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Chapter 8

Distances between algebraic functions

Let $K = k(t)$. In section 8.1 we give a lower bound for the distance between two roots of a polynomial $f \in k[t][X]$, and in section 8.3 we derive such a lower bound between roots of different polynomials. We follow [9], [10] where similar results have been derived over number fields.

8.1 Root separation of polynomials

Let $K = k(t)$ and let $f \in K[X]$ be a polynomial of degree $n \geq 4$ with splitting field $L$ and non-zero discriminant. Assume that $f = a \prod_{i=1}^{n} (X - \gamma_i)$ with $a \in K^*$ and $\gamma_i \in L$ for $i = 1, \ldots, n$. Let $S$ be a finite set of valuations on $K$ and let $T$ be the set of valuations on $L$ above those in $S$. For each $\nu \in S$ fix a prolongation of $| \cdot |_{\nu}$ to $L$, also denoted by $| \cdot |_{\nu}$. Define

$$\Delta_S(f) := \prod_{\nu \in S} \min_{1 \leq i < j \leq n} \frac{|\gamma_i - \gamma_j|_{\nu}}{\max(1, |\gamma_i|_{\nu}) \max(1, |\gamma_j|_{\nu})}.$$ 

Since $L/K$ is a Galois extension, this quantity $\Delta_S(f)$ is independent of the choices of the extensions of $| \cdot |_{\nu}$ to $L$. To be specific, by (1.4.3) we have
for \( \omega \in \mathcal{A}(\nu) \) and \( \sigma \in \mathcal{E}(\omega|\nu) \) that

\[
\min_{1 \leq i < j \leq n} \frac{|\gamma_i - \gamma_j|_\omega}{\max(1, |\gamma_i|_\omega) \max(1, |\gamma_j|_\omega)} = \left( \min_{1 \leq i < j \leq n} \frac{|\sigma(\gamma_i) - \sigma(\gamma_j)|_\nu}{\max(1, |\sigma(\gamma_i)|_\nu) \max(1, |\sigma(\gamma_j)|_\nu)} \right)^{g_\nu} \leq \left( \min_{1 \leq i < j \leq n} \frac{|\gamma_i - \gamma_j|_\nu}{\max(1, |\gamma_i|_\nu) \max(1, |\gamma_j|_\nu)} \right)^{g_\nu},
\]

since \( \sigma \in \text{Gal}(L/K) \) acts on \( 1, \ldots, n \) as a permutation and \( g_\nu = [L_\omega : K_\nu] \) is independent of \( \omega \). Hence

\[
\Delta_S(f) = \prod_{\omega \in T} \left( \min_{1 \leq i < j \leq n} \frac{|\gamma_i - \gamma_j|_\omega}{\max(1, |\gamma_i|_\omega) \max(1, |\gamma_j|_\omega)} \right)^{1/[L:K]}.
\] (8.1.1)

Put \( H(f) = \prod_{\nu \in M_K} |f|_{\nu} \). Then clearly \( H(f) \geq 1 \).

**Theorem 8.1.1.** Let \( c_4(n) = \exp\left( \frac{(n-1)(n+11)\#S-S}{20+1/n} \right) \). We have

\[
\Delta_S(f) \geq c_4(n)^{-1} H(f)^{n+1+\frac{n}{\sum_{\nu} |\omega|_{\nu}}_\nu}.
\]

**Proof.** Homogenize \( f = a_0X^n + a_1X^{n-1} + \cdots + a_n \) and choose

\[ F(X,Y) = b(a_0X^n + a_1X^{n-1}Y + \cdots + a_nY^n) \]

with \( b \in K^* \) such that

\[ |b|_\infty = |f|_\infty^{-1}H(f), \quad |b|_\nu = |f|_\nu^{-1} \quad \text{for} \ \nu \neq \nu_\infty. \]

The existence of \( b \) is guaranteed because \( \prod_{\nu \in M_K} |f|_{\nu}^{-1}H(f) = 1 \). So we get \( F \in \mathcal{O}_S[X,Y], \ |F|_\infty = H(f) \) and hence

\[ H^*(F) = \max(1, |F|_\infty) = H(f). \]

Factor \( F \) in \( L \) as \( F = \prod_{i=1}^{n} (\alpha_i X + \beta_i Y) \). Then \( \gamma_i = -\frac{\beta_i}{\alpha_i} \). Put

\[ \delta_\omega = \min_{1 \leq i < j \leq n} \frac{|\alpha_i \beta_j - \alpha_j \beta_i|_\omega}{|\alpha_i|_\omega |\beta_j|_\omega} \quad (\omega \in T). \]
Then
\[ \Delta_S(f) = \prod_{\omega \in T} \delta_{ij/[L:K]}^{\omega}. \]

