Chapter 7

Lower bounds for resultants

Evertse and Győry deduced some semi-effective lower bounds of resultants over number fields in [10], [12]. Apart from two theorems mentioned in the introduction, they have also established the following:

Theorem (Evertse, Győry). *Let $F, G \in \mathbb{Z}[X, Y]$ be two binary forms of degree $m > 1, n > 2$ such that FG has splitting field L over $\mathbb{Q}$ and is square-free, and $F(1,0) = G(1,0) = 1$. Then*

$$|R(F,G)| \geqslant C(m,n,L)\max(|D(F)|_{S_1}^{\frac{n}{7(m-1)}}, |D(G)|_{S_1}^{\frac{m}{7(n-1)}}),$$

where $C(m,n,L)$ depends on m, n and L.

The constant $C(m,n,L)$ cannot be effectively computed from their method of proof. In this chapter, we deduce effective analogous results over function fields, with the help of outcome derived before.

7.1 Monic binary forms

Recall that $K = k(t)$ and S is a finite set of valuations of K containing ν_∞. Let L be a finite extension of $K = k(t)$ of genus g_L. Let T be a finite set of places of L above those in S. Denote by $\mathcal{O}_T$ the integral closure of $\mathcal{O}_S$.

A binary form of degree n is called X-monic if the leading coefficient of X^n is 1. We call two X-monic quadratic forms related if the coefficients of the term XY are the same, and unrelated if otherwise.

Lemma 7.1.1. *Let F, G be two binary quadratic forms over the ring $\mathcal{O}_T$ satisfying $F(1,0) = G(1,0) = 1$, and suppose that FG is square-free and splits into linear forms over L. Then we have*

(i) $|D(F)|_T \leqslant e^{2(\#T+\max(0,2g_L-2))}|R(F,G)|_T|D(G)|_T$, *if F, G are related;*

(ii) $|D(F)|_T \leqslant e^{6(\#T+\max(0,2g_L-2))}|R(F,G)|_T^2$, *if F, G are unrelated.*

Proof. Put $g' = \max(0, 2g_L - 2)$. Since $F(1,0) = G(1,0) = 1$ and FG splits into linear factors over L, we may assume that

$$F(X,Y) = (X - \alpha_1 Y)(X - \alpha_2 Y),$$

$$G(X,Y) = (X - \beta_1 Y)(X - \beta_2 Y).$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are distinct elements of L.

We actually have $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{O}_T$, since $\mathcal{O}_T$ is integrally closed.

Now, we have

$$D(F) = (\alpha_1 - \alpha_2)^2, D(G) = (\beta_1 - \beta_2)^2,$$

$$R(F,G) = (\alpha_1 - \beta_1)(\alpha_1 - \beta_2)(\alpha_2 - \beta_1)(\alpha_2 - \beta_2).$$

If F, G are related, i.e., $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$, then $\alpha_1 - \alpha_2 = (\beta_1 - \alpha_2) + (\beta_2 - \alpha_2)$. Considering the identity $(\beta_1 - \alpha_2) - (\beta_2 - \alpha_2) - (\beta_1 - \beta_2) = 0$, and applying Corollary 2.2.11, we have

$$H_T(\beta_1 - \alpha_2, -(\beta_2 - \alpha_2), -(\beta_1 - \beta_2)) \leqslant e^{\#T+g'}|(\beta_1 - \alpha_2)(\beta_2 - \alpha_2)(\beta_1 - \beta_2))|_T,$$

hence

$$\begin{aligned} |D(F)|_T^{\frac{1}{2}} &= |\alpha_1 - \alpha_2|_T \\ &= |\beta_1 - \alpha_2 + \beta_2 - \alpha_2|_T \\ &\leqslant e^{\#T+g'}|(\beta_1 - \alpha_2)(\beta_2 - \alpha_2)(\beta_1 - \beta_2))|_T \\ &= e^{\#T+g'}|(\beta_1 - \alpha_2)(\beta_2 - \alpha_2)|_T|D(G)|_T^{\frac{1}{2}}. \end{aligned}$$

