**Author**: Zhuang, Weidong
**Title**: Symmetric diophantine approximation over function fields
**Issue Date**: 2015-12-03

# Abstract

Let $F \in \mathbb{Z}[X,Y]$ be a *binary form*, i.e., a homogeneous polynomial in two variables. We denote the discriminant of $F$ by $D(F)$ and its height, i.e., the maximum of the absolute values of its coefficients, by $H(F)$. Two binary forms $F, G \in \mathbb{Z}[X,Y]$ are called $\mathrm{GL}(2,\mathbb{Z})$-equivalent if $G = \pm F_U$ for some matrix $U \in \mathrm{GL}(2,\mathbb{Z})$. Here $F_U(X,Y) = F(aX+bY, cX+dY)$ for $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Two $\mathrm{GL}(2,\mathbb{Z})$-equivalent binary forms have the same discriminant. A binary form $F \in \mathbb{Z}[X,Y]$ is called reduced if its height cannot be made smaller by replacing it by a $\mathrm{GL}(2,\mathbb{Z})$-equivalent form. A conjecture formulated by Evertse but probably much older asserts that if $F \in \mathbb{Z}[X,Y]$ is a reduced binary form of degree $n \geqslant 2$ and non-zero discriminant, then $H(F) \leqslant c_1(n)|D(F)|^{c_2(n)}$ where $c_1(n), c_2(n)$ depend on $n$ only. This conjecture follows from work of Lagrange (1773) and Gauss (1801) for $n = 2$ and Hermite (1851) for $n = 3$, but for $n \geqslant 4$ it is still open. The best known result towards this conjecture is due to Evertse [9] who derived a similar inequality but with $c_1$ depending on $n$ and the splitting field of $F$. This constant $c_1$ cannot be computed effectively from Evertse's method of proof. Further, Evertse and Győry [11] obtained an inequality $H(F) \leqslant \exp\left(c_1(n)|D(F)|^{c_2(n)}\right)$.

In this thesis, we consider binary forms with coefficients in the polynomial ring $k[t]$, where $k$ is an algebraically closed field of characteristic 0. If we define an absolute value $|\cdot|$ on $k[t]$ by setting $|f| := e^{\deg f}$ for $f \in k[t]$, we can formulate an analogue of Evertse's conjecture for binary forms in $k[t][X,Y]$. In this thesis, we give a proof of this analogue. To achieve this, we first generalized Mason's ABC-theorem using work of Brownawell

and Masser [6], Zannier [26] and J.T.-Y. Wang [25], then we developed an analogue over function fields of a theorem of Evertse from the geometry of numbers and subsequently a reduction theory for binary forms over function fields. As an application, we then derived results on the root separation problem over function fields, which is another interesting problem from Diophantine approximation. An elementary inequality of Mahler (1964) states that if $f \in \mathbb{Z}[X]$ is a polynomial of degree $n \geqslant 2$ of non-zero discriminant, then for any two distinct roots $\alpha, \beta \in \mathbb{C}$ of $f$ we have $\alpha - \beta| \geqslant c(n)H(f)^{1-n}$ where $c(n) > 0$ depends on $n$ only. The root separation problem is to prove a similar inequality with instead of $1 - n$ a larger exponent. This is still open. In this thesis, we consider the analogous problem for polynomials in $k[t][X]$, and in this setting we managed to solve the root separation problem.

This thesis is organized as follows. In Chapter 1, we introduce standard notation and collect some results needed later. In Chapter 2, we recall Mason's ABC-theorem and deduce a generalization. Then in Chapter 3, we develop an analogue of the geometry of numbers over function fields. This is applied in Chapter 4 to develop a reduction theory for binary forms over function fields. Combining the results of Chapter $1-4$, we prove in Chapter 5 a function field analogue, in fully effective form, of Evertse's conjecture mentioned above. In Chapter 6, we consider the number of equivalence classes of binary forms of given discriminant, under certain conditions. In the last two chapters, we derive an effective inequality concerning the resultant of this binary forms and derive an effective lower bound for the distance between two algebraic functions, where we make a distinction between the cases that they are conjugate over $k(t)$ or not.