Let \( F^*(X, Y) = F(aX + bY, cX + dY) \) with \((a, b, c, d) \in \text{GL}(2, \mathcal{O}_S)\) be such that \( F^* \) is reduced. Then \( F^*(X, Y) = \prod_{i=1}^n (\alpha_i^*X + \beta_i^*Y) \) where \((\alpha_i^*, \beta_i^*) = (\alpha_i, \beta_i) (a, b) \, i = 1, \ldots, n. \)

Now for \( \omega \in T \) put \( f_{i\omega} := |\alpha_i, \beta_i|_\omega, \) \( f_{i\omega}^* := |\alpha_i^*, \beta_i^*|_\omega \) and \( \zeta_{ij\omega} := |\alpha_i \beta_j - \alpha_j \beta_i|_\omega. \) Then \( \prod_{i=1}^n \sum_{\omega} f_{i\omega} = |F|_\omega, \prod_{i=1}^n f_{i\omega}^* = |F^*|_\omega \) and \( \prod_{1 \leq i < j \leq n} \zeta_{ij\omega} = |D(F)|_\omega^{1/2}. \)

By the ultrametric inequality we have \( \zeta_{ij\omega} \leq f_{i\omega} f_{j\omega}, \) and
\[ \zeta_{ij\omega} \leq \min(f_{i\omega} f_{j\omega}, |ad - bc|_\omega^{-1} f_{i\omega}^* f_{j\omega}^*) \text{ for } 1 \leq i < j \leq n, \omega \in T. \tag{8.1.2} \]

We are going to bound \( \delta_\omega \) from below for each \( \omega \in T. \) Let \( \omega \in T, \) and assume, without loss of generality, that \( \delta_\omega = \frac{\zeta_{12\omega}}{f_{1\omega} f_{2\omega}}. \) Then
\[ \delta_\omega \geq \frac{\zeta_{12\omega}}{f_{1\omega} f_{2\omega}} \prod_{1 \leq i < j \leq n} \min_{(i, j) \neq (1, 2)} \frac{\zeta_{ij\omega}}{\min(f_{i\omega} f_{j\omega}, |ad - bc|_\omega^{-1} f_{i\omega}^* f_{j\omega}^*)} = \frac{|D(F)|_\omega^{1/2}}{\Lambda_\omega}, \]
with \( \Lambda_\omega = f_{1\omega} f_{2\omega} \prod_{1 \leq i < j \leq n} \min_{(i, j) \neq (1, 2)} (f_{i\omega} f_{j\omega}, |ad - bc|_\omega^{-1} f_{i\omega}^* f_{j\omega}^*). \)

We claim that
\[ \Lambda_\omega \leq |F|_\omega |F^*|_\omega^{n-2} |ad - bc|_\omega^{-n(n-2)/2}. \tag{8.1.3} \]

Then
\[ \delta_\omega \geq \frac{|D(F)|_\omega^{1/2} |ad - bc|_\omega^{-n(n-2)/2}}{|F|_\omega |F^*|_\omega^{n-2}}. \]

By the Main Theorem, we have
\[ |D(F)|_S^{1/2} \geq H^*(F^*)^{n/(40n+2)} e^{\frac{(1-n) \binom{n+1}{2} (S-1)}{20n+7n}}. \tag{8.1.4} \]
Using \(ad - bc \in \mathcal{O}_S^*\), \(H_S(F) \leq H^*(F), H_S(F^*) = H^*(F^*) \leq H^*(F) = H(f)\), we deduce that

\[
\Delta_S(f) \geq \left( \prod_{\omega \in \mathcal{T}} \frac{|D(F)|^{1/2}|ad - bc|^{(n-2)/2}}{|F_\omega|^n} \right)^{1/[L:K]} \\
= \frac{|D(F)|^{1/2}}{H_S(F)H_S(F^*)^{n-2}} \\
\geq \exp \left( -\frac{(n-1)((n+1)\#S - 5)}{20 + 1/n} \right) \frac{1}{H(f)} H(F^*)^{-\frac{n}{4n+2}}^{-n+2} \\
\geq \exp \left( -\frac{(n-1)((n+1)\#S - 5)}{20 + 1/n} \right) H(f)^{-n+1+\frac{n}{4n+2}}. \tag{8.1.5}
\]

