 \text{ g/cm}^3$ [2]). Neutron stars are even more compact: their typical radius is 10 km and their typical density $10^{14} \text{ g/cm}^3$. Black holes come in three types:

- end-state of stellar evolution, with a typical radius of a few kilometers. Typical densities are of the order of $10^{17} \text{ g/cm}^3$.
- supermassive black holes at the center of galaxies. The mass of black holes of this type can be as large as $10^9 M_\odot$ (with a radius of the order of $10^9 \text{ km}$), but since the density decreases with the size of the black hole, supermassive black holes have a moderate density: only a few grams per centimeter cubed.
- primordial black holes, a hypothetical type of black hole that was formed in the early universe [3]. Their mass would be of the order of $10^{-19} M_\odot$.

The study of compact objects is relevant for astrophysics because the three types are all possible end states of the evolution of stars. A couple of interesting processes might take place when a star has exhausted its hydrogen fuel:

- the helium in the star is converted to carbon from which (depending on the star’s mass) even heavier nuclei are formed. This process is called nucleosynthesis.
• ejection of the outer shells in planetary nebulae or (for very heavy stars) supernovae.

• gravitational collapse. During the formation of a white dwarf, the gravitational collapse is halted by the degeneracy pressure of the electrons. When a neutron star is formed, even the electron degeneracy pressure is not strong enough to balance the collapse and electrons and protons are forced to form neutrons. The neutron degeneracy pressure is enough to counteract the gravitational forces. If even the neutron degeneracy pressure is not large enough, nothing can halt the gravitational collapse and a black hole is formed.

In order to study these interesting phenomena, we need to understand the motion of compact objects in great detail.

Another reason to study black holes in particular is their influence on the evolution of galaxies. In this case, the compact object is used as a probe to study the black hole. It is believed that most galaxies have a supermassive black hole at their center \([4]\), with a mass ranging from millions to billions solar masses. In 1974 the radio source Sagittarius A* was discovered at the center of our galaxy \([5]\) and in 2002 it was shown that this object must be a very heavy black hole \([6]\). Because of their extremely large mass, the black holes at the centers of galaxies influence the evolution of these galaxies to a great extent.

### 1.2 Black Holes

Black holes were predicted by Einstein’s theory of General Relativity, but for a long time it was generally assumed that they did not exist in reality \([7]\). Let us first have a look at black holes in the theory of general relativity and then at their role in astrophysics.

Shortly after Einstein published his theory of General Relativity in 1915, the German astronomer Karl Schwarzschild published an exact solution of Einstein’s equations for a “mass point”\([8]\). The line element of this famous Schwarzschild solution in Droste coordinates* is given by:

\[
\text{\textit{ds}}^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{1 - \frac{2M}{r}} \text{dr}^2 + r^2 \text{d\theta}^2 + r^2 \sin^2 \theta \text{d\phi}^2,
\]

for a metric with \((-\, +\, +\, +\, +\, +\)-signature. The radius \(r_S = 2M\) is called the Schwarzschild radius. Units have been chosen such that \(c = 1\) and Newton’s constant, \(G = 1\).

---

*The Schwarzschild solution was independently and almost simultaneously found by Johannes Droste \([9]\). Droste used simpler coordinates, but he published his results too late. Even though the solution is known as the ‘Schwarzschild solution’, the formulation that is generally used is the one found by Droste.
It follows from Birkhoff’s theorem, that any spherically symmetric part of spacetime that is a solution to Einstein’s equation in vacuum has a Schwarzschild geometry [10], [11]. This means that the exterior of any massive, spherically symmetric, electrically neutral object can be described by the Schwarzschild metric. Birkhoff’s theorem can be generalized to the exterior of a massive, spherically symmetric, charged object, which can be described by the Reissner-Nordström metric, that will be used in this thesis [12]. An object is called a black hole if all of its mass is confined within its Schwarzschild radius.

It took more than forty years after the formulation of the Schwarzschild metric, to understand the significance of the coordinate singularity at \( r = r_S \). In 1958 David Finkelstein realised that the surface \( r = r_S \) is an event horizon and that a black hole is a region of space from which nothing (not even light) can escape [13]. From the efforts of Hawking, Israel and Carter the no-hair theorem emerged [11], which states that the gravitational and electromagnetic field of a black hole are completely determined by only three parameters: the mass \( M \), the angular momentum \( J \) and the charge \( Q \).

Since black holes do not emit radiation, they can only be detected by their interaction with their environment. Black holes that are the end state of some star can be detected when they are moving in a binary system together with a normal star (approximately two thirds of all stars are members of a binary pair [3]). If the binary system has a small orbit, material of the normal star is transferred to the black hole, forming an accretion disk around it. The material that falls inwards loses gravitational energy, thereby emitting gravitational and electromagnetic waves. Gravitational waves are very hard to detect, but the electromagnetic radiation that is produced in this process can be detected by X-ray detectors.

However, the process described above is not unique for black holes; material from a normal star in a binary system can also form an accretion disk around a neutron star. Neutron stars can not be more massive than a few solar masses (Tolman-Oppenheimer-Volkoff-limit), so if the invisible object has a larger mass, it can be concluded that it is a black hole.

The first observation of a system that seemed to contain a black hole was in 1972 [7]. The system was an X-ray binary source called Cygnus X-1, that was composed of a giant star and an invisible companion.

The observation of supermassive black holes at galactic centers is a different challenge: the observation is not only hindered by the fact that black holes are black, but also by the fact that the radius of the black hole is typically 10 million times smaller than the radius of the entire galaxy [3]. The presence of a black hole is generally inferred from the motion of the objects orbiting around it: if these orbits require a very large mass in a small volume it is assumed that this object must be a black hole. In figure 1.1 the orbits of six stars orbiting the black hole in the Milky Way are shown.

Supermassive black holes in galactic centers are associated with the phenomenon of Active Galactic Nuclei (AGN): intense radiation that is produced at the center
of galaxies. This radiation produced at the core can be as luminous as the total radiation of all the stars in the galaxy together. It is assumed that black holes are responsible for this process, since an enormous amount of energy is released when particles fall into a black hole.

1.3 Objective

This research project fits in the larger objective of being able to compute the motion of compact objects in a black hole geometry. We are working in the test-particle limit: the particle mass $m$ is much smaller than the mass $M$ of the black hole and the compact object can be fully characterised by its mass, spin and charge. The goal of the research [15] and of this thesis, is to determine the motion of spinning test particles.

The equations of motion of classical spinning test particles in general relativity were found by Papapetrou in 1951 [16]. In 2015, d’Ambrosi et al. introduced a formalism in which the equations of motion of spinning particles are much simplified [15]. In their article, d’Ambrosi et al., use a parametrization for which the spin tensor is conserved along the world line of the test particle. This allows them to give a full characterization of circular orbits in terms of the radius and the total angular momentum. They also study the equations of motion for a non-minimal hamiltonian that contains a spin-spin interaction (equivalent to the Stern-Gerlach force) via the gravitational field.

Figure 1.1: Orbits of six stars orbiting Sagittarius A* [14]
In the field of quantum mechanics, spin has always been a topic of extensive investigation. Soon after Bohr’s introduction of the hydrogen model, the effect of spin-orbit coupling on the fine structure was already calculated [17]. Spin-orbit coupling also plays an important role in atomic and molecular physics [18], [19]. It becomes clear that spin deserves a lot of attention in general relativity as well if we look at our own universe that contains many spinning objects; consider for example our own Earth, or neutron stars, that can perform hundreds rotations per second [20]. Using the same formalism as in [15], it is possible to study the dynamics of spinning objects for more general spacetimes than the Schwarzschild spacetime.