Similarly, we have

$$|D(F)|_T^{\frac{1}{2}} \leqslant e^{\#T+g'}|(\beta_1-\alpha_1)(\beta_2-\alpha_1)|_T|D(G)|_T^{\frac{1}{2}}.$$

Hence

$$|D(F)|_T \leqslant e^{2(\#T+g')}|R(F,G)|_T|D(G)|_T.$$

If F, G are unrelated, i.e., $\alpha_1+\alpha_2 \neq \beta_1+\beta_2$, then we consider the identity

$$(\alpha_1-\beta_1)-(\alpha_1-\beta_2)-(\alpha_2-\beta_1)+(\alpha_2-\beta_2)=0,$$

which satisfies the condition of Corollary 2.2.11. We derive that

$$H_T(\alpha_1-\beta_1,\alpha_1-\beta_2,\alpha_2-\beta_1,\alpha_2-\beta_2) \leqslant e^{3(\#T+g')}|R(F,G)|_T.$$

Hence

$$\begin{aligned} |D(F)|_T^{\frac{1}{2}} &= |\alpha_1-\alpha_2|_T \\ &= |(\alpha_1-\beta_1)-(\alpha_2-\beta_1)|_T \\ &\leqslant H_T(\alpha_1-\beta_1,\alpha_1-\beta_2,\alpha_2-\beta_1,\alpha_2-\beta_2) \\ &\leqslant e^{3(\#T+g')}|R(F,G)|_T, \end{aligned}$$

and

$$|D(F)|_T \leqslant e^{6(\#T+g')}|R(F,G)|_T^2.$$

□

Theorem 7.1.2. *Let K_1 be a finite extension of $K=k(t)$, S_1 a finite set of valuations of K_1. Let $F, G \in \mathcal{O}_{S_1}[X,Y]$ be binary forms satisfying the following conditions:*

$$\begin{cases} \deg F = m \geqslant 2, \deg G = n \geqslant 3, F(1,0)=G(1,0)=1, \\ FG \textit{ has splitting field } L \textit{ over } K_1 \textit{ and } FG \textit{ is square-free.} \end{cases}$$

Then we have

$$|R(F,G)|_{S_1} \geqslant e^{-\frac{mn}{2}(\#S_1+\frac{\max(0,2g_L-2)}{[L:K_1]})}\max(|D(F)|_{S_1}^{\frac{n}{6(m-1)}}, |D(G)|_{S_1}^{\frac{m}{6(n-1)}}).$$

Proof. Let $T \subset M_L$ be the set of valuations over those in S_1. Over L, we have

$$F(X,Y)=\prod_{i=1}^{m}(X-\alpha_iY),\ G(X,Y)=\prod_{j=1}^{n}(X-\beta_jY).$$

Since $F,G \in \mathcal{O}_{S_1}[X,Y]$, we have $\alpha_i,\beta_j \in \mathcal{O}_T$, $i=1,\dots,m, j=1,\dots,n$.

Let

$$F_{pq}(X,Y)=(X-\alpha_pY)(X-\alpha_qY)\,(1\leqslant p<q\leqslant m),$$

$$G_{ij}(X,Y)=(X-\beta_iY)(X-\beta_jY)\,(1\leqslant i<j\leqslant n).$$

Now fix a pair $p<q$. Let I_{pq} be the collection of pairs $i<j$ such that G_{ij} is related to F_{pq}: $\alpha_p+\alpha_q=\beta_i+\beta_j$. Then each two pairs in I_{pq} are disjoint since FG is square-free, and hence $\#I_{pq}\leqslant[\frac{n}{2}]\leqslant\frac{n(n-1)}{6}$.

