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Chapter 8

Distances between algebraic functions

Let $K = k(t)$. In section 8.1 we give a lower bound for the distance tetween two roots of a polynomial $f \in k[t][X]$, and in section 8.3 we derive such a lower bound between roots of different polynomials. We follow [9], [10] where similar results have been derived over number fields.

8.1 Root separation of polynomials

Let $K = k(t)$ and let $f \in K[X]$ be a polynomial of degree $n \geqslant 4$ with splitting field L and non-zero discriminant. Assume that $f = a\prod_{i=1}^{n}(X-\gamma_i)$ with $a \in K^*$ and $\gamma_i \in L$ for $i = 1, \ldots, n$. Let S be a finite set of valuations on K and let T be the set of valuations on L above those in S. For each $\nu \in S$ fix a prolongation of $|\cdot|_\nu$ to L, also denoted by $|\cdot|_\nu$. Define

$$\Delta_S(f) := \prod_{\nu\in S} \min_{1\leqslant i<j\leqslant n} \frac{|\gamma_i - \gamma_j|_\nu}{\max(1, |\gamma_i|_\nu)\max(1, |\gamma_j|_\nu)}.$$

Since L/K is a Galois extension, this quantity $\Delta_S(f)$ is independent of the choices of the extensions of $|\cdot|_\nu$ to L. To be specific, by (1.4.3) we have for $\omega \in \mathcal{A}(\nu)$ and $\sigma \in \mathcal{E}(\omega|\nu)$ that

$$\begin{aligned}
&\min_{1\leqslant i<j\leqslant n} \frac{|\gamma_i-\gamma_j|_\omega}{\max(1,|\gamma_i|_\omega)\max(1,|\gamma_j|_\omega)} \\
&= \left(\min_{1\leqslant i<j\leqslant n} \frac{|\sigma(\gamma_i-\gamma_j)|_\nu}{\max(1,|\sigma(\gamma_i)|_\nu)\max(1,|\sigma(\gamma_j)|_\nu)}\right)^{g_\nu} \\
&= \left(\min_{1\leqslant i<j\leqslant n} \frac{|\gamma_{\sigma(i)}-\gamma_{\sigma(j)}|_\nu}{\max(1,|\gamma_{\sigma(i)}|_\nu)\max(1,|\gamma_{\sigma(j)}|_\nu)}\right)^{g_\nu} \\
&= \left(\min_{1\leqslant i<j\leqslant n} \frac{|\gamma_i-\gamma_j|_\nu}{\max(1,|\gamma_i|_\nu)\max(1,|\gamma_j|_\nu)}\right)^{g_\nu},
\end{aligned}$$

since $\sigma \in \mathrm{Gal}(L/K)$ acts on $1,\dots,n$ as a permutation and $g_\nu = [L_\omega : K_\nu]$ is independent of ω. Hence

$$\Delta_S(f) = \prod_{\omega\in T}\left(\min_{1\leqslant i<j\leqslant n} \frac{|\gamma_i-\gamma_j|_\omega}{\max(1,|\gamma_i|_\omega)\max(1,|\gamma_j|_\omega)}\right)^{1/[L:K]}. \tag{8.1.1}$$

Put $H(f) = \prod_{\nu\in M_K} |f|_\nu$. Then clearly $H(f) \geqslant 1$.

Theorem 8.1.1. *Let* $c_4(n) = \exp\left(\frac{(n-1)\left((n+11)\#S-5\right)}{20+1/n}\right)$. *We have*

$$\Delta_S(f) \geqslant c_4(n)^{-1} H(f)^{-n+1+\frac{n}{40n+2}}.$$

Proof. Homogenize $f = a_0X^n + a_1X^{n-1} + \cdots + a_n$ and choose

$$F(X,Y) = b(a_0X^n + a_1X^{n-1}Y + \cdots + a_nY^n)$$

with $b \in K^*$ such that

$$|b|_\infty = |f|_\infty^{-1}H(f),\ |b|_\nu = |f|_\nu^{-1} \text{ for } \nu \neq \nu_\infty.$$

The existence of b is guaranteed because $\prod_{\nu\in M_K} |f|_\nu^{-1}H(f) = 1$. So we get $F \in \mathcal{O}_S[X,Y]$, $|F|_\infty = H(f)$ and hence

$$H^*(F) = \max(1, |F|_\infty) = H(f).$$

Factor F in L as $F = \prod_{i=1}^{n}(\alpha_iX + \beta_iY)$. Then $\gamma_i = -\frac{\beta_i}{\alpha_i}$. Put

$$\delta_\omega = \min_{1\leqslant i<j\leqslant n} \frac{|\alpha_i\beta_j - \alpha_j\beta_i|_\omega}{|\alpha_i,\beta_i|_\omega |\alpha_j,\beta_j|_\omega}\ (\omega \in T).$$

Then

$$\Delta_S(f) = \prod_{\omega\in T} \delta_\omega^{1/[L:K]}.$$

Let $F^*(X,Y) = F(aX+bY, cX+dY)$ with $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}\,(2,\mathcal{O}_S)$ be such that F^* is reduced. Then $F^*(X,Y) = \prod_{i=1}^{n}(\alpha_i^* X + \beta_i^* Y)$ where $(\alpha_i^*, \beta_i^*) = (\alpha_i,\beta_i)\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), i = 1,\dots,n$.