In this thesis we will restrict our attention to charged non-rotating (Reissner-Nordström) black holes. We study the motion of test particles that have spin as well as charge.

1.4 The Reissner-Nordström geometry

The geometry of a charged, non-rotating black hole is described by a Reissner-Nordström metric. The most general Reissner-Nordström black hole has an electric charge as well as a magnetic charge, but we will study the slightly simpler case of electric charge \( Q \) only. This is a very reasonable assumption, since magnetic monopoles have never been observed. The Reissner-Nordström metric is given by:

\[
ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + \frac{1}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{1.2}
\]

with \( M \) the mass of the black hole. Units are again chosen such that \( c = 1, G = 1 \) and no factor of \( 4\pi \) appears in Coulomb’s law.

The only nonzero components of the electromagnetic field strength tensor are \( F_{rt} \) and \( F_{tr} \) [21]:

\[
F_{rt} = -F_{tr} = \frac{Q}{r^2}. \tag{1.3}
\]

The black hole has event horizons at

\[
r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \tag{1.4}
\]

Black holes with \( Q^2 > M^2 \) do not have an event horizon and the singularity at \( r = 0 \) would thus be a naked singularity, which means that particles could travel all the way to \( r = 0 \) and then return safely. This does not only violate Penrose’s cosmic censorship conjecture [22], but it would also imply that the mass constituting the black hole would be negative [21]. We will only consider the case \( Q^2 \leq M^2 \). A black hole with \( Q^2 = M^2 \) has only one horizon and is called an extremal black hole.

The Christoffel symbols and the non-zero components of the Riemann curvature tensor can be found in Appendix A.
Phase-space structure

In analogy with [15], we will specify the dynamics by defining a phase space, a set of Poisson-Dirac brackets and a hamiltonian. Because of the presence of the gravitational and electromagnetic gauge fields, the canonical momenta in the hamiltonian formalism become gauge dependent. This difficulty can be avoided by the use of covariant momentum (analogous to the use of covariant derivatives in quantum field theory), which allows us to find gauge covariant equations of motion. The use of covariant momenta is discussed in [23].

In addition to the position and momentum degrees of freedom, $x^\mu$ and $\pi_\mu$, the spin degrees of freedom can be described by an antisymmetric spin tensor $\Sigma^{\mu\nu}$. The tensor can be decomposed into two four-vectors

$$S^\mu = \frac{1}{2\sqrt{-g}}\varepsilon^{\mu\nu\kappa\lambda} u_\nu \Sigma_{\kappa\lambda} \quad \text{and} \quad Z^\mu = \Sigma^{\mu\nu} u_\nu, \quad (2.1)$$

with the time-like vector $u$, for which $u_\mu u^\mu = -1$. $S$ is the Pauli-Lubanski pseudo-vector that can be associated with a magnetic dipole moment or internal angular momentum and $Z$ is the Pirani vector that can be associated with an electric or mass dipole moment [24], [25]. Both vectors are spacelike.

The set of Dirac-Poisson brackets that we will use is almost identical to the set used in [15], with only one addition to the momentum-momentum term to account for the electric field.

$$\left\{ x^\mu, x^\nu \right\} = 0$$

$$\left\{ \pi_\mu, \pi_\nu \right\} = \frac{1}{2} \Sigma^{\kappa\lambda} R_{\kappa\lambda\mu\nu} + qF_{\mu\nu}$$

$$\left\{ \Sigma^{\mu\nu}, \Sigma^{\kappa\lambda} \right\} = g^{\mu\kappa} \Sigma^{\nu\lambda} - g^{\mu\lambda} \Sigma^{\nu\kappa} - g^{\nu\kappa} \Sigma^{\mu\lambda} + g^{\nu\lambda} \Sigma^{\mu\kappa} \quad (2.2)$$

$$\left\{ x^\mu, \pi_\nu \right\} = \delta^\mu_\nu$$

$$\left\{ x^\mu, \Sigma^{\kappa\lambda} \right\} = 0$$

$$\left\{ \Sigma^{\mu\nu}, \pi_\lambda \right\} = \Gamma^\mu_\lambda \chi^{\nu\kappa} - \Gamma^\nu_\lambda \chi^{\mu\kappa}$$
In Appendix B we will prove that the Jacobi identities hold for the brackets that are given above.

As a last ingredient of the phase-space structure we need to specify a hamiltonian. In the main part of this thesis we will work with the minimal hamiltonian defined by

\[ H_0 = \frac{1}{2m} g^{\mu\nu} \pi_\mu \pi_\nu. \] \hspace{1cm} (2.3)

In the last part of this thesis we will also consider a non-minimal hamiltonian:

\[ H = H_0 + \frac{\kappa}{4} R_{\mu\nu\kappa\lambda} \Sigma^{\mu\nu} \Sigma^{\kappa\lambda} + \lambda F_{\mu\nu} \Sigma^{\mu\nu}. \] \hspace{1cm} (2.4)
Chapter 3

Equations of Motion and Conserved quantities

3.1 Equations of Motion

For functions $A$ (that can be scalars as well as tensors) that are parametrized in terms of the proper time $\tau$, the evolution equation can be found from the bracket with the Hamiltonian:

$$\frac{dA}{d\tau} = \{A, H_0\}. \quad (3.1)$$

We can find the equations of motion for $x^\mu$, $\pi_\mu$ and $\Sigma^{\mu\nu}$:

- $\dot{x}^\mu = \{x^\mu, H_0\} = \frac{1}{m} g^{\mu\nu} \pi_\nu$, \quad (3.2)
  which means that the momentum is proportional to the four-velocity $u^\mu$:
  $$\pi_\mu = mg_{\mu\nu} \dot{x}^\nu. \quad (3.3)$$

- $\dot{\pi}_\mu = \{\pi_\mu, H_0\} = \Gamma^\nu_\mu_\lambda \pi_\nu \dot{x}^\lambda + \frac{1}{m} (\frac{1}{2} \Sigma^{\kappa\lambda} R_{\kappa\lambda\mu} \pi_\nu + q F^{\nu}_\mu \pi_\nu)$. \quad (3.4)
  This can be rewritten in terms of the component of the covariant derivative parallel to the world line $x^\mu(\tau)$:
  $$D_\tau \pi_\mu = \frac{1}{m} (\frac{1}{2} \Sigma^{\kappa\lambda} R_{\kappa\lambda\mu} \pi_\nu + q F^{\nu}_\mu \pi_\nu). \quad (3.5)$$

- $\dot{\Sigma}^{\mu\nu} = \{\Sigma^{\mu\nu}, H_0\} = \dot{x}^\sigma (\Gamma^\mu_\rho_\sigma \Sigma^{\nu\rho} - \Gamma^\nu_\rho_\sigma \Sigma^{\mu\rho})$, \quad (3.6)
  from which it follows that the spin-tensor is covariantly conserved along the world line:
  $$D_\tau \Sigma^{\mu\nu} = 0. \quad (3.7)$$
By substitution of equation 3.3 into equation 3.5, we obtain

\[ D^2 \tau x^\mu = \ddot{x}^\mu + \Gamma^\mu_{\lambda \nu} \dot{x}^\lambda \dot{x}^\nu = \frac{1}{m^2} \left( \frac{1}{2} \Sigma^\kappa\lambda R^\mu_{\kappa\lambda \nu} \dot{x}^\nu + q F^\mu_{\nu} \dot{x}^\nu \right), \] (3.8)

which reduces to the geodesic equation for \( \Sigma = 0 \) and \( F = 0 \) or \( q = 0 \).