Put $g'=\max(0,2g_L-2)$. By Lemma 7.1.1 we get

$$|D(F_{pq})|_T\leqslant e^{6(\#T+g')}|R(F_{pq},G_{ij})|_T^2 \text{ for } (i,j)\notin I_{pq}.$$

Hence we have

$$\begin{aligned} |D(F_{pq})|_T &\leqslant e^{6(\#T+g')}\Big(\prod_{(i,j)\notin I_{pq}}|R(F_{pq},G_{ij})|_T\Big)^{\frac{2}{\frac{n(n-1)}{2}-\#I_{pq}}} && (7.1.1)\\ &\leqslant e^{6(\#T+g')}\Big(\prod_{(i,j)\notin I_{pq}}|R(F_{pq},G_{ij})|_T\Big)^{\frac{6}{n(n-1)}} && (7.1.2)\\ &\leqslant e^{6(\#T+g')}\Big(\prod_{1\leqslant i<j\leqslant n}|R(F_{pq},G_{ij})|_T\Big)^{\frac{6}{n(n-1)}} && (7.1.3)\\ &= e^{6(\#T+g')}|R(F_{pq},G)|_T^{\frac{6}{n}}. && (7.1.4)\end{aligned}$$

So

$$\begin{aligned} |D(F)|_T &= \prod_{1\leqslant p<q\leqslant m}|D(F_{pq})|_T\\ &\leqslant e^{3m(m-1)(\#T+g')}\Big(\prod_{1\leqslant p<q\leqslant m}|R(F_{pq},G)|_T\Big)^{\frac{6}{n}}\\ &\leqslant e^{3m(m-1)(\#T+g')}|R(F,G)|_T^{\frac{6(m-1)}{n}}.\end{aligned}$$

Similarly,

$$|D(G)|_T \leqslant e^{3n(n-1)(\#T+g')}|R(F,G)|_T^{\frac{6(n-1)}{m}}.$$

Then it follows from $|x|_T = |x|_{S_1}^{[L:K_1]}, x \in K_1$ and $\#T \leqslant [L:K_1]\#S_1$. $\square$

Remark 7.1.3. *If $m = n = 2$, the results above are not valid. Simply take $K = k(t), S = \nu_\infty$. Let $u, v \in k[t]$ be a solution of $x^2 - (t^2-1)y^2 = 1$ and put $F(X,Y) = X^2 - u^2Y^2, G(X,Y) = X^2 - (t^2-1)v^2Y^2$. Then it is easy to check that $D(F) = 4u^2, D(G) = 4v^2(t^2-1), R(F,G) = 1$ and FG is square free with splitting field $K(\sqrt{t^2-1})$. However, since $u = t, v = 1$ is a solution of $x^2 - (t^2-1)y^2 = 1$, we can find infinitely many solutions $u, v \in k[t]$ satisfying $u + \sqrt{t^2-1}v = (t+\sqrt{t^2-1})^j$ with $|u|_\infty$ goes to infinity.*

7.2 Results for binary cubic forms

Recall that K_1 is a finite extension of $K = k(t)$ with genus g_{K_1}, S_1 a finite set of valuations on K_1. Consider two binary forms $F, G \in K_1[X,Y]$ such that

$$F(X,Y) = \prod_{i=1}^{3}(\alpha_i X - \beta_i Y),$$

$$G(X,Y) = \prod_{i=1}^{3}(\gamma_i X - \delta_i Y),$$

where $\alpha_i, \beta_i, \gamma_j, \delta_j \in K_1, i, j = 1,2,3$, and FG is square-free. In this section we prove

Proposition 7.2.1. *With the same setting as above, we have*

$$|R(F,G)|_{S_1} \geqslant e^{-\frac{90}{17}(2g_{K_1}-1+\#S_1)}|D(F)D(G)|_{S_1}^{\frac{3}{34}}|FG|_{S_1}^{\frac{45}{17}}.$$

Before proving this result we start with some preliminaries and a lemma.

Put $\Delta_{ij} = \alpha_i\delta_j - \beta_i\gamma_j, F_{ij} = \alpha_i\beta_j - \alpha_j\beta_i, G_{ij} = \gamma_i\delta_j - \gamma_j\delta_i$ for $i, j = 1,2,3$. Then by direct calculation

$$\det \begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{pmatrix} = 0.$$

Put

$$u_1 = \Delta_{11}\Delta_{22}\Delta_{33}, \; u_2 = -\Delta_{11}\Delta_{23}\Delta_{32},$$
$$u_3 = \Delta_{12}\Delta_{23}\Delta_{31}, \; u_4 = -\Delta_{12}\Delta_{21}\Delta_{33},$$
$$u_5 = \Delta_{13}\Delta_{21}\Delta_{32}, \; u_6 = -\Delta_{13}\Delta_{22}\Delta_{31}.$$