Now for $\omega\in T$ put $f_{i\omega} := |\alpha_i,\beta_i|_\omega, f_{i\omega}^* := |\alpha_i^*,\beta_i^*|_\omega$ and $\zeta_{ij\omega} := |\alpha_i\beta_j - \alpha_j\beta_i|_\omega$. Then $\prod_{i=1}^n f_{i\omega} = |F|_\omega, \prod_{i=1}^n f_{i\omega}^* = |F^*|_\omega$ and $\prod_{1\leqslant i<j\leqslant n}\zeta_{ij\omega} = |D(F)|_\omega^{1/2}$.

By the ultrametric inequality we have $\zeta_{ij\omega}\leqslant f_{i\omega}f_{j\omega}$, and

$$\zeta_{ij\omega} = |ad-bc|_\omega^{-1}|\alpha_i^*\beta_j^* - \alpha_j^*\beta_i^*|_\omega \leqslant |ad-bc|_\omega^{-1} f_{i\omega}^* f_{j\omega}^*,$$

So

$$\zeta_{ij\omega}\leqslant \min(f_{i\omega}f_{j\omega}, |ad-bc|_\omega^{-1}f_{i\omega}^*f_{j\omega}^*) \text{ for } 1\leqslant i<j\leqslant n, \omega\in T. \tag{8.1.2}$$

We are going to bound δ_ω from below for each $\omega\in T$. Let $\omega\in T$, and assume, without loss of generality, that $\delta_\omega = \frac{\zeta_{12\omega}}{f_{1\omega}f_{2\omega}}$. Then

$$\delta_\omega \geqslant \frac{\zeta_{12\omega}}{f_{1\omega}f_{2\omega}} \prod_{\substack{1\leqslant i<j\leqslant n\\ (i,j)\neq(1,2)}} \frac{\zeta_{ij\omega}}{\min(f_{i\omega}f_{j\omega}, |ad-bc|_\omega^{-1}f_{i\omega}^*f_{j\omega}^*)} = \frac{|D(F)|_\omega^{1/2}}{\Lambda_\omega},$$

with $\Lambda_\omega = f_{1\omega}f_{2\omega}\prod_{\substack{1\leqslant i<j\leqslant n\\ (i,j)\neq(1,2)}}\min(f_{i\omega}f_{j\omega}, |ad-bc|_\omega^{-1}f_{i\omega}^*f_{j\omega}^*)$.

We claim that

$$\Lambda_\omega \leqslant |F|_\omega |F^*|_\omega^{n-2}|ad-bc|_\omega^{-n(n-2)/2}. \tag{8.1.3}$$

Then

$$\delta_\omega \geqslant \frac{|D(F)|_\omega^{1/2}|ad-bc|_\omega^{n(n-2)/2}}{|F|_\omega|F^*|_\omega^{n-2}}.$$

By the Main Theorem, we have

$$|D(F)|_S^{1/2} \geqslant H^*(F^*)^{n/(40n+2)} e^{\frac{(1-n)\left((n+11)\#S-5\right)}{20+1/n}}. \tag{8.1.4}$$

Using $ad-bc \in \mathcal{O}_S^*$, $H_S(F) \leqslant H^*(F), H_S(F^*) = H^*(F^*) \leqslant H^*(F) = H(f)$, we deduce that

$$\begin{aligned}\Delta_S(f) &\geqslant \left(\prod_{\omega\in T} \frac{|D(F)|_\omega^{1/2}|ad-bc|_\omega^{n(n-2)/2}}{|F|_\omega|F^*|_\omega^{n-2}}\right)^{1/[L:K]}\\ &= \frac{|D(F)|_S^{1/2}}{H_S(F)H_S(F^*)^{n-2}}\\ &\geqslant \exp\left(-\frac{(n-1)\big((n+11)\#S-5\big)}{20+1/n}\right)\frac{1}{H(f)}H^*(F^*)^{\frac{n}{40n+2}-n+2}\\ &\geqslant \exp\left(-\frac{(n-1)\big((n+11)\#S-5\big)}{20+1/n}\right)H(f)^{-n+1+\frac{n}{40n+2}}. \qquad (8.1.5)\end{aligned}$$