Even though the spin tensor is covariantly constant, the Pauli-Lubanski and Pirani vectors are not:

\[ D^\tau S^\mu = \frac{1}{2m \sqrt{-g}} \epsilon^{\mu \nu \kappa \lambda} \Sigma^\kappa \lambda u^\sigma \left( \frac{1}{2} \Sigma^\alpha \beta R^\mu_{\alpha \beta \nu \sigma} + q F^\mu_{\nu \sigma} \right), \] (3.9)

and

\[ D^\tau Z^\mu = \frac{1}{m} \Sigma^\mu \nu u^\sigma \left( \frac{1}{2} \Sigma^\alpha \beta R^\mu_{\alpha \beta \nu \sigma} + q F^\mu_{\nu \sigma} \right). \] (3.10)

### 3.2 Constants of Motion

By construction, the hamiltonian is a constant of motion with value

\[ H_0 = -\frac{m}{2}. \] (3.11)

It follows from equation 3.7, that the total spin is also conserved:

\[ I = \frac{1}{2} g^\kappa \mu g^\lambda \nu \Sigma^\kappa \lambda \Sigma^\mu \nu = S^\mu \mu + Z^\mu \mu. \] (3.12)

Additional constants of motion \( J \) are solutions to the equation:

\[ \{ J, H_0 \} = \frac{1}{m} g^\mu \nu \tau_\mu \left( \frac{\partial f}{\partial \tau^\mu} + \Gamma^\mu_{\kappa \lambda} \tau_\kappa \frac{\partial f}{\partial \tau_\lambda} + \left( \frac{1}{2} \Sigma^\alpha \beta R^\kappa \beta \lambda \mu + q F^\kappa \lambda \mu \right) \frac{\partial f}{\partial \tau_\lambda} \right) + \Gamma^\mu_{\kappa \lambda} \Sigma^\kappa \lambda \frac{\partial f}{\partial \Sigma^\kappa \lambda} = 0. \] (3.13)

Consequently, constants of motion that are linear in the phase space variables are of the form:

\[ J = \gamma + \alpha^\sigma \tau_\sigma + \frac{1}{2} \beta_{\sigma \tau} \Sigma^{\sigma \tau}, \] (3.14)

with

\[ \nabla_\mu \alpha_\nu + \nabla_\nu \alpha_\mu = 0 \]

\[ \nabla_\lambda \beta_{\mu \nu} = R^\kappa_{\mu \nu \lambda} \alpha_\kappa \quad \text{or} \quad \beta_{\mu \nu} = \frac{1}{2} (\nabla_\mu \alpha_\nu - \nabla_\nu \alpha_\mu) \] (3.15)

\[ \partial_\mu \gamma = q F^\mu \lambda \alpha^\lambda. \]

\( \alpha^\mu \) is an isometry of the metric (a so-called Killing vector). The metric is invariant under a general coordinate transformation in the direction of \( \alpha^\mu \). The Reissner-Nordström metric has the same Killing vectors as the Schwarzschild metric: \( \partial_t \),
\[ \partial_\phi, \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi \text{ and } \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi. \]

The four corresponding constants of motion are given by:

\[
E = \frac{qQ}{r} - \pi_t - \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right) \Sigma^{tr},
\]

\[
J_1 = -\sin \phi \pi_\theta - \frac{\cos \theta \cos \phi}{\sin \theta} \pi_\phi - r \sin \phi \Sigma^{r\theta}
- r \sin \theta \cos \theta \cos \phi \Sigma^{r\phi} + r^2 \sin^2 \theta \cos \theta \Sigma^{\theta \phi},
\]

\[
J_2 = \cos \phi \pi_\theta - \frac{\cos \theta \sin \phi}{\sin \theta} \pi_\phi + r \cos \phi \Sigma^{r\theta}
- r \sin \theta \cos \theta \sin \phi \Sigma^{r\phi} + r^2 \sin^2 \theta \sin \phi \Sigma^{\theta \phi},
\]

\[
J_3 = \pi_\phi + r \sin^2 \theta \Sigma^{r\phi} + r^2 \sin \theta \cos \theta \Sigma^{\theta \phi}.
\]

These constants of motion correspond to conservation of energy and conservation of total angular momentum.
Planar Orbits

The motion of spinless particles in a spherically symmetric geometry is always confined to a plane. In [15] it is noted that this statement no longer holds for particles with nonzero spin. Precession of spin can be compensated by precession of the orbital angular momentum, such that the total angular momentum is conserved, but the orbit is not planar anymore. However, planar motion is possible under certain conditions.

In this chapter, we will closely follow the analysis of [15]. For planar motion, the direction of the angular momentum must be conserved, which results in the constraint \( J_1 = J_2 = 0 \) if the plane of motion is the plane \( \theta = \frac{\pi}{2} \). The momentum in the direction perpendicular to the plane should be zero, \( \pi_\theta = 0 \). From the expressions for \( J_1 \) and \( J_2 \), it follows that \( \Sigma^{r\theta} = 0 \) and \( \Sigma^{\theta\phi} = 0 \), so the spin must be aligned with the orbital angular momentum. From the absence of acceleration in the plane perpendicular to \( \theta = \frac{\pi}{2} \), it follows that \( \Sigma^{\theta t} = 0 \) as well.

The remaining constants of motion simplify to:

\[
E = \frac{qQ}{r} - \pi_t - \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right) \Sigma^{tr} \quad \text{and} \quad J = \pi_\phi + r \Sigma^{r\phi}. \tag{4.1}
\]

The simplest case of planar motion is circular motion, for which \( r = R \) and \( \pi_r = 0 \). In this case, the Hamiltonian constraint reduces to:

\[
(1 - \frac{2M}{R} + \frac{Q^2}{R^2})u_t^{t^2} = 1 + R^2u^{\phi^2}. \tag{4.2}
\]

In order to stay at fixed \( r \), the radial acceleration should also be zero:

\[
mu_t^{t^2} \frac{2MR^3 - 3M^2R^2 + 6MRQ^2 - 3R^2Q^2 - 2Q^4}{R^3(MR - Q^2)} + mu^{\phi^2} \frac{-R^3 + 3MR^2 - 2RQ^2}{R^2 - 2MR + Q^2} = -u_t \left( \frac{qQ}{R} \frac{-2MR + 3Q^2}{R(MR - Q^2)} + \frac{qQ}{R^2} \right) + u^{\phi} \frac{J}{R} \frac{MR - Q^2}{R^2 - 2MR + Q^2} \tag{4.3}
\]
Together, equations 4.2 and 4.3 fully determine $u^t$ and $u^\phi$, so they must be constants and their derivatives should thus be zero. For $u^t$ we find:

$$\frac{du^t}{d\tau} = \frac{1}{mR^2} \Sigma^{t\phi}(MR - Q^2)u^\phi = 0 \quad (4.4)$$

and for $u^\phi$:

$$\frac{du^\phi}{d\tau} = \frac{1}{mR^6} \Sigma^{t\phi}(R^2 - 2MR + Q^2)(MR - Q^2)u^t. \quad (4.5)$$

From both equations it follows that $\Sigma^{t\phi}$ must be zero. Since $\Sigma^{t\phi}$ must always remain zero, we obtain another equation in terms of $u^t$ and $u^\phi$ from the vanishing rate of change:

$$0 = R(R^2 - 2MR + Q^2)(MR - Q^2)\dot{\Sigma}^{t\phi}$$

$$= mu^t u^\phi R(-3M^2 R^2 - R^4 + 4MR^3 + 2MRQ^2 - 2R^2Q^2)$$

$$- u^t \frac{J}{R}(MR - Q^2)^2 + u^\phi (ER^3 - qQR^2)(R^2 - 2MR + Q^2). \quad (4.6)$$