Finally, to prove (8.1.3), we have to distinguish two cases. First let \(n \geq 4\) be even. Take \(I = \{(1, 2), \ldots, (n - 1, n)\}\). Then

\[
\Lambda_\omega \leq \prod_{i=1}^{n} f_{i\omega} \prod_{1 \leq i < j \leq n \atop (i,j) \notin I} |ad - bc|_{\omega}^{-1} f_{i\omega} f_{j\omega}^* \\
= \prod_{i=1}^{n} f_{i\omega} \left( \prod_{i=1}^{n} f_{i\omega}^* \right)^{n-2} |ad - bc|_{\omega}^{-(n-2)/2} \\
= |F_\omega|^n |F^*|^{n-2} |ad - bc|_{\omega}^{-(n-2)/2}.
\]

Next let \(n \geq 5\) be odd. Take

\[
I = \{(1, 2), \ldots, (n - 2, n - 1), (n - 2, n), (n - 1, n)\}.
\]

Then

\[
\Lambda_\omega \leq \prod_{i=1}^{n-3} f_{i\omega} \prod_{n-2 \leq i < j \leq n \atop (i,j) \notin I} (f_{i\omega} f_{j\omega} f_{i\omega}^* f_{j\omega}^* |ad - bc|_{\omega}^{-1})^{1/2} \prod_{1 \leq i < j \leq n \atop (i,j) \notin I} |ad - bc|_{\omega}^{-1} f_{i\omega} f_{j\omega}^* \\
= \prod_{i=1}^{n} f_{i\omega} \left( \prod_{i=1}^{n} f_{i\omega}^* \right)^{n-2} |ad - bc|_{\omega}^{-(n-2)/2} \\
= |F_\omega|^n |F^*|^{n-2} |ad - bc|_{\omega}^{-(n-2)/2}.
\]
As a direct consequence, we obtain the following result on simultaneous root separation for various absolute values.

**Corollary 8.1.2.** We have

\[
\prod_{\nu \in S} \min_{1 \leq i < j \leq n} |\gamma_i - \gamma_j|_{\nu} \geq \exp\left(-\frac{(n-1)(n+11)\#S - 5}{20}\right) H(f)^{-n+1 + \frac{n}{\min\{1, \#S\}}}. 
\]

**Proof.** Since the denominator of \( \Delta_S(f) \) is at least 1, this is a direct consequence of Theorem 8.1.1 and the fact \( |x|_S^{[L:K]} = |x|_T \).

**Corollary 8.1.3.**

\[
\Delta_S(f) \geq \exp\left(-\frac{n-1}{100}(5n(n+7)\#S + \frac{2g_L-1}{[L:K]})\right) H(f)^{-n+1 + \frac{n}{2}}.
\]

**Proof.** It is similar with proof of Theorem 8.1.1, but replace (8.1.4) by using Theorem 5.3.2.

### 8.2 Two lemmas

We need some preparations for the next section where we consider distances between algebraic function that are roots of different polynomials.

Let \( K = k(t) \). Let \( H^*(\gamma) = \prod_{\omega \in M_L} \max(1, |\gamma|_\omega)^{1/[L:K]} \) for any \( \gamma \in L \) algebraic over \( K \). This is independent of the choice of \( L \).

Let \( \xi, \eta \) be distinct and algebraic over \( K \). Let \( L = K(\xi, \eta) \) and \( T \) a finite set of valuations on \( L \). Define

\[
\Delta_T(\xi, \eta) := \left( \prod_{\omega \in T} \frac{|\xi - \eta|_\omega}{\max(1, |\xi|_\omega) \max(1, |\eta|_\omega)} \right)^{1/[L:K]}.
\]

Then clearly

\[
\Delta_T(\xi, \eta) = \left( \prod_{\omega \in T} \frac{\max(1, |\xi|_\omega) \max(1, |\eta|_\omega)}{|\xi - \eta|_\omega} \right)^{1/[L:K]} H^*(\xi)^{-1} H^*(\eta)^{-1} \geq H^*(\xi)^{-1} H^*(\eta)^{-1}.
\]
This is a type of Liouville-type inequality. Recall that for a matrix $A = (a_{ij})_{i,j}$, we have defined its $\nu$-value $|A|_{\nu} = \max_{i,j}(|a_{ij}|_{\nu})$ for $\nu \in M_K$. In this way, we also define

$$H_S(A) = \prod_{\nu \in S} |A|_{\nu}.$$ 

**Lemma 8.2.1.** Let $F(X,Y) \in \mathcal{O}_S[X,Y]$ be a binary form of degree $n \geq 3$ with non-zero discriminant. Then for any $U \in \text{GL}(2, \mathcal{O}_S)$, we have

$$\frac{H_S(F_U)}{H_S(F)} \leq H_S(U) \leq \left(\frac{H_S(F)H_S(F_U)}{H_S(U)}\right)^{3/n}.$$ 