Finally, to prove (8.1.3), we have to distinguish two cases. First let $n \geqslant 4$ be even. Take $I = \{(1,2),\ldots,(n-1,n)\}$. Then

$$\begin{aligned}\Lambda_\omega &\leqslant \prod_{i=1}^n f_{i\omega} \prod_{\substack{1\leqslant i<j\leqslant n\\(i,j)\notin I}} |ad-bc|_\omega^{-1} f_{i\omega}^* f_{j\omega}^*\\ &= \prod_{i=1}^n f_{i\omega}\Big(\prod_{i=1}^n f_{i\omega}^*\Big)^{n-2}|ad-bc|_\omega^{-n(n-2)/2}\\ &= |F|_\omega |F^*|_\omega^{n-2}|ad-bc|_\omega^{-n(n-2)/2}.\end{aligned}$$

Next let $n \geqslant 5$ be odd. Take

$$I = \{(1,2),\ldots,(n-2,n-1),(n-2,n),(n-1,n)\}.$$

Then

$$\begin{aligned}\Lambda_\omega &\leqslant \prod_{i=1}^{n-3} f_{i\omega} \prod_{n-2\leqslant i<j\leqslant n}\big(f_{i\omega}f_{j\omega}f_{i\omega}^*f_{j\omega}^*|ad-bc|_\omega^{-1}\big)^{1/2} \prod_{\substack{1\leqslant i<j\leqslant n\\(i,j)\notin I}} |ad-bc|_\omega^{-1} f_{i\omega}^* f_{j\omega}^*\\ &= \prod_{i=1}^n f_{i\omega}\Big(\prod_{i=1}^n f_{i\omega}^*\Big)^{n-2}|ad-bc|_\omega^{-n(n-2)/2}\\ &= |F|_\omega |F^*|_\omega^{n-2}|ad-bc|_\omega^{-n(n-2)/2}.\end{aligned}$$

□

As a direct consequence, we obtain the following result on simultaneous root separation for various absolute values.

Corollary 8.1.2. *We have*

$$\prod_{\nu\in S}\min_{1\leqslant i<j\leqslant n}|\gamma_i-\gamma_j|_\nu \geqslant \exp(\frac{-(n-1)\Big((n+11)\#S-5\Big)}{20})H(f)^{-n+1+\frac{n}{40n+2}}.$$

Proof. Since the denominator of $\Delta_S(f)$ is at least 1, this is a direct consequence of Theorem 8.1.1 and the fact $|x|_S^{[L:K]}=|x|_T$. □

Corollary 8.1.3.

$$\Delta_S(f)\geqslant \exp\Big(-\frac{n-1}{100}\big(5n(n+7)\#S+\tfrac{2g_L-1}{[L:K]}\big)\Big)H(f)^{-n+1+\frac{n}{42}}.$$

Proof. It is similar with proof of Theorem 8.1.1, but replace (8.1.4) by using Theorem 5.3.2. □

8.2 Two lemmas

We need some preparations for the next section where we consider distances between algebraic function that are roots of different polynomials.

Let $K=k(t)$. Let $H^*(\gamma)=\prod_{\omega\in M_L}\max(1,|\gamma|_\omega)^{1/[L:K]}$ for any $\gamma\in L$ algebraic over K. This is independent of the choice of L.

Let ξ,η be distinct and algebraic over K. Let $L=K(\xi,\eta)$ and T a finite set of valuations on L. Define

$$\Delta_T(\xi,\eta):=\left(\prod_{\omega\in T}\frac{|\xi-\eta|_\omega}{\max(1,|\xi|_\omega)\max(1,|\eta|_\omega)}\right)^{1/[L:K]}.$$

Then clearly

$$\begin{aligned}\Delta_T(\xi,\eta) &= \left(\prod_{\omega\notin T}\frac{\max(1,|\xi|_\omega)\max(1,|\eta|_\omega)}{|\xi-\eta|_\omega}\right)^{1/[L:K]} H^*(\xi)^{-1}H^*(\eta)^{-1}\\ &\geqslant H^*(\xi)^{-1}H^*(\eta)^{-1}.\end{aligned}$$

This is a type of Liouville-type inequality. Recall that for a matrix $A = (a_{ij})_{i,j}$, we have defined its ν-value $|A|_\nu = \max_{i,j}(|a_{ij}|_\nu)$ for $\nu \in M_K$. In this way, we also define

$$H_S(A) = \prod_{\nu \in S} |A|_\nu.$$

Lemma 8.2.1. *Let $F(X,Y) \in \mathcal{O}_S[X,Y]$ be a binary form of degree $n \geqslant 3$ with non-zero discriminant. Then for any $U \in \mathrm{GL}\,(2, \mathcal{O}_S)$, we have*

$$\frac{H_S(F_U)}{H_S(F)} \leqslant H_S(U) \leqslant \big(H_S(F)H_S(F_U)\big)^{3/n}.$$