Finally, using equations 4.2 and 4.3, it is possible to eliminate $E$ and $u^t$ from equation 4.6. After some rearrangement, we obtain:

$$J(Q^2 - MR)(-2MR + 3Q^2 - R^2u^\phi^2(R^2 - 2Q^2))$$

$$= qQR^2 u^\phi(R^2 - 2MR + Q^2)\sqrt{(1 + R^2u^\phi^2)(R^2 - 2MR + Q^2)}$$

$$+ mR^2 u^\phi((Q^2 - MR)(2Q^2 - R^2) - R^3u^\phi^2(-5MQ^2 + 6M^2R + 4Q^2R$$

$$- 6MR^2 + R^3)),$$  

which fully determines $u^\phi$ in terms of $R$, $J$, $Q$ and $M$. $u^t$ can then be determined from equation 4.2.
Chapter 5

Dependence of the radius of the ISCO on the charge and spin of the particle

In Newtonian gravity, a massive test particle can move in a stable orbit around a gravitating object at any radius. For spherically symmetric black holes in general relativity, this is no longer the case; there is an innermost stable circular orbit (ISCO).

Figure 5.1: Effective potential for different values of $\frac{L}{M}$ for a massive particle orbiting a Schwarzschild black hole [26]

In a Schwarzschild geometry, the motion of a test particle can be described in terms of an effective potential [21]:

$$\frac{1}{2} u'^2 + V(r) = \frac{1}{2} E^2,$$  \hspace{1cm} (5.1)

with

$$V(r) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3},$$  \hspace{1cm} (5.2)
where \( L = R^2 \mu \), the orbital angular momentum per unit mass. In figure 5.1 the effective gravitational potential of a particle without spin is drawn as a function of the radial distance, for different values of the orbital angular momentum. For an angular momentum \( L > \sqrt{12} M \), the potential has a maximum, which corresponds to an unstable circular orbit, and a minimum, that corresponds to a stable orbit. For \( L = L_{ISCO} = \sqrt{12} M \), the potential has only one critical point, which corresponds to a marginally stable orbit. In fact, the name ISCO is not completely correct, since only orbits with \( r > r_{ISCO} \) can be stable and the ISCO itself is only marginally stable. As becomes clear from figure 5.1, for values of the angular momentum smaller than the value at the ISCO, the potential has no minimum and stable orbits are not possible.

ISCO\’s are relevant for astrophysics, since they are the inner radius of the accretion disk of a black hole.

We will find the radius of the ISCO by setting the derivative of \( L \) with respect to \( R \) in equation 4.7 equal to zero, while keeping \( \sigma \) and \( q \) constant. Substitution of the value of the found expression for \( L \) into equation 4.7 itself then gives an expression for \( R_{ISCO} \) in terms of the spin and charge of the particle.

5.1 The spinless Schwarzschild case

To make the procedure clear, let\’s look at a spinless particle orbiting a Schwarzschild black hole first. It can be shown that for Schwarzschild the angular momentum \( L \equiv R^2 \mu \) per unit mass is related to the radius of the orbit in the following way [11]:

\[
L^2 (R - 3M) - MR^2 = 0. \tag{5.3}
\]

By taking the derivative with respect to \( R \) and demanding that \( \frac{dL}{dR} = 0 \), we find that

\[
L_{ISCO} = \sqrt{2MR} \quad \text{and} \quad R_{ISCO} = 6M. \tag{5.4}
\]

5.2 The Reissner-Nordström case

To find the radius of the ISCO for a spinning particle in a Reissner-Nordström geometry, we take the derivative of equation 4.7 with respect to \( R \). We first split the constant of motion \( J \) into an orbital and internal angular momentum:

\[
J = mL + R\sigma \tag{5.5}
\]

with

\[
L = R^2 \mu. \tag{5.6}
\]

We simplify the notation by introducing three dimensionless quantities:

\[
x \equiv \frac{R}{M'} \quad y \equiv \frac{L}{M'} \quad w \equiv \frac{Q}{M}. \tag{5.7}
\]
In these variables, equation 4.7 can be rewritten as:

\[
\frac{\sigma}{m} x (x - w^2) (x^2 (2x - 3w^2) + y^2 (x^2 - 2w^2)) = y(x^2 - 2x + w^2) [(x^2 (x - w^2) - y^2 (-3x + x^2 + 2w^2)) - \frac{qw}{m} x \sqrt{(x^2 + y^2)(x^2 - 2x + w^2)}].
\]

(5.8)

5.2.1 The case \(\sigma = 0\)

Before we look at the most general case of a particle with spin and charge, we will first look at the case without spin. For \(\sigma = 0\), equation 5.8 simplifies to

\[
x^2 (x - w^2) - y^2 (-3x + x^2 + 2w^2) = \frac{qw}{m} x \sqrt{(x^2 + y^2)(x^2 - 2x + w^2)}.
\]

(5.9)

Taking the derivative with respect to \(x\), while demanding that \(\frac{dy}{dx} = 0\) and that \(w\) is a constant, yields:

\[
x^2 + (3 - 2x)y^2 + 2x(x - w^2) = \frac{qw (3x^4 - 5x^3 + 2x^2 (y^2 + w^2) - 3xy^2 + y^2 w^2)}{m \sqrt{(x^2 + y^2)(x^2 - 2x + w^2)}}.
\]

(5.10)

These two equations determine the radius of the ISCO as a function of the charge of the particle and the charge of the black hole. The analytic solution can be found in Appendix C. In figure 5.2 the radius of the innermost stable circular orbit is plotted as a function of the charge of the test particle, for four different black

![Figure 5.2: Dependence of the ISCO radius on the test particle’s charge](image-url)
holes. Elimination of $y^2$ yields two solutions (see equation C.3). The right (left) part of the graph corresponds to the solution with the $+(-)$-sign. We only look at the interval $|\frac{\alpha}{\beta}| \leq 1$, since larger charge-to-mass ratios would be unphysical (as was explained in chapter 1).

From the figure it becomes clear that for a weakly to intermediately charged black hole ($|\frac{Q}{M}| < 0.5$), the radius of the ISCO only depends on the charge very weakly. The ISCO radius remains close to the Schwarzschild value ($6M$). For more strongly charged black holes we notice that for particles with a charge opposite to that of the black hole the radius of the ISCO increases with the magnitude of the particle’s charge. Due to the extra electrical attraction, the particle needs to stay further from the black hole to move in a stable orbit. For not too large test charges with the same sign, the effect is opposite, the charged particle has a smaller ISCO than a neutral one. However, for large enough values of the charge of the test particle, the ISCO becomes larger again. For extremal test particles (with $\frac{q_m}{M} = 1$) moving around an extremal black hole $R_{ISCO} \to \infty$, so there is no stable orbit possible.

Let’s try to explain the graph by looking at the effective potential [27]

$$V = \frac{qQ}{mr} + \sqrt{(1 + \frac{L^2}{r^2})(1 - \frac{2M}{r} + \frac{Q^2}{r^2})}.$$  

(5.11)

In figure 5.3 a graph of the effective potential of a black hole with $\frac{Q}{M} = 0.8$ and a test particle with $\frac{q}{m} = 0.8$ is shown for several values of the angular momentum $L$. The graphs shows an inflection point for $L = 4.11$, the angular momentum corresponding to the ISCO of figure 5.2. If we compare equation 5.11 to figure 5.2, we must conclude that for a large part of the graph, the $\frac{L^2}{mr^2}$ term dominates the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{potential_graph.png}
\caption{Effective potential for $\frac{Q}{M} = 0.8$ and $\frac{q}{m} = -0.8$ for several values of $L$.}
\end{figure}
Coulomb term. This explains why opposite charges are repelled, instead of attracted. Figure 5.4 confirms this picture: the decrease of the ISCO radius is caused by a decrease in $L$. However, for large enough $q$, the angular momentum becomes so small that the term is no longer dominant and the ISCO radius increases due to Coulomb repulsion.