**Proof.** Let $T$ be the set of valuations on the splitting field $L$ lying above the valuations in $S$, write $F(X,Y) = a_0 \prod_{i=1}^n (\alpha_i X + \beta_i Y)$ with $a_0 \in K^*$, $\alpha_i, \beta_i \in \mathcal{O}_T$ and $F_U(X,Y) = a_0 \prod_{i=1}^n (\alpha_i^* X + \beta_i^* Y)$ with

$$(\alpha_i^*, \beta_i^*) = (\alpha_i, \beta_i)U, \ i = 1, \ldots, n.$$ 

Let $U = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$. Then

$$\begin{cases}
  a\alpha_i + c\beta_i = \alpha_i^* \\
  b\alpha_i + d\beta_i = \beta_i^*
\end{cases} \quad \text{for } i = 1, \ldots, n.$$ 

From the non-archimedean property, it easily follows that

$$\max(|\alpha_i^*|_\omega, |\beta_i^*|_\omega) \leq |U|_\omega \max(|\alpha_i|_\omega, |\beta_i|_\omega) \text{ for } \omega \in T,$$

therefore by Gauss’ lemma we have

$$H_T(F_U) \leq |U|_T H_T(F),$$

which gives

$$H_S(F_U) \leq |U|_S H_S(F).$$

Take any three indices $i, j, l$ and consider the system of equations

$$Ax = 0,$$  \hspace{1cm} (8.2.1)
where $\mathbf{x} = (x_1, \ldots, x_7)^T$ and

$$A = \begin{pmatrix}
\alpha_i & \beta_i & 0 & 0 & \alpha_i^* & 0 & 0 \\
0 & 0 & \alpha_i & \beta_i & \beta_i^* & 0 & 0 \\
\alpha_j & \beta_j & 0 & 0 & 0 & \alpha_j^* & 0 \\
0 & 0 & \alpha_j & \beta_j & 0 & \beta_j^* & 0 \\
\alpha_l & \beta_l & 0 & 0 & 0 & 0 & \alpha_l^* \\
0 & 0 & \alpha_l & \beta_l & 0 & \beta_l^* & 0
\end{pmatrix}.$$  

Put $X = (x_1 \ x_5 \ x_3 \ x_4 \ x_6)$. Then

$$-x_5(\alpha_i^*, \beta_i^*) = (\alpha_i, \beta_i)X,$$

$$-x_6(\alpha_j^*, \beta_j^*) = (\alpha_j, \beta_j)X,$$

$$-x_7(\alpha_l^*, \beta_l^*) = (\alpha_l, \beta_l)X.$$

However, $D(F) \neq 0$, so $X$ maps three pairwise non-parallel vectors to three other pairwise non-parallel vectors. Such a matrix $X$ is unique up to a scalar if it exists. But we already know that $X = U$ with $x_5 = x_6 = x_7 = -1$ is a solution, therefore the solution space of (8.2.1) is one-dimensional and hence for any solution there exists $\lambda$ such that $U = \lambda X$. Let $\Delta_s$ be the determinant of the matrix obtained by removing the $s$-th column of $A$. We claim that $(\Delta_1, -\Delta_2, \ldots, -\Delta_7)$ is a solution of the system of linear equations. To see this, we make an extra seventh row by copying an row and thus obtain a square matrix with determinant 0. By Laplace’s formula, expanding this determinant along the seventh row, we immediately get the result. So

$$U = \lambda \left( \frac{\Delta_1}{-\Delta_2 - \Delta_4} \right).$$

By the ultrametric inequality and again Laplace’s formula, it is easy to see that

$$|\Delta_r|_\omega \leq \prod_{s=i,j,h} \max(|\alpha_s^*|_\omega, |\beta_s^*|_\omega) \max(|\alpha_s|_\omega, |\beta_s|_\omega), \omega \in M_L \text{ for } r = 1, 2, 3, 4.$$  