Proof. Let T be the set of valuations on the splitting field L lying above the valuations in S, write $F(X,Y) = a_0 \prod_{i=1}^{n}(\alpha_i X + \beta_i Y)$ with $a_0 \in K^*, \alpha_i, \beta_i \in \mathcal{O}_T$ and $F_U(X,Y) = a_0 \prod_{i=1}^{n}(\alpha_i^* X + \beta_i^* Y)$ with

$$(\alpha_i^*, \beta_i^*) = (\alpha_i, \beta_i)U, \ i = 1, \ldots, n.$$

Let $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then

$$\begin{cases} a\alpha_i + c\beta_i = \alpha_i^* \\ b\alpha_i + d\beta_i = \beta_i^* \end{cases} \quad \text{for } i = 1, \ldots, n.$$

From the non-archimedean property, it easily follows that

$$\max(|\alpha_i^*|_\omega, |\beta_i^*|_\omega) \leqslant |U|_\omega \max(|\alpha_i|_\omega, |\beta_i|_\omega) \text{ for } \omega \in T,$$

hence by Gauss' lemma we have

$$H_T(F_U) \leqslant |U|_T H_T(F),$$

which gives

$$H_S(F_U) \leqslant |U|_S H_S(F).$$

Take any three indices i, j, l and consider the system of equations

$$A\mathbf{x} = \mathbf{0}, \tag{8.2.1}$$

where $\mathbf{x} = (x_1, \dots, x_7)^{\mathrm{T}}$ and

$$A = \begin{pmatrix} \alpha_i & \beta_i & 0 & 0 & \alpha_i^* & 0 & 0 \\ 0 & 0 & \alpha_i & \beta_i & \beta_i^* & 0 & 0 \\ \alpha_j & \beta_j & 0 & 0 & 0 & \alpha_j^* & 0 \\ 0 & 0 & \alpha_j & \beta_j & 0 & \beta_j^* & 0 \\ \alpha_l & \beta_l & 0 & 0 & 0 & 0 & \alpha_l^* \\ 0 & 0 & \alpha_l & \beta_l & 0 & 0 & \beta_l^*. \end{pmatrix}$$

Put $X = \left(\begin{smallmatrix} x_1 & x_3 \\ x_2 & x_4 \end{smallmatrix} \right)$. Then

$$-x_5(\alpha_i^*, \beta_i^*) = (\alpha_i, \beta_i)X,$$
$$-x_6(\alpha_j^*, \beta_j^*) = (\alpha_j, \beta_j)X,$$
$$-x_7(\alpha_l^*, \beta_l^*) = (\alpha_l, \beta_l)X.$$

However, $D(F) \neq 0$, so X maps three pairwise non-parallel vectors to three other pairwise non-parallel vectors. Such a matrix X is unique up to a scalar if it exists. But we already know that $X = U$ with $x_5 = x_6 = x_7 = -1$ is a solution, therefore the solution space of (8.2.1) is one-dimensional and hence for any solution there exists λ such that $U = \lambda X$. Let Δ_s be the determinant of the matrix obtained by removing the s-th column of A. We claim that $(\Delta_1, -\Delta_2, \dots, \Delta_7)$ is a solution of the system of linear equations. To see this, we make an extra seventh row by copying an row and thus obtain a square matrix with determinant 0. By Laplace's formula, expanding this determinant along the seventh row, we immediately get the result. So $U = \lambda \left(\begin{smallmatrix} \Delta_1 & \Delta_3 \\ -\Delta_2 & -\Delta_4 \end{smallmatrix} \right)$. By the ultrametric inequality and again Laplace's formula, it is easy to see that

$$|\Delta_r|_\omega \leqslant \prod_{s=i,j,h} \max(|\alpha_s^*|_\omega, |\beta_s^*|_\omega) \max(|\alpha_s|_\omega, |\beta_s|_\omega), \omega \in M_L \text{ for } r = 1, 2, 3, 4.$$

Hence

$$|U|_\omega \leqslant |\lambda|_\omega \prod_{s=i,j,h} \max(|\alpha_s^*|_\omega, |\beta_s^*|_\omega) \max(|\alpha_s|_\omega, |\beta_s|_\omega) \ (\omega \in M_L).$$

Therefore, by taking the product over $\omega \in M_L$,

$$\prod_{\omega\in M_L} |U|_\omega \leqslant \prod_{s=i,j,h} H_L(\alpha_s,\beta_s)H_L(\alpha_s^*,\beta_s^*).$$

By taking the geometric means over all triples (i,j,h) and going back from L to K, we obtain that

$$\begin{aligned}\prod_{\nu\in M_K} |U|_\nu &= \Big(\prod_{\omega\in M_L} |U|_\omega\Big)^{1/[L:K]}\\ &\leqslant \Big(H_K(F)H_K(F_U)\Big)^{\binom{n-1}{2}/\binom{n}{3}}\\ &= \Big(H_K(F)H_K(F_U)\Big)^{3/n}.\end{aligned}$$