![Graph](image)

**Figure 5.4:** Angular momentum on the ISCO as a function of the test particle charge for $\frac{Q}{M} = 0.8$

### 5.2.2 The case $\sigma \neq 0$

Taking the derivative of equation 5.8 with respect to $x$, without setting $\sigma = 0$ yields:

$$
\frac{\sigma}{m}(10x^4 + 4x^3y^2 - 20x^3w^2 - 4xy^2w^2 - 3x^2y^2w^2 + 9x^2w^4 + 2y^2w^4)
= y((x^2 - 2x + w^2)(3x^2 + 3y^2 - 2x(y^2 + w^2)) + 2(x - 1)(x^3
+ 3xy^2 - 2y^2w^2 - x^2(y^2 + w^2)) - \frac{qw(x^2 - 2x + w^2)}{m\sqrt{(x^2 + y^2)(x^2 - 2x + w^2)}})
(5x^4 - 7x^3 + 2x^2(2y^2 + w^2) - 5xy^2 + y^2w^2).
$$

Equations 5.8 and 5.12 allow the elimination of $y$, so that $x_{\text{ISCO}}$ can be plotted as a function of $\frac{\sigma}{m}$, $\frac{q}{m}$ and $w$. Elimination of $\frac{qw}{m}$ yields a fifth order polynomial in $y$ and elimination of $\frac{\sigma}{m}$ yields a fifth order polynomial in $y^2$, so we used Wolfram Mathematica to solve the problem numerically.

In figure 5.5 the radius of the ISCO is plotted as a function of the test particle’s spin. In figure 5.5a the dependence is shown for a weakly charged black hole, for which the charge of the test particle almost has no influence. If the spin of
the particle is aligned with its orbital angular momentum ($\frac{a}{m} > 0$), the ISCO radius becomes smaller than the Schwarzschild value, asymptotically reaching the value of the outer horizon. If the spin of the particle is anti-aligned with its orbital angular momentum, the ISCO radius becomes larger.

The spin dependence of the ISCO radius of a strongly charged black hole is qualitatively the same as for a weakly charged black hole, but in this case, the particle’s charge does influence the ISCO radius. For positive or slightly negative spin values, a particle with charge opposite to the charge of the black hole has the largest ISCO radius. This is the same effect as in the left part of figure 5.2. However, around $\frac{a}{m} = -0.6$ the opposite effect takes over: a particle with opposite charge has a smaller ISCO than the neutral particle and the particle. For positive spin, the effect of the particle’s charge is more significant than for negative spin, because the Coulomb force falls off as $\frac{1}{r}$. 

**Figure 5.5:** Dependence of the ISCO radius on the test particle’s spin
Chapter 6

Non-minimal hamiltonian

We now look at the case of a hamiltonian that is no longer minimal:

\[ H = H_0 + H_1, \]  

with \( H_1 \) given by

\[ H_1 = \frac{\kappa}{4} R_{\mu\nu\lambda\sigma} \Sigma^{\mu\nu} \Sigma^{\lambda\sigma} + \lambda F_{\mu\nu} \Sigma^{\mu\nu}. \]  

The first term of equation 6.2, that was also introduced in [15], corresponds to spin-spin coupling via the gravitational field and the second term corresponds to coupling of the spin tensor to the electromagnetic field. The equation of motion for \( x^\mu \) does not change, but the equation of motion for \( \pi_\mu \) is modified to:

\[ \dot{\pi}_\mu = \{ \pi_\mu, H \} = \Gamma_\mu^{\nu \lambda} \pi_\nu x^\lambda + \frac{1}{m} (\frac{1}{2} \Sigma^{\kappa\lambda} R_{\kappa\lambda\mu} \pi_\nu + q F_{\mu\nu} \pi_\nu) \]

or

\[ D_\tau \pi_\mu = \frac{1}{m} (\frac{1}{2} \Sigma^{\kappa\lambda} R_{\kappa\lambda\mu} \pi_\nu + q F_{\mu\nu} \pi_\nu) - \frac{\kappa}{4} \Sigma^{\kappa\lambda} \Sigma^{\rho\sigma} \nabla_\rho R_{\kappa\lambda\rho\sigma} - \lambda \Sigma^{\kappa\lambda} \nabla_\mu F_{\kappa\lambda}. \]  

The last term is the Stern-Gerlach force and the third term is its gravitational equivalent.

The equation of motion for \( \Sigma^{\mu\nu} \) is modified to:

\[ \dot{\Sigma}^{\mu\nu} = \{ \Sigma^{\mu\nu}, H \} = x^\sigma (\Gamma^{\mu}_{\sigma \rho} \Sigma^{\nu\rho} - \Gamma^{\nu}_{\sigma \rho} \Sigma^{\mu\rho}) + \kappa \Sigma^{\kappa\lambda} (R_{\kappa\lambda}^{\mu} \pi^\nu \Sigma - R_{\kappa\lambda}^{\nu} \pi^\mu \Sigma + 2\lambda (F^{\mu}_{\nu} \Sigma^{\nu\lambda} - F^{\nu}_{\mu} \Sigma^{\mu\lambda}). \]  

or

\[ D_\tau \Sigma^{\mu\nu} = \kappa \Sigma^{\kappa\lambda} (R_{\kappa\lambda}^{\mu} \sigma \Sigma^{\nu\sigma} - R_{\kappa\lambda}^{\nu} \sigma \Sigma^{\mu\sigma}) + 2\lambda (F_{\nu}^{\mu} \Sigma^{\nu\lambda} - F_{\mu}^{\nu} \Sigma^{\mu\lambda}), \]  

so the spin tensor is no longer conserved along the world line of the particle. The equations of motion for the Pauli-Lubanski and Pirani vectors are given by:

\[ D_\tau \Sigma^{\mu} = \frac{1}{2 \sqrt{-g}} \epsilon^{\mu\nu\kappa\lambda} \Sigma_{\kappa\lambda} \frac{1}{m} (\frac{1}{2} \Sigma^{\kappa\lambda} R_{\kappa\lambda\rho\epsilon} u_\rho + q F_{\mu\rho} u_\rho) - \frac{\kappa}{4} \Sigma^{\kappa\lambda} \Sigma^{\rho\sigma} \nabla_\rho R_{\kappa\lambda\rho\sigma} - \lambda \Sigma^{\kappa\lambda} \nabla_\nu F_{\kappa\lambda} \]

or

\[ u_\nu (\kappa \Sigma^{\kappa\lambda} (R_{\kappa\lambda}^{\nu} \sigma \Sigma_{\nu\sigma} - R_{\kappa\lambda}^{\nu} \sigma \Sigma_{\nu\lambda}) + 2\lambda (F_{\kappa}^{\nu} \Sigma_{\nu\lambda} - F_{\kappa}^{\nu} \Sigma_{\nu\lambda})), \]  

\[ D_\tau \Sigma^{\mu} = \frac{1}{2 \sqrt{-g}} \epsilon^{\mu\nu\kappa\lambda} \Sigma_{\kappa\lambda} \frac{1}{m} (\frac{1}{2} \Sigma^{\kappa\lambda} R_{\kappa\lambda\rho\epsilon} u_\rho + q F_{\mu\rho} u_\rho) - \frac{\kappa}{4} \Sigma^{\kappa\lambda} \Sigma^{\rho\sigma} \nabla_\rho R_{\kappa\lambda\rho\sigma} - \lambda \Sigma^{\kappa\lambda} \nabla_\nu F_{\kappa\lambda} \]