Then

$$u_1 + u_2 + u_3 + u_4 + u_5 + u_6 = 0, \tag{7.2.1}$$

$$u_1 u_3 u_5 = -u_2 u_4 u_6 = R(F, G) \neq 0. \tag{7.2.2}$$

Hence $u_i \neq 0$ for $i = 1, \ldots, 6$. Also

$$R(F, G) = \prod_{i=1}^{3} \prod_{j=1}^{3} \Delta_{ij}, D(F) = (F_{12}F_{23}F_{13})^2, D(G) = (G_{12}G_{23}G_{13})^2.$$

Similarly as in [12], we have

$$(D(F)D(G))^{\frac{3}{2}} = \pm R(F, G)^{-1} \prod_{\substack{1 \leqslant p < q \leqslant 6 \\ p \not\equiv q (\operatorname{mod} 2)}} (u_p + u_q). \tag{7.2.3}$$

Hence

$$u_p + u_q \neq 0 \text{ for } 1 \leqslant p < q \leqslant 6, p \not\equiv q (\operatorname{mod} 2). \tag{7.2.4}$$

Put $c_n = e^{\binom{n}{2}(\max(2g_{K_1} - 2 + \# S_1, 0)}$. Analogously to Lemma 5, [12], we have

Lemma 7.2.2. *For* $(u_1, \ldots, u_6)$ *satisfying (7.2.1), (7.2.2), (7.2.4), we have*

$$\prod_{\substack{1 \leqslant p < q \leqslant 6 \\ p \not\equiv q (\operatorname{mod} 2)}} H_{S_1}(u_p, u_q) \leqslant c_5^9 |R(F, G)|_{S_1}^{18} |FG|_{S_1}^{-45}.$$

Proof. We adapt the idea in the proof of Lemma 5 of [12].

By symmetry, we have to consider only the following four cases:

(i) $\sum_{i=1}^{6} u_i$ has no vanishing proper subsum;

(ii) $u_1 + u_3 = 0, u_2 + u_4 + u_5 + u_6 = 0$ with no vanishing proper subsum;

(iii) $u_1 + u_2 + u_3 = u_4 + u_5 + u_6 = 0$ with no vanishing proper subsum;

(iv) $u_1 + u_3 + u_5 = u_2 + u_4 + u_6 = 0$ with no vanishing proper subsum.

First, since

$$|\Delta_{ij}|_\nu = |\alpha_i\delta_j - \beta_i\gamma_j|_\nu \leqslant \max(|\alpha_i|_\nu, |\beta_i|_\nu)\max(|\gamma_j|_\nu, |\delta_j|_\nu) \text{ for } \nu \in M_{K_1}, i, j = 1, 2, 3,$$

we have

$$|u_1|_\nu \leqslant \prod_{i=1}^{3} \max(|\alpha_i|_\nu, |\beta_i|_\nu) \prod_{j=1}^{3} \max(|\gamma_j|_\nu, |\delta_j|_\nu) = |FG|_\nu,$$

and similarly for $i = 1, \ldots, 6$

$$|u_i|_\nu \leqslant |FG|_\nu \text{ for } \nu \in M_{K_1}. \tag{7.2.5}$$

For case (i), by applying Corollary 2.2.11, we get for $p < q$ with $p \not\equiv q (\bmod 2)$ that

$$\begin{aligned} H_{S_1}(u_p, u_q) &\leqslant H_{S_1}(u_1, \ldots, u_6) \\ &\leqslant c_5 \prod_{i=1}^{6} |u_i|_{S_1} \left(\prod_{\nu \notin S_1} \max_{1 \leqslant i \leqslant 6} (|u_i|_\nu) \right)^5 \\ &\leqslant c_5 |R(F,G)|_{S_1}^2 |FG|_{S_1}^{-5}. \end{aligned}$$

hence

$$\prod_{\substack{1 \leqslant p < q \leqslant 6 \\ p \not\equiv q (\bmod 2)}} H_{S_1}(u_p, u_q) \leqslant c_5^9 |R(F,G)|_{S_1}^{18} |FG|_{S_1}^{-45}.$$