Since $U \in \mathrm{GL}\,(2,\mathcal{O}_S)$, we have $|U|_\nu = 1$ for $\nu \notin S$. Further, $F, F_U \in \mathcal{O}_S[X,Y]$. Hence

$$H_S(U) \leqslant \Big(H_S(F)H_S(F_U)\Big)^{3/n}.$$

□

Lemma 8.2.2. *Let L be a finite extension of K of degree n and T the set of valuations on L above those in S. For $x \in L$, denote by $\sigma_i, i = 1,\ldots,n$ the K-embeddings of L into its algebraic closure, with σ_1 the identity. Then for $x \in K^*$, there exists $\alpha,\beta \in \mathcal{O}_T$ such that $\frac{\alpha}{\beta} = x$ and for $F = \prod_{i=1}^{n}(\sigma_i(\alpha)X+\sigma_i(\beta)Y)$ we have*

$$e^{-\frac{2g_L}{n}} H_S(F)^{\frac{1}{n}} \leqslant H^*(x) \leqslant H_S(F)^{\frac{1}{n}}.$$

Proof. First pick $\alpha',\beta' \in L$ such that $x = \frac{\alpha'}{\beta'}$. By Lemma 3.2.3, there is $\theta \in L^*$ such that

$$|\theta|_\omega \leqslant \min\left(\tfrac{1}{|\alpha'|_\omega}, \tfrac{1}{|\beta'|_\omega}\right) \text{ for } \omega \notin T$$
$$|\theta|_\omega \leqslant A_\omega \text{ for } \omega \in T,$$

where $A_\omega \in e^{\mathbb{Z}}, \omega \in T$ satisfy $\prod_{\omega\in T} A_\omega = e^{2g_L} \prod_{\omega\notin T} \max(|\alpha'|_\omega, |\beta'|_\omega)$.

Let $\alpha = \theta\alpha', \beta = \theta\beta'$. Then $\alpha, \beta \in \mathcal{O}_T$ and so $F \in \mathcal{O}_S[X, Y]$ and $x = \frac{\alpha}{\beta}$. Also, we have

$$\begin{aligned}
1 &\geqslant \prod_{\omega \notin T} \max(|\alpha|_\omega, |\beta|_\omega) \\
&= \prod_{\omega \notin T} |\theta|_\omega \prod_{\omega \notin T} \max(|\alpha'|_\omega, |\beta'|_\omega) \\
&= \frac{1}{\prod_{\omega \in T} |\theta|_\omega} \prod_{\omega \notin T} \max(|\alpha'|_\omega, |\beta'|_\omega) \\
&\geqslant \frac{1}{\prod_{\omega \in T} A_\omega} \prod_{\omega \notin T} \max(|\alpha'|_\omega, |\beta'|_\omega) \\
&= e^{-2g_L}. \qquad (8.2.2)
\end{aligned}$$

Let M be a normal extension of K containing L, and U the set of valuations above those in S. By Lemma 1.4.1 we have

$$\begin{aligned}
\prod_{\nu \notin S} |F|_\nu &= \Big(\prod_{\omega \notin U} |F|_\mu \Big)^{\frac{1}{[M:K]}} \\
&= \Big(\prod_{\mu \notin U} \prod_{i=1}^{n} \max(|\sigma_i(\alpha)|_\mu, |\sigma_i(\beta)|_\mu) \Big)^{\frac{1}{[M:K]}} \\
&= \Big(\prod_{\mu \notin U} \max(|\alpha|_\mu, |\beta|_\mu) \Big)^{\frac{n}{[M:K]}} \\
&= \Big(\prod_{\omega \notin T} \max(|\alpha|_\omega, |\beta|_\omega) \Big)^{\frac{n[M:L]}{[M:K]}} \\
&= \prod_{\omega \notin T} \max(|\alpha|_\omega, |\beta|_\omega). \qquad (8.2.3)
\end{aligned}$$

Combining (8.2.2) with (8.2.3) we derive that

$$e^{-2g_L} \leqslant \frac{H(F)}{H_S(F)} \leqslant 1.$$

By the product formula we have

$$\begin{aligned} H^*(x) &= \Big(\prod_{i=1}^n H^*(\sigma_i(x))\Big)^{\frac{1}{n}} \\ &= \Big(\prod_{i=1}^n \prod_{\omega\in M_L} \max(|\sigma_i(\alpha)|_\omega, |\sigma_i(\beta)|_\omega)\Big)^{\frac{1}{n[L:K]}} \\ &= \Big(\prod_{\omega\in M_L} |F|_\omega\Big)^{\frac{1}{n[L:K]}} \\ &= H(F)^{\frac{1}{n}}. \end{aligned}$$