or

\[ u_\nu (\kappa \Sigma^{\kappa\lambda} (R_{\kappa\lambda}^{\nu} \sigma \Sigma_{\nu\sigma} - R_{\kappa\lambda}^{\nu} \sigma \Sigma_{\nu\lambda}) + 2\lambda (F_{\kappa}^{\nu} \Sigma_{\nu\lambda} - F_{\kappa}^{\nu} \Sigma_{\nu\lambda})). \]
and

$$D_{\tau}Z^{\mu} = u_{\nu}(\kappa \Sigma^{\kappa \lambda}(R_{\kappa \lambda}^{\mu \nu} - R_{\kappa \lambda}^{\nu \sigma} \Sigma^{\mu \sigma}) + 2\lambda(F_{\mu}^{\nu} \Sigma^{\nu \lambda} - F_{\nu}^{\nu} \Sigma^{\mu \lambda}))$$

$$+ \frac{1}{m} \Sigma^{\mu \nu}(\frac{1}{2} \Sigma^{\kappa \lambda} R_{\chi \lambda \nu}^{\sigma} u_{\sigma} + q F_{\nu}^{\sigma} u_{\sigma} - \kappa \Sigma^{\kappa \lambda} \beta^{\rho \sigma} \nabla_{\nu} R_{\kappa \lambda \rho \sigma} - \lambda \Sigma^{\kappa \lambda} \nabla_{\nu} F_{\kappa \lambda})$$

(6.8)

We will show that constants of motion of the form 3.14 are still conserved. In [15] it is already shown that the Poisson bracket of \(J\) with the spin-spin part of \(H_1\) is zero. The Poisson Bracket of \(J\) with the part of \(H_1\) containing the Field-Strength tensor is given by:

\[
\{ J, \lambda F_{\kappa \lambda} \Sigma^{\kappa \lambda} \} = \frac{\partial J}{\partial \Sigma^{\mu \nu}} \{ \Sigma^{\mu \nu}, \lambda F_{\kappa \lambda} \Sigma^{\kappa \lambda} \} + \frac{\partial J}{\partial \pi_{\mu}} \{ \pi_{\mu}, \lambda F_{\kappa \lambda} \Sigma^{\kappa \lambda} \}
\]

\[
= \lambda (\beta^{\mu \nu} F_{\kappa \lambda} (g^{\nu \lambda} \Sigma^{\nu \lambda} - g^{\nu \lambda} \Sigma^{\nu \lambda} - g^{\nu \lambda} \Sigma^{\nu \lambda} + g^{\nu \lambda} \Sigma^{\nu \lambda}))
\]

\[
+ \alpha^{\mu} (-\partial_{\mu} F_{\kappa \lambda} \Sigma^{\kappa \lambda} + F_{\kappa \lambda} (\Gamma_{\mu}^{\lambda} \Sigma^{\rho \sigma} - \Gamma_{\mu}^{\rho} \Sigma^{\lambda \sigma}))
\]

\[
= \lambda (4\beta^{\mu \nu} F_{\kappa \lambda} \Sigma^{\nu \lambda} - \alpha^{\mu} \nabla_{\mu} F_{\kappa \lambda} \Sigma^{\kappa \lambda})
\]

(6.9)

where we used equation 3.15 in the fourth line and the Ricci identity of the electromagnetic field in the fifth line.

To see that this is zero, we need to look at the expressions of \(\alpha^{\mu}\) and \(F_{\mu \nu}\) for the Reissner-Nordström black hole. Let's first look at the first term:

The only nonzero component of the field strength tensor is the \(tr\)-component, so the only nonzero contribution of \(\nabla_{\kappa} \alpha^{\mu} F_{\mu \lambda}\) is given by the product of the covariant derivative of the \(t\)-component of the timelike Killing vector with \(F_{tr}\). However, the only nonzero component of the covariant derivative of \(\alpha^{t}\) is the \(r\)-component. Since the term is contracted with the antisymmetric spin tensor, which has \(\Sigma^{rr} = 0\), the first term is zero.

For the second term, the only non-zero component of \(\nabla_{\kappa} F_{\lambda \mu}\) is for \(\kappa = r\). Again, only contraction with the time component of \(\alpha\) yields a nonzero contribution. However, this again has to be contracted with \(\Sigma^{rr}\), so this term also yields zero.

This shows that the conserved quantities found in chapter 3 are still constants of motion.
In this thesis, we generalized the work done in [15]: we found the equations of motion and the conserved quantities of a charged spinning particle in a Reissner-Nordström geometry. We found that the inclusion of the electric field gave rise to a Lorentz force in the equation for the momentum, but it did not influence the equation of motion of the spin.

We found that energy and the total angular momentum were constants of motion. Focusing on circular orbits, we found an expression for $u^\phi$ in terms of the radius $R$ and the total angular momentum $J$. $u^t$ and the energy $E$ could be calculated in terms of $u^\phi$.

We found that the effect of spin on the radius of the innermost stable circular orbit was dominant over the effect of charge for negative spin, but for positive spin the Coulomb force gives a significant contribution. We found the interesting result that a particle with charge $|q/m| = 1$ with the same sign as the charge of an extremal black hole can never find a stable orbit.

In the last chapter we added two more interactions to the hamiltonian: a spin-spin interaction via the gravitational field and an interaction between the spin and the electric field. We showed that the spin was no longer conserved along the world line, but the other conserved quantities were still constants of motion.
Appendix A

Christoffel symbols and Riemann Curvature tensor

The Christoffel symbols of the Reissner-Nordström metric are given by:

\begin{align*}
\Gamma^r_{rr} &= -\frac{Mr - Q^2}{r(r^2 - 2Mr + Q^2)} \\
\Gamma^t_{tr} &= \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left(\frac{M}{r^2} - \frac{Q^2}{r^3}\right) \\
\Gamma^r_{tr} &= \frac{Mr - Q^2}{r(r^2 - 2Mr + Q^2)} \\
\Gamma^t_{tr} &= \frac{1}{r} \\
\Gamma^t_{rr} &= -(r - 2M + \frac{Q^2}{r}) \\
\Gamma^t_{\theta \theta} &= \frac{1}{r} \\
\Gamma^t_{\phi \phi} &= \frac{1}{r} \\
\Gamma^\theta_{r \phi} &= -\sin^2 \theta (r - 2M + \frac{Q^2}{r}) \\
\Gamma^\phi_{t \phi} &= -\sin \theta \cos \theta \\
\Gamma^\phi_{\theta \phi} &= \frac{\cos \theta}{\sin \theta}
\end{align*}

(A.1)

The non-zero components of the Riemann curvature tensor are given by:

\begin{align*}
R_{\theta t \theta} &= \left(\frac{M}{r} - \frac{Q^2}{r^2}\right) \\
R_{\theta \phi \theta} &= -\frac{2M}{r} + \frac{Q^2}{r^2} \\
R_{\theta r \theta} &= \frac{M}{r} - \frac{Q^2}{r^2} \\
R_{r t r} &= \frac{-2Mr + 3Q^2}{r^2(r^2 - 2Mr + Q^2)} \\
R_{r \phi r} &= \frac{Mr - Q^2}{r^2(r^2 - 2Mr + Q^2)} \\
R_{t \phi t} &= -\frac{1}{r} \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left(\frac{M}{r^2} - \frac{Q^2}{r^3}\right)
\end{align*}

(A.2)
Appendix B

Jacobi Identities

In this chapter we will prove that the Jacobi identities

\[ \{ A, \{ B, C \} \} + \{ B, \{ C, A \} \} + \{ C, \{ A, B \} \} = 0 \tag{B.1} \]

hold for the Dirac-Poisson brackets defined in Chapter 2.