For case (ii), we apply Corollary 2.2.11 to $u_2 + u_4 + u_5 + u_6 = 0$ and derive that for $(p,q)=(2,5),(4,5),(5,6)$,

$$\begin{aligned}
H_{S_1}(u_p,u_q) &\leqslant H_{S_1}(u_2,u_4,u_5,u_6)\\
&\leqslant c_3|u_2u_4u_5u_6|_{S_1}\left(\prod_{\nu\notin S_1}\max(|u_2|_\nu,|u_4|_\nu,|u_5|_\nu,|u_6|_\nu)\right)^3\\
&\leqslant c_3\prod_{i=1}^{6}|u_i|_{S_1}\left(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant 6}(|u_i|_\nu)\right)^5\\
&\leqslant c_3|R(F,G)|_{S_1}^2|FG|_{S_1}^{-5}, \qquad (7.2.6)
\end{aligned}$$

where in the penultimate inequality we have used the consequence of the product formula that $|u_j|_{S_1}\left(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant 6}(|u_i|_\nu)\right)\geqslant 1$ for $j=1,3$, while in the last inequality we used (7.2.2) and (7.2.5).

For $(p,q)=(1,2),(1,4),(1,6),(2,3),(3,4),(3,6)$, we combine $u_1+u_3=0$ with (7.2.2) and get, for example in the case $(p,q)=(1,2)$, using $(u_1^2,u_2^2)=\frac{u_2}{u_5}(u_4u_6,u_2u_5)$,

$$\begin{aligned}
H_{S_1}(u_1,u_2)^2 &= H_{S_1}(u_1^2,u_2^2)\\
&\leqslant \left|\frac{u_2}{u_5}\right|_{S_1} H_{S_1}(u_4,u_2)H_{S_1}(u_6,u_5)\\
&\leqslant \left|\frac{u_2}{u_5}\right|_{S_1} H_{S_1}(u_2,u_4,u_5,u_6)^2.
\end{aligned}$$

By Corollary 2.2.11, this is at most

$$\begin{aligned}
&\left|\frac{u_2}{u_5}\right|_{S_1} c_3^2|u_2u_4u_5u_6|_{S_1}^2\left(\prod_{\nu\notin S_1}\max(|u_2|_\nu,|u_4|_\nu,|u_5|_\nu,|u_6|_\nu)\right)^6\\
=\ & c_3^2|R(F,G)|_{S_1}^2|u_2u_5|_{S_1}\left(\prod_{\nu\notin S_1}\max(|u_2|_\nu,|u_4|_\nu,|u_5|_\nu,|u_6|_\nu)\right)^6\\
\leqslant\ & c_3^2|R(F,G)|_{S_1}^2|u_1u_2u_3u_4u_5u_6|_{S_1}\left(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant 6}(|u_i|_\nu)\right)^{10}\\
\leqslant\ & c_3^2|R(F,G)|_{S_1}^4|FG|_{S_1}^{-10},
\end{aligned}$$

where in the penultimate inequality we have used the inequality that

$|u_j|_{S_1}\left(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant 6}(|u_i|_\nu)\right)\geqslant 1$ for $j=1,3,4,6$, and in the last inequality again (7.2.2) and (7.2.5).

This also gives

$$H_{S_1}(u_1,u_2)\leqslant c_3|R(F,G)|_{S_1}^2|FG|_{S_1}^{-5}.$$

In the same way, this inequality holds true for $(p,q)=(1,4),(1,6),(2,3),(3,4),(3,6)$, and therefore

$$\prod_{\substack{1\leqslant p<q\leqslant 6\\ p\not\equiv q(\mathrm{mod}\,2)}} H_{S_1}(u_p,u_q)\leqslant c_3^9|R(F,G)|_{S_1}^{18}|FG|_{S_1}^{-45}.$$