Hence

$$|U|_\omega \leq |\lambda|_\omega \prod_{s=i,j,h} \max(|\alpha_s^*|_\omega, |\beta_s^*|_\omega) \max(|\alpha_s|_\omega, |\beta_s|_\omega) \ (\omega \in M_L).$$
Therefore, by taking the product over $\omega \in M_L$,
\[
\prod_{\omega \in M_L} |U|_{\omega} \leq \prod_{s=i,j,h} H_L(\alpha_s, \beta_s) H_L(\alpha^*_s, \beta^*_s).
\]

By taking the geometric means over all triples $(i, j, h)$ and going back from $L$ to $K$, we obtain that
\[
\prod_{\nu \in M_K} |U|_{\nu} = \left( \prod_{\omega \in M_L} |U|_{\omega} \right)^{1/[L:K]}
\leq \left( H_K(F) H_K(F_U) \right)^{\binom{n-1}{2}/\binom{n}{3}}
= \left( H_K(F) H_K(F_U) \right)^{3/n}.
\]

Since $U \in \text{GL}(2, \mathcal{O}_S)$, we have $|U|_{\nu} = 1$ for $\nu \notin S$. Further, $F, F_U \in \mathcal{O}_S[X, Y]$. Hence
\[
H_S(U) \leq \left( H_S(F) H_S(F_U) \right)^{3/n}.
\]

\[
\square
\]

**Lemma 8.2.2.** Let $L$ be a finite extension of $K$ of degree $n$ and $T$ the set of valuations on $L$ above those in $S$. For $x \in L$, denote by $\sigma_i, i = 1, \ldots, n$ the $K$-embeddings of $L$ into its algebraic closure, with $\sigma_1$ the identity. Then for $x \in K^*$, there exists $\alpha, \beta \in \mathcal{O}_T$ such that $\frac{\alpha}{\beta} = x$ and for $F = \prod_{i=1}^n (\sigma_i(\alpha)X + \sigma_i(\beta)Y)$ we have
\[
e^{-2g_L} H_S(F)^{\frac{1}{n}} \leq H^*(x) \leq H_S(F)^{\frac{1}{n}}.
\]

**Proof.** First pick $\alpha', \beta' \in L$ such that $x = \frac{\alpha'}{\beta'}$. By Lemma 3.2.3, there is $\theta \in L^*$ such that
\[
|\theta|_{\omega} \leq \min\left(\frac{1}{|\alpha'|_{\omega}}, \frac{1}{|\beta'|_{\omega}}\right) \text{ for } \omega \notin T
\]
\[
|\theta|_{\omega} \leq A_\omega \text{ for } \omega \in T,
\]
where $A_\omega \in e^{2g}, \omega \in T$ satisfy $\prod_{\omega \in T} A_\omega = e^{2g_L} \prod_{\omega \notin T} \max(|\alpha'|_{\omega}, |\beta'|_{\omega})$. 


Let $\alpha = \theta \alpha'$, $\beta = \theta \beta'$. Then $\alpha, \beta \in O_T$ and so $F \in O_S[X,Y]$ and $x = \frac{\alpha}{\beta}$. Also, we have

\[1 \geq \prod_{\omega \notin T} \max(|\alpha|_{\omega}, |\beta|_{\omega}) \]

\[= \prod_{\omega \notin T} |\theta|_{\omega} \prod_{\omega \notin T} \max(|\alpha'|_{\omega}, |\beta'|_{\omega}) \]

\[= \frac{1}{\prod_{\omega \notin T} |\theta|_{\omega}} \prod_{\omega \notin T} \max(|\alpha'|_{\omega}, |\beta'|_{\omega}) \]

\[\geq \prod_{\omega \notin T} A_\omega \prod_{\omega \notin T} \max(|\alpha'|_{\omega}, |\beta'|_{\omega}) \]

\[= e^{-2g_L}. \tag{8.2.2} \]

Let $M$ be a normal extension of $K$ containing $L$, and $U$ the set of valuations above those in $S$. By Lemma 1.4.1 we have

\[
\prod_{\nu \notin S} |F|_{\nu} = \left( \prod_{\omega \notin U} |F|_{\mu} \right)^{\frac{1}{[M:K]}} \\
= \left( \prod_{\mu \notin U} \prod_{i=1}^{n} \max(|\sigma_i(\alpha)|_\mu, |\sigma_i(\beta)|_\mu) \right)^{\frac{1}{[M:K]}} \\
= \left( \prod_{\mu \notin U} \max(|\alpha|_\mu, |\beta|_\mu) \right)^{\frac{n}{[M:K]}} \\
= \left( \prod_{\omega \notin T} \max(|\alpha|_{\omega}, |\beta|_{\omega}) \right)^{\frac{1}{[M:K]}} \\
= \prod_{\omega \notin T} \max(|\alpha|_{\omega}, |\beta|_{\omega}). \tag{8.2.3} \]