For the bracket of the spin tensor with two momenta, rearranging the terms shows that all of the Jacobi identity terms cancel:

\[
\begin{align*}
\{ \pi_\mu, \{ \pi_\nu, \Sigma^{\kappa\lambda} \} \} &+ \{ \pi_\nu, \{ \Sigma^{\kappa\lambda}, \pi_\mu \} \} + \{ \Sigma^{\kappa\lambda}, \{ \pi_\mu, \pi_\nu \} \} \\
&= \{ \pi_\mu, \Gamma_\nu^\lambda \Sigma^{\kappa\rho} - \Gamma_\nu^\kappa \Sigma^{\lambda\rho} \} + \{ \pi_\nu, \Gamma_\mu^\kappa \Sigma^{\lambda\rho} - \Gamma_\mu^\lambda \Sigma^{\kappa\rho} \} \\
&+ \{ \Sigma^{\kappa\lambda}, \frac{1}{2} \Sigma^{\rho\sigma} R_{\rho\sigma\mu\nu} + q F_{\mu\nu} \} \\
&= \Gamma_\nu^\lambda \rho (\Gamma_\mu^\rho \Sigma^{\kappa\sigma} - \Gamma_\mu^\kappa \Sigma^{\lambda\sigma}) - \Sigma^{\kappa\rho} \partial_\mu \Gamma_\nu^\lambda \rho - \Gamma_\nu^\kappa \rho (\Gamma_\mu^\rho \Sigma^{\lambda\sigma} - \Gamma_\mu^\lambda \Sigma^{\kappa\rho}) \\
&+ \Sigma^{\lambda\rho} \partial_\mu \Gamma_\nu^\kappa \rho + \Gamma_\mu^\kappa \rho (\Gamma_\nu^\rho \Sigma^{\lambda\sigma} - \Gamma_\nu^\lambda \Sigma^{\rho\sigma}) - \Sigma^{\lambda\rho} \partial_\nu \Gamma_\mu^\kappa \rho \\
&- \Gamma_\mu^\lambda \rho (\Gamma_\nu^\rho \Sigma^{\kappa\sigma} - \Gamma_\nu^\kappa \Sigma^{\rho\sigma}) + \Sigma^{\rho\sigma} \partial_\nu \Gamma_\mu^\lambda \rho \\
&+ \frac{1}{2} R_{\rho\sigma\mu\nu} (g^{\kappa\rho} \Sigma^{\lambda\sigma} - g^{\kappa\sigma} \Sigma^{\lambda\rho} - g^{\lambda\rho} \Sigma^{\kappa\sigma} + g^{\lambda\sigma} \Sigma^{\kappa\rho}) \\
&= \Sigma^{\kappa\sigma} (-\partial_\mu \Gamma_\nu^\lambda \sigma + \partial_\nu \Gamma_\mu^\lambda \sigma + \Gamma_\mu^\rho \sigma \Gamma_\nu^\lambda \rho - \Gamma_\nu^\rho \sigma \Gamma_\mu^\lambda \rho + R_{\mu\nu\sigma}^\lambda) \\
&+ \Sigma^{\lambda\sigma} (\partial_\mu \Gamma_\nu^\kappa \sigma - \partial_\nu \Gamma_\mu^\kappa \sigma - \Gamma_\mu^\rho \sigma \Gamma_\nu^\kappa \rho + \Gamma_\nu^\rho \sigma \Gamma_\mu^\kappa \rho - R_{\mu\nu\sigma}^\lambda) \\
&+ \Sigma^{\rho\sigma} (-\Gamma_\nu^\lambda \rho \Gamma_\mu^\kappa \sigma + \Gamma_\nu^\kappa \rho \Gamma_\mu^\lambda \sigma - \Gamma_\mu^\kappa \rho \Gamma_\nu^\lambda \sigma + \Gamma_\mu^\lambda \rho \Gamma_\nu^\kappa \sigma) = 0. \tag{B.2}
\end{align*}
\]

The Jacobi identity for momentum with two spin tensors vanishes due to the
fact that the covariant derivative of the metric is zero.

\[
\begin{align*}
\{\pi_\mu, \{\pi_\nu, \pi_\lambda\}\} + \{\pi_\nu, \{\pi_\lambda, \pi_\mu\}\} + \{\pi_\lambda, \{\pi_\mu, \pi_\nu\}\} & = \{\pi_\mu, \Sigma^{\rho\sigma}, \Sigma_\lambda\} + \{\Sigma^{\rho\sigma}, \pi_\mu\} + \{\Sigma^{\rho\sigma}, \Sigma_\lambda\} \\
& = \{\pi_\mu, 8^{\rho\sigma}\Sigma_\lambda - 8^{\rho\sigma}\Sigma_\lambda + 8^{\rho\sigma}\Sigma_\lambda\} + \{\Sigma^{\rho\sigma}, \Gamma_\mu^{\rho} \Sigma^{\sigma\tau} - \Gamma_\mu^{\rho} \Sigma^{\sigma\tau}\} + \{\Sigma^{\rho\sigma}, \Gamma_\mu^{\rho} \Sigma^{\lambda\tau} - \Gamma_\mu^{\rho} \Sigma^{\lambda\tau}\} \\
& = -8^{\rho\sigma}\partial_\mu 8^{\rho\sigma} + 8^{\rho\sigma}(\Gamma_\mu^{\rho} \Sigma^{\sigma\tau} - \Gamma_\mu^{\rho} \Sigma^{\sigma\tau}) + \{\Sigma^{\rho\sigma}, \Gamma_\mu^{\rho} \Sigma^{\lambda\tau} - \Gamma_\mu^{\rho} \Sigma^{\lambda\tau}\}
\end{align*}
\]

(B.3)

Finally, by rearranging terms, it can be shown that the Jacobi identity also

For three momenta, the Jacobi identity holds due to the Bianchi identities of the Maxwell and Riemann tensor.

\[
\begin{align*}
\{\pi_\mu, \{\pi_\nu, \pi_\lambda\}\} + \{\pi_\nu, \{\pi_\lambda, \pi_\mu\}\} + \{\pi_\lambda, \{\pi_\mu, \pi_\nu\}\} & = \{\pi_\mu, 1/2 \Sigma^{\rho\sigma} R_{\sigma\rho\lambda} + q F_{\nu\lambda}\} + \{\pi_\nu, 1/2 \Sigma^{\rho\sigma} R_{\sigma\rho\mu} + q F_{\mu}\} + \\
& = \{\pi_\mu, 1/2 \Sigma^{\rho\sigma} R_{\sigma\rho\mu} + q F_{\mu}\}
\end{align*}
\]

(B.4)

Finally, by rearranging terms, it can be shown that the Jacobi identity also
holds for three spin tensors.