This implies that

$$e^{-\frac{2g_L}{n}} H_S(F)^{\frac{1}{n}} \leqslant H^*(x) \leqslant H_S(F)^{\frac{1}{n}}.$$

□

8.3 A symmetric improvement of the Liouville-type inequality

Theorem 8.3.1. *Suppose ξ, η are algebraic over K. Let $L = K(\xi, \eta)$ and assume*

$$[K(\xi) : K] \geqslant 3,\ [K(\eta) : K] \geqslant 3,\ [L : K] = [K(\xi) : K][K(\eta) : K].$$

Let S be a finite set of valuations on K, T_0 the set of valuations on L lying above those in S and $T \subset T_0$ such that

$$\varpi := \max_{\nu\in S} \frac{1}{[L : K]} \sum_{\substack{\omega|\nu \\ \omega\in T}} [L_\omega : K_\nu] < \frac{1}{3}.$$

Let g_1, g_2 be the genera of $K(\xi)$ and $K(\eta)$ respectively. Then

$$\Delta_T(\xi, \eta) \geqslant C_5^{-1}\big(H^*(\xi)H^*(\eta)\big)^{-1+\vartheta},$$

where $\vartheta = \frac{1-3\varpi}{717(1+3\varpi)}$ *and*

$$C_5 = \exp\Big(\frac{422(m+n-5+2g_1+2g_2)}{717}+(4m+4n+433)\frac{\#S}{717}+(m+n)(m+n-5)(1-\vartheta)\Big).$$

Proof. Assume $[K(\xi):K]=m$, $[K(\eta):K]=n$. Then $[L:K]=mn$. Without loss of generality, suppose $\nu_\infty \in S$. For if $\nu_\infty \notin S$, then adding ν_∞ to S does not affect ϖ. Let $\sigma_1,\dots,\sigma_m$ and $\tau_1,\dots,\tau_n$ be the K-isomorphic embeddings of $K(\xi)$ and $K(\eta)$ respectively into M.

By Lemma 8.2.2 there are $\alpha,\beta \in K(\xi)$ and $\gamma,\delta \in K(\eta)$ that are integral over $\mathcal{O}_S$ such that $\xi = \frac{\alpha}{\beta}, \eta = \frac{\gamma}{\delta}$, and the corresponding binary forms $F(X,Y) = \prod_{i=1}^{m}\big(\sigma_i(\alpha)X+\sigma_i(\beta)Y\big), G(X,Y)=\prod_{j=1}^{n}\big(\tau_j(\gamma)X+\tau_j(\delta)Y\big)$ satisfy

$$\begin{aligned} e^{-\frac{2g_L}{m}}H_S(F)^{\frac{1}{m}} &\leqslant H^*(\xi) \leqslant H_S(F)^{\frac{1}{m}},\\ e^{-\frac{2g_L}{n}}H_S(G)^{\frac{1}{n}} &\leqslant H^*(\eta) \leqslant H_S(G)^{\frac{1}{n}}. \end{aligned} \tag{8.3.1}$$

Moreover, the assumption implies that ξ,η are not conjugate over K and hence F,G are irreducible and FG is square-free. By Theorem 7.5.1, there exists $U \in \mathrm{GL}\,(2,\mathcal{O}_S)$ such that

$$|R(F,G)|_S \geqslant C' H_S(G_U)^{\frac{m}{717}} H_S(F_U)^{\frac{n}{717}}, \tag{8.3.2}$$

where

$$C' = \exp\Big(-\frac{422mn(m+n-5+2g_1+2g_2)}{717} - mn(4m+4n+433)\frac{\#S}{717}\Big).$$

Notice that

$$F_U(X,Y) = \prod_{i=1}^{m}\big(\sigma_i(\alpha')X+\sigma_i(\beta')Y\big),$$

$$G_U(X,Y) = \prod_{j=1}^{n}\big(\tau_j(\gamma')X+\tau_j(\delta')Y\big),$$

where

$$(\alpha',\beta') = (\alpha,\beta)U,\ (\gamma',\delta') = (\gamma,\delta)U.$$

Let $V \in \mathrm{GL}(2, \mathcal{O}_S)$ be the inverse of U. Then

$$\begin{aligned}
\alpha\delta - \beta\gamma &= (\det V)(\alpha'\delta' - \beta'\gamma'),\\
\max(|\alpha|_\omega, |\beta|_\omega) &\leqslant |V|_\omega \max(|\alpha'|_\omega, |\beta'|_\omega),\\
\max(|\gamma|_\omega, |\delta|_\omega) &\leqslant |V|_\omega \max(|\gamma'|_\omega, |\delta'|_\omega).
\end{aligned}$$