Combining (8.2.2) with (8.2.3) we derive that

\[e^{-2g_L} \leq \frac{H(F)}{H_S(F)} \leq 1. \]
By the product formula we have

\[ H^*(x) = \left( \prod_{i=1}^{n} H^*(\sigma_i(x)) \right)^{\frac{1}{n}} \]

\[ = \left( \prod_{i=1}^{n} \prod_{\omega \in M_L} \max(|\sigma_i(\alpha)|_\omega, |\sigma_i(\beta)|_\omega) \right)^{\frac{1}{[L:K]}} \]

\[ = \left( \prod_{\omega \in M_L} |F|_\omega \right)^{\frac{1}{[L:K]}} \]

\[ = H(F)^{\frac{1}{2}}. \]

This implies that

\[ e^{-\frac{2g_L}{n}} H_S(F)^{\frac{1}{2}} \leq H^*(x) \leq H_S(F)^{\frac{1}{2}}. \]

\[ \square \]

### 8.3 A symmetric improvement of the Liouville-type inequality

**Theorem 8.3.1.** Suppose \( \xi, \eta \) are algebraic over \( K \). Let \( L = K(\xi, \eta) \) and assume

\[ [K(\xi) : K] \geq 3, [K(\eta) : K] \geq 3, [L : K] = [K(\xi) : K][K(\eta) : K]. \]

Let \( S \) be a finite set of valuations on \( K \), \( T_0 \) the set of valuations on \( L \) lying above those in \( S \) and \( T \subset T_0 \) such that

\[ \varpi := \max_{\nu \in S} \frac{1}{[L : K]} \sum_{\omega | \nu, \omega \in T} [L_\omega : K_\nu] < \frac{1}{3}. \]

Let \( g_1, g_2 \) be the genera of \( K(\xi) \) and \( K(\eta) \) respectively. Then

\[ \Delta_T(\xi, \eta) \geq C_5^{-1}(H^*(\xi)H^*(\eta))^{-1+\theta}, \]
where $\vartheta = \frac{1-3\varpi}{717(1+3\varpi)}$ and

$$C_5 = \exp\left(\frac{422(m+n-5+2g_1+2g_2)}{717} + (4m+4n+433)\frac{\#S}{717} + (m+n)(m+n-5)(1-\vartheta)\right).$$

Proof. Assume $[K(\xi) : K] = m$, $[K(\eta) : K] = n$. Then $[L : K] = mn$. Without loss of generality, suppose $\nu_\infty \in S$. For if $\nu_\infty \notin S$, then adding $\nu_\infty$ to $S$ does not affect $\varpi$. Let $\sigma_1, \ldots, \sigma_m$ and $\tau_1, \ldots, \tau_n$ be the $K$-isomorphic embeddings of $K(\xi)$ and $K(\eta)$ respectively into $M$.

By Lemma 8.2.2 there are $\alpha, \beta \in K(\xi)$ and $\gamma, \delta \in K(\eta)$ that are integral over $O_S$ such that $\xi = \frac{\alpha}{\beta}, \eta = \frac{\gamma}{\delta}$, and the corresponding binary forms $F(X,Y) = \prod_{i=1}^m (\sigma_i(\alpha)X + \sigma_i(\beta)Y), G(X,Y) = \prod_{j=1}^n (\tau_j(\gamma)X + \tau_j(\delta)Y)$ satisfy

$$e^{-2\mu m} H_S(F)^{\frac{1}{m}} \leq H^*(\xi) \leq H_S(F)^{\frac{1}{m}},$$

$$e^{-2\mu n} H_S(G)^{\frac{1}{n}} \leq H^*(\eta) \leq H_S(G)^{\frac{1}{n}}. \quad (8.3.1)$$

Moreover, the assumption implies that $\xi, \eta$ are not conjugate over $K$ and hence $F, G$ are irreducible and $FG$ is square-free. By Theorem 7.5.1, there exists $U \in \text{GL}(2, O_S)$ such that

$$|R(F,G)|_S \geq C'H_S(G_U)^{\frac{m}{m'}} H_S(F_U)^{\frac{n}{n'}}, \quad (8.3.2)$$

where

$$C' = \exp\left(\frac{422mn(m+n-5+2g_1+2g_2)}{717} - mn(4m+4n+433)\frac{\#S}{717}\right).$$