\[
\{\Sigma^{\mu\nu}, \{\Sigma^{\nu}, \Sigma^{\mu\sigma}\}\} + \{\Sigma^{\lambda\nu}, \{\Sigma^{\mu\rho}, \Sigma^{\mu\nu}\}\} + \{\Sigma^{\mu\rho}, \{\Sigma^{\lambda\nu}, \Sigma^{\nu}\}\} =
\{\Sigma^{\mu\nu}, g^{\lambda\rho} \Sigma^{\lambda\sigma} - g^{\lambda\sigma} \Sigma^{\lambda\rho} - g^{\lambda\rho} \Sigma^{\lambda\sigma} + g^{\lambda\sigma} \Sigma^{\lambda\rho}\}
+ \{\Sigma^{\lambda\nu}, g^{\mu\rho} \Sigma^{\mu\sigma} - g^{\mu\sigma} \Sigma^{\mu\rho} - g^{\mu\rho} \Sigma^{\mu\sigma} + g^{\mu\sigma} \Sigma^{\mu\rho}\}
+ \{\Sigma^{\mu\rho}, g^{\lambda\nu} \Sigma^{\lambda\sigma} - g^{\lambda\sigma} \Sigma^{\lambda\nu} - g^{\lambda\nu} \Sigma^{\lambda\sigma} + g^{\lambda\sigma} \Sigma^{\lambda\nu}\}
+ \{\Sigma^{\mu\nu}, g^{\lambda\rho} \Sigma^{\lambda\sigma} - g^{\lambda\sigma} \Sigma^{\lambda\rho} - g^{\lambda\rho} \Sigma^{\lambda\sigma} + g^{\lambda\sigma} \Sigma^{\lambda\rho}\}
+ \{\Sigma^{\lambda\nu}, g^{\mu\rho} \Sigma^{\mu\sigma} - g^{\mu\sigma} \Sigma^{\mu\rho} - g^{\mu\rho} \Sigma^{\mu\sigma} + g^{\mu\sigma} \Sigma^{\mu\rho}\}
+ \{\Sigma^{\mu\rho}, g^{\lambda\nu} \Sigma^{\lambda\sigma} - g^{\lambda\sigma} \Sigma^{\lambda\nu} - g^{\lambda\nu} \Sigma^{\lambda\sigma} + g^{\lambda\sigma} \Sigma^{\lambda\nu}\}
+ \{\Sigma^{\lambda\nu}, g^{\mu\rho} \Sigma^{\mu\sigma} - g^{\mu\sigma} \Sigma^{\mu\rho} - g^{\mu\rho} \Sigma^{\mu\sigma} + g^{\mu\sigma} \Sigma^{\mu\rho}\}
+ \{\Sigma^{\mu\rho}, g^{\lambda\nu} \Sigma^{\lambda\sigma} - g^{\lambda\sigma} \Sigma^{\lambda\nu} - g^{\lambda\nu} \Sigma^{\lambda\sigma} + g^{\lambda\sigma} \Sigma^{\lambda\nu}\}
+ \{\Sigma^{\lambda\nu}, g^{\mu\rho} \Sigma^{\mu\sigma} - g^{\mu\sigma} \Sigma^{\mu\rho} - g^{\mu\rho} \Sigma^{\mu\sigma} + g^{\mu\sigma} \Sigma^{\mu\rho}\}
+ \{\Sigma^{\mu\rho}, g^{\lambda\nu} \Sigma^{\lambda\sigma} - g^{\lambda\sigma} \Sigma^{\lambda\nu} - g^{\lambda\nu} \Sigma^{\lambda\sigma} + g^{\lambda\sigma} \Sigma^{\lambda\nu}\}
+ \{\Sigma^{\lambda\nu}, g^{\mu\rho} \Sigma^{\mu\sigma} - g^{\mu\sigma} \Sigma^{\mu\rho} - g^{\mu\rho} \Sigma^{\mu\sigma} + g^{\mu\sigma} \Sigma^{\mu\rho}\}
+ \{\Sigma^{\mu\rho}, g^{\lambda\nu} \Sigma^{\lambda\sigma} - g^{\lambda\sigma} \Sigma^{\lambda\nu} - g^{\lambda\nu} \Sigma^{\lambda\sigma} + g^{\lambda\sigma} \Sigma^{\lambda\nu}\}
= g^{\mu\nu} (g^{\lambda\rho} \Sigma^{\lambda\sigma} - g^{\lambda\sigma} \Sigma^{\lambda\rho} - g^{\lambda\rho} \Sigma^{\lambda\sigma} + g^{\lambda\sigma} \Sigma^{\lambda\rho})
- g^{\mu\rho} (g^{\lambda\nu} \Sigma^{\lambda\sigma} - g^{\lambda\sigma} \Sigma^{\lambda\nu} - g^{\lambda\nu} \Sigma^{\lambda\sigma} + g^{\lambda\sigma} \Sigma^{\lambda\nu})
- g^{\lambda\nu} (g^{\mu\rho} \Sigma^{\mu\lambda} - g^{\mu\lambda} \Sigma^{\mu\rho} - g^{\mu\rho} \Sigma^{\mu\lambda} + g^{\mu\lambda} \Sigma^{\mu\rho})
+ g^{\lambda\rho} (g^{\mu\nu} \Sigma^{\mu\lambda} - g^{\mu\lambda} \Sigma^{\mu\nu} - g^{\mu\nu} \Sigma^{\mu\lambda} + g^{\mu\lambda} \Sigma^{\mu\nu})
\]

The proofs of the other Jacobi identities are trivial.
Appendix C

Analytic solution for the spinless ISCO

Eliminating $\frac{qw}{m}$ from equations 5.9 and 5.10, gives the following second order equation in $y^2$:

\begin{align*}
x^2(x - w^2)(3x^4 - 5x^3 + 2x^2w^2) - x^3(x^2 + 2x(x - w^2))(x^2 - 2x + w^2) \\
y^2((3x - x^2 - 2w^2)(3x^4 - 5x^3 + 2x^2w^2) + x^2(x - w^2)(2x^2 - 3x + w^2) \\
- x^3(3 - 2x)(x^2 - 2x + w^2) - x(x^2 + 2x(x - w^2))(x^2 - 2x + w^2)) \\
+ y^4((3x - x^2 - 2w^2)(2x^2 - 3x + w^2) - x(3 - 2x)(x^2 - 2x + w^2)) = 0,
\end{align*}

which can be simplified to

\begin{align*}
(1 - w^2)x^6 + y^2(-x^2(3w^4 + 2w^2x(-5 + 3x) + x^2(6 - 6x + x^2)) \\
+ y^4(-2w^4 - 3w^2(-2 + x)x + x^2(-3 + 2x)) = 0.
\end{align*}

We can solve this equation to obtain an expression for $y^2$:

\begin{align*}
y^2 &= \frac{x^2}{2(2w^4 + 3w^2(x - 2)x - x^2(2x - 3))} \cdot \frac{(2w^2(5 - 3x)x - 3w^4 - x^2(6 + x(x - 6)) \pm \sqrt{9w^2 + (x - 6)x(w^2 + (x - 2)x)^2})}{x^2(6 + x(x - 6)) \pm \sqrt{9w^2 + (x - 6)x(w^2 + (x - 2)x)^2}}.
\end{align*}

Substituting this expression into equation 5.9 yields an equation in terms of $x$, $w$ and $q/m$:

\begin{align*}
x^2(x - w^2) - (-3x + x^2 + 2w^2) &= \frac{x^2}{2(2w^4 + 3w^2(x - 2)x - x^2(2x - 3))} \\
(2w^2(5 - 3x)x - 3w^4 - x^2(6 + x(x - 6)) \pm \sqrt{9w^2 + (x - 6)x(w^2 + (x - 2)x)^2}) &= \frac{x^2}{2(2w^4 + 3w^2(x - 2)x - x^2(2x - 3))} \cdot \frac{(2w^2(5 - 3x)x - 3w^4 - x^2(6 + x(x - 6)) \pm \sqrt{9w^2 + (x - 6)x(w^2 + (x - 2)x)^2})}{x^2(6 + x(x - 6)) \pm \sqrt{9w^2 + (x - 6)x(w^2 + (x - 2)x)^2}}.
\end{align*}
References