For $\omega \in M_L$, put

$$\Delta_\omega(\xi, \eta) := \frac{|\xi - \eta|_\omega}{\max(1, |\xi|_\omega)\max(1, |\eta|_\omega)},$$

$$\Delta'_\omega(\xi, \eta) := \frac{|\alpha'\delta' - \beta'\gamma'|_\omega}{\max(|\alpha'|_\omega, |\beta'|_\omega)\max(|\gamma'|_\omega, |\delta'|_\omega)}.$$

Then $\Delta_\omega(\xi, \eta) \leqslant 1, \Delta'_\omega(\xi, \eta) \leqslant 1$. From what we mentioned above we have

$$\begin{aligned}
\Delta_\omega(\xi, \eta) &= \frac{|\alpha\delta - \beta\gamma|_\omega}{\max(|\alpha|_\omega, |\beta|_\omega)\max(|\gamma|_\omega, |\delta|_\omega)}\\
&\geqslant \frac{|\det V|_\omega |\alpha'\delta' - \beta'\gamma'|_\omega}{|V|_\omega^2 \max(|\alpha'|_\omega, |\beta'|_\omega)\max(|\gamma'|_\omega, |\delta'|_\omega)}\\
&= \frac{|\det V|_\omega}{|V|_\omega^2}\Delta'_\omega(\xi, \eta)\\
&= \frac{|\det V|_\nu^{[L_\omega:K_\nu]}}{|V|_\nu^{2[L_\omega:K_\nu]}}\Delta'_\omega(\xi, \eta).
\end{aligned}$$

Since $|\det V|_\nu \leqslant |V|_\nu^2$ for any $\nu \in M_K$ and $V \in \mathrm{GL}(2, \mathcal{O}_S)$, we derive that

$$\begin{aligned}
\prod_{\omega \in T} \Delta_\omega(\xi, \eta) &\geqslant \prod_{\nu \in S} \prod_{\substack{\omega \in T\\ \omega | \nu}} \Big(\frac{|\det V|_\nu}{|V|_\nu^2}\Big)^{[L_\omega:K_\nu]} \prod_{\omega \in T} \Delta'_\omega(\xi, \eta)\\
&\geqslant \prod_{\nu \in S} \Big(\frac{|\det V|_\nu}{|V|_\nu^2}\Big)^{[L:K]\varpi} \prod_{\omega \in T} \Delta'_\omega(\xi, \eta)\\
&= \frac{1}{H_S(V)^{2[L:K]\varpi}} \prod_{\omega \in T} \Delta'_\omega(\xi, \eta).
\end{aligned}$$

By Lemma 8.2.1 we have

$$\begin{aligned}
H_S(V) &\leqslant \big(H_S(F_U)H_S(F_{UV})\big)^{3/m}\\
&= \big(H_S(F)H_S(F_U)\big)^{3/m},
\end{aligned}$$

and

$$H_S(V) \leqslant \big(H_S(G)H_S(G_U)\big)^{3/n},$$

and from these inequalities we deduce that

$$\prod_{\omega\in T}\Delta_\omega(\xi,\eta) \geqslant \Big(\frac{1}{H_S(F)^{1/m}H_S(G)^{1/n}H_S(F_U)^{1/m}H_S(G_U)^{1/n}}\Big)^{3[L:K]\varpi}\prod_{\omega\in T}\Delta'_\omega(\xi,\eta).$$

By taking $\varepsilon = \frac{1}{717(1+3\varpi)} < 1$ and

$$H = H_S(F)^{1/m}H_S(G)^{1/n},\ H' = H_S(F_U)^{1/m}H_S(G_U)^{1/n},$$

we conclude that

$$\begin{aligned}\prod_{\omega\in T}\Delta_\omega(\xi,\eta) &\geqslant (HH')^{-3[L:K]\varepsilon\varpi}\prod_{\omega\in T}\Big(\Delta_\omega(\xi,\eta)^{1-\varepsilon}\Delta'_\omega(\xi,\eta)^{\varepsilon}\Big)\\ &\geqslant (HH')^{-3mn\varepsilon\varpi}\prod_{\omega\in T_0}\Big(\Delta_\omega(\xi,\eta)^{1-\varepsilon}\Delta'_\omega(\xi,\eta)^{\varepsilon}\Big) \qquad (8.3.3)\end{aligned}$$

However, since $[L:K]=[K(\xi):K][K(\eta):K]$ we have

$$R(F,G)=\prod_{i=1}^{m}\prod_{j=1}^{n}\big(\sigma_i(\alpha)\tau_j(\delta)-\sigma_i(\beta)\tau_j(\gamma)\big)=N_{L/K}(\alpha\delta-\beta\gamma).$$

This implies that

$$|R(F,G)|_\nu=\prod_{\omega|\nu}|\alpha\delta-\beta\gamma|_\nu \text{ for } \nu\in M_K.$$

Similarly to (8.2.3), we have $H_S(F)=H_{T_0}(\alpha,\beta)^{\frac{m}{[L:K]}}, H_S(G)=H_{T_0}(\gamma,\delta)^{\frac{n}{[L:K]}}$.