Notice that

$$F_U(X,Y) = \prod_{i=1}^m (\sigma_i(\alpha')X + \sigma_i(\beta')Y),$$

$$G_U(X,Y) = \prod_{j=1}^n (\tau_j(\gamma')X + \tau_j(\delta')Y),$$

where

$$(\alpha', \beta') = (\alpha, \beta)U, (\gamma', \delta') = (\gamma, \delta)U.$$
Let $V \in \text{GL}(2, \mathcal{O}_S)$ be the inverse of $U$. Then

$$\alpha \delta - \beta \gamma = (\det V)(\alpha' \delta' - \beta' \gamma'),$$

$$\max(|\alpha|_\omega, |\beta|_\omega) \leq |V|_\omega \max(|\alpha'|_\omega, |\beta'|_\omega),$$

$$\max(|\gamma|_\omega, |\delta|_\omega) \leq |V|_\omega \max(|\gamma'|_\omega, |\delta'|_\omega).$$

For $\omega \in M_L$, put

$$\Delta_\omega(\xi, \eta) := \frac{|\xi - \eta|_\omega}{\max(1, |\xi|_\omega) \max(1, |\eta|_\omega)},$$

$$\Delta'_\omega(\xi, \eta) := \frac{|\alpha' \delta' - \beta' \gamma'|_\omega}{\max(|\alpha'|_\omega, |\beta'|_\omega) \max(|\gamma'|_\omega, |\delta'|_\omega)}.$$

Then $\Delta_\omega(\xi, \eta) \leq 1$, $\Delta'_\omega(\xi, \eta) \leq 1$. From what we mentioned above we have

$$\Delta_\omega(\xi, \eta) = \frac{|\alpha \delta - \beta \gamma|_\omega}{\max(|\alpha|_\omega, |\beta|_\omega) \max(|\gamma|_\omega, |\delta|_\omega)} \geq \frac{|\det V|_\omega |\alpha' \delta' - \beta' \gamma'|_\omega}{|V|_\omega^2 \max(|\alpha'|_\omega, |\beta'|_\omega) \max(|\gamma'|_\omega, |\delta'|_\omega)} = \frac{|\det V|_\omega}{|V|_\omega^2 L_\omega: K_\nu} \Delta'_\omega(\xi, \eta).$$

Since $|\det V|_\nu \leq |V|_\nu^2$ for any $\nu \in M_K$ and $V \in \text{GL}(2, \mathcal{O}_S)$, we derive that

$$\prod_{\omega \in T} \Delta_\omega(\xi, \eta) \geq \prod_{\nu \in S} \prod_{\omega \in T} \left( \frac{|\det V|_\nu}{|V|_\nu^2 L_\omega: K_\nu} \right) \prod_{\omega \in T} \Delta'_\omega(\xi, \eta) \geq \prod_{\nu \in S} \left( \frac{|\det V|_\nu}{|V|_\nu^2} \right)^{L: K} \prod_{\omega \in T} \Delta'_\omega(\xi, \eta) = \frac{1}{H_S(V)^2 L: K} \prod_{\omega \in T} \Delta'_\omega(\xi, \eta).$$

By Lemma 8.2.1 we have

$$H_S(V) \leq (H_S(F_U) H_S(F_{UV}))^{3/m} = (H_S(F) H_S(F_U))^{3/m},$$
and
\[ H_S(V) \leq (H_S(G)H_S(G_U))^{3/n}, \]
and from these inequalities we deduce that
\[
\prod_{\omega \in T} \Delta_\omega(\xi, \eta) \geq \left( \frac{1}{H_S(F)^{1/m}H_S(G)^{1/n}H_S(F_U)^{1/m}H_S(G_U)^{1/n}} \right)^{3[L:K]^{1} \prod_{\omega \in T} \Delta'_\omega(\xi, \eta)}.
\]
By taking \( \varepsilon = \frac{1}{117(1+3\varpi)} < 1 \) and
\[ H = H_S(F)^{1/m}H_S(G)^{1/n}, \quad H' = H_S(F_U)^{1/m}H_S(G_U)^{1/n}, \]
we conclude that
\[
\prod_{\omega \in T} \Delta_\omega(\xi, \eta) \geq (HH')^{-3[L:K]^{1}\prod_{\omega \in T} \Delta'_\omega(\xi, \eta)} \prod_{\omega \in T} \left( \Delta_\omega(\xi, \eta)^{1-\varepsilon} \Delta'_\omega(\xi, \eta)^{\varepsilon} \right) \quad (8.3.3)
\]
However, since \([L : K] = [K(\xi) : K][K(\eta) : K]\) we have
\[
R(F, G) = \prod_{i=1}^{m} \prod_{j=1}^{n} (\sigma_i(\alpha)\tau_j(\delta) - \sigma_i(\beta)\tau_j(\gamma)) = N_{L/K}(\alpha\delta - \beta\gamma).
\]
This implies that
\[
|R(F, G)|_{'\nu} = \prod_{\omega|\nu} |\alpha\delta - \beta\gamma|_{'\nu} \text{ for } \nu \in M_K.
\]
Similarly to (8.2.3), we have \( H_S(F) = H_{T_0}(\alpha, \beta)\frac{m}{[L:K]}, H_S(G) = H_{T_0}(\gamma, \delta)\frac{n}{[L:K]} \).
Combining this with (8.3.2) we deduce that
\[
\prod_{\omega \in T_0} \Delta(\xi, \eta) = \frac{|R(F,G)|_S}{H_{T_0}(\alpha, \beta)H_{T_0}(\gamma, \delta)} \]
\[
= \frac{|R(F,G)|_S}{(H_S(F)^{1/m}H_S(G)^{1/n})^{[L,K]}} \]
\[
\geq \exp \left( - \frac{422mn(m+n-5+2g_1+2g_2)}{717} - mn(4m + 4n + 433)\frac{\#S}{717} \right) \times \]
\[
\times \left( \frac{H_S(G_U)^{1/n}H_S(F_U)^{1/m}}{H_S(F)^{1/m}H_S(G)^{1/n}} \right)^{mn} \]  

(8.3.4)

Similarly, we have
\[
\prod_{\omega \in T_0} \Delta'(\xi, \eta) = \frac{|R(F_U,G_U)|_S}{H_{T_0}(\alpha', \beta')H_{T_0}(\gamma', \delta')} \]
\[
= \frac{|R(F,G)|_S}{(H_S(F_U)^{1/m}H_S(G_U)^{1/n})^{[L,K]}} \]
\[
\geq \exp \left( - \frac{422mn(m+n-5+2g_1+2g_2)}{717} - mn(4m + 4n + 433)\frac{\#S}{717} \right) \times \]
\[
\times \left( \frac{H_S(G_U)^{1/n}H_S(F_U)^{1/m}}{H_S(F)^{1/m}H_S(G)^{1/n}} \right)^{mn} \]  

(8.3.5)

Substituting (8.3.4) and (8.3.5) into (8.3.3), we conclude that
\[
\prod_{\omega \in T} \Delta(\xi, \eta) \geq (HH')^{-3mn\varepsilon\varrho} \frac{H'^{mn}}{H^{mn(1-\varepsilon)H^{mn\varepsilon}}} \times \]
\[
\times \exp \left( - \frac{422mn(m+n-5+2g_1+2g_2)}{717} - mn(4m + 4n + 433)\frac{\#S}{717} \right) H^{mn(-1+\vartheta)} \]
\[
= \exp \left( - \frac{422mn(m+n-5+2g_1+2g_2)}{717} - mn(4m + 4n + 433)\frac{\#S}{717} \right) \times \]
\[
\times \left( H'^*(\xi)H'^*(\eta)e^{2g_L\left(\frac{1}{m} + \frac{1}{n}\right)} \right)^{mn(-1+\vartheta)}. \]
where the equality is because of the choice of $\varepsilon$, which makes the exponent of $H'$ to be 0, and the last inequality is due to (8.3.1). This implies that

$$\Delta_T(\xi, \eta) \geq D^{-1}\left(H^*(\xi)H^*(\eta)\right)^{-1+\vartheta},$$

where

$$D = \exp\left(\frac{422(m+n+5+2g_1+2g_2)}{717} + (4m + 4n + 433)\#S_{717} + 2g_L\left(\frac{1}{m} + \frac{1}{n}\right)(1 - \vartheta)\right).$$

Notice that $\vartheta < 1$ and by (5.1.4),

$$\frac{2g_L - 2}{mn} \leq m + n - 6,$$

we conclude that $D \leq C_5$ where

$$C_5 = \exp\left(\frac{422(m+n+5+2g_1+2g_2)}{717} + (4m + 4n + 433)\#S_{717} + (m+n)(m+n-5)(1 - \vartheta)\right).$$