Combining this with (8.3.2) we deduce that

$$\begin{aligned}
\prod_{\omega\in T_0}\Delta_\omega(\xi,\eta) &= \frac{|R(F,G)|_S}{H_{T_0}(\alpha,\beta)H_{T_0}(\gamma,\delta)}\\
&= \frac{|R(F,G)|_S}{\Big(H_S(F)^{1/m}H_S(G)^{1/n}\Big)^{[L:K]}}\\
&\geqslant \exp\Big(-\frac{422mn(m+n-5+2g_1+2g_2)}{717}-mn(4m+4n+433)\frac{\#S}{717}\Big)\times\\
&\quad\times\frac{H_S(G_U)^{\frac{m}{717}}H_S(F_U)^{\frac{n}{717}}}{\Big(H_S(F)^{1/m}H_S(G)^{1/n}\Big)^{[L:K]}}\\
&= \exp\Big(-\frac{422mn(m+n-5+2g_1+2g_2)}{717}-mn(4m+4n+433)\frac{\#S}{717}\Big)\times\\
&\quad\times\left(\frac{\Big(H_S(G_U)^{\dfrac{1}{n}}H_S(F_U)^{\dfrac{1}{m}}\Big)^{\frac{1}{717}}}{H_S(F)^{1/m}H_S(G)^{1/n}}\right)^{mn}. \qquad (8.3.4)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\prod_{\omega\in T_0}\Delta'_\omega(\xi,\eta) &= \frac{|R(F_U,G_U)|_S}{H_{T_0}(\alpha',\beta')H_{T_0}(\gamma',\delta')}\\
&= \frac{|R(F,G)|_S}{\Big(H_S(F_U)^{1/m}H_S(G_U)^{1/n}\Big)^{[L:K]}}\\
&\geqslant \exp\Big(-\frac{422mn(m+n-5+2g_1+2g_2)}{717}-mn(4m+4n+433)\frac{\#S}{717}\Big)\times\\
&\quad\times\left(\frac{\Big(H_S(G_U)^{\dfrac{1}{n}}H_S(F_U)^{\dfrac{1}{m}}\Big)^{\frac{1}{717}}}{H_S(F_U)^{1/m}H_S(G_U)^{1/n}}\right)^{mn}. \qquad (8.3.5)
\end{aligned}$$

Substituting (8.3.4) and (8.3.5) into (8.3.3), we conclude that

$$\begin{aligned}
\prod_{\omega\in T}\Delta_\omega(\xi,\eta) &\geqslant (HH')^{-3mn\varepsilon\varpi}\frac{H'^{\frac{mn}{717}}}{H^{mn(1-\varepsilon)}H'^{mn\varepsilon}}\times\\
&\quad\times\exp\Big(-\frac{422mn(m+n-5+2g_1+2g_2)}{717}-mn(4m+4n+433)\frac{\#S}{717}\Big)\\
&= \exp\Big(-\frac{422mn(m+n-5+2g_1+2g_2)}{717}-mn(4m+4n+433)\frac{\#S}{717}\Big)H^{mn(-1+\vartheta)}\\
&\geqslant \exp\Big(-\frac{422mn(m+n-5+2g_1+2g_2)}{717}-mn(4m+4n+433)\frac{\#S}{717}\Big)\times\\
&\quad\times\Big(H^*(\xi)H^*(\eta)e^{2g_L(\frac{1}{m}+\frac{1}{n})}\Big)^{mn(-1+\vartheta)}.
\end{aligned}$$

where the equality is because of the choice of ε, which makes the exponent of H' to be 0, and the last inequality is due to (8.3.1). This implies that

$$\Delta_T(\xi,\eta) \geqslant D^{-1}\Big(H^*(\xi)H^*(\eta)\Big)^{-1+\vartheta},$$

where

$$D = \exp\Big(\tfrac{422(m+n-5+2g_1+2g_2)}{717} + (4m+4n+433)\tfrac{\#S}{717} + 2g_L(\tfrac{1}{m}+\tfrac{1}{n})(1-\vartheta)\Big).$$

□

Notice that $\vartheta < 1$ and by (5.1.4),

$$\frac{2g_L - 2}{mn} \leqslant m+n-6,$$

we conclude that $D \leqslant C_5$ where

$$C_5 = \exp\Big(\tfrac{422(m+n-5+2g_1+2g_2)}{717} + (4m+4n+433)\tfrac{\#S}{717} + (m+n)(m+n-5)(1-\vartheta)\Big).$$