For case (iii), first we apply Corollary 2.2.11 to $u_1+u_2+u_3=0$ and $u_4+u_5+u_6=0$ and obtain

$$\begin{aligned}
&H_{S_1}(u_1,u_2)H_{S_1}(u_2,u_3)H_{S_1}(u_4,u_5)H_{S_1}(u_5,u_6)\\
\leqslant\ & H_{S_1}(u_1,u_2,u_3)^2H_{S_1}(u_4,u_5,u_6)^2\\
\leqslant\ & c_2^4\prod_{i=1}^{6}|u_i|_{S_1}^2\left(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant 3}(|u_i|_\nu)\right)^4\left(\prod_{\nu\notin S_1}\max_{4\leqslant i\leqslant 6}(|u_i|_\nu)\right)^4\\
\leqslant\ & c_2^4|R(F,G)|_{S_1}^4|FG|_{S_1}^{-8}. \qquad (7.2.7)
\end{aligned}$$

We estimate $H_{S_1}(u_p,u_q)$ for $(p,q)=(1,4),(1,6),(3,4),(3,6)$. When $(p,q)=(1,4)$, we have by (7.2.2), for instance in the case $(p,q)=(1,4)$, that $(u_1,u_4)=\frac{u_1u_4}{R(F,G)}(-u_2u_6,u_3u_5)$. Hence by corollary 2.2.11 we have

$$\begin{aligned}
H_{S_1}(u_1,u_4)\ &\leqslant\ |u_1u_4|_{S_1}|R(F,G)|_{S_1}^{-1}H_{S_1}(u_2,u_3)H_{S_1}(u_6,u_5)\\
&\leqslant\ |u_1u_4|_{S_1}|R(F,G)|_{S_1}^{-1}H_{S_1}(u_1,u_2,u_3)H_{S_1}(u_4,u_5,u_6)\\
&\leqslant\ |u_1u_4|_{S_1}|R(F,G)|_{S_1}^{-1}c_2^2|u_1u_2u_3|_{S_1}\left(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant 3}(|u_i|_\nu)\right)^2\\
&\qquad\times|u_4u_5u_6|_{S_1}\left(\prod_{\nu\notin S_1}\max_{4\leqslant i\leqslant 6}(|u_i|_\nu)\right)^2\\
&\leqslant\ c_2^2|u_1u_4|_{S_1}|R(F,G)|_{S_1}\left(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant 6}(|u_i|_\nu)\right)^4.
\end{aligned}$$

For $(p,q)=(1,6),(3,4),(3,6)$ we obtain similar estimates. Therefore

$$\begin{aligned}
& H_{S_1}(u_1,u_4)H_{S_1}(u_1,u_6)H_{S_1}(u_3,u_4)H_{S_1}(u_3,u_6) \\
\leqslant\ & c_2^8|u_1u_4u_3u_6|_{S_1}^2|R(F,G)|_{S_1}^4\left(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant 6}(|u_i|_\nu)\right)^{16} \\
\leqslant\ & c_2^8\left(\prod_{i=1}^6|u_i|_{S_1}^2\right)|R(F,G)|_{S_1}^4\left(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant 6}(|u_i|_\nu)\right)^{20} \\
=\ & c_2^8|R(F,G)|_{S_1}^8|FG|_{S_1}^{-20}, \qquad (7.2.8)
\end{aligned}$$

where in the penultimate inequality we have used that

$$|u_j|_{S_1}\left(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant 6}(|u_i|_\nu)\right)\geqslant 1 \text{ for } j=2,5.$$

We still have to estimate $H_{S_1}(u_2,u_5)$. Since

$$(u_2,u_5)=R(F,G)^{-1}(-u_2^2u_4u_6,u_1u_3u_5^2),$$

we obtain in a similar way, using corollary 2.2.11, that

$$\begin{aligned}
H_{S_1}(u_2,u_5) &\leqslant |R(F,G)|_{S_1}^{-1}H_{S_1}(u_2,u_1)H_{S_1}(u_2,u_3)H_{S_1}(u_4,u_5)H_{S_1}(u_6,u_5) \\
&\leqslant |R(F,G)|_{S_1}^{-1}H_{S_1}(u_1,u_2,u_3)^2H_{S_1}(u_4,u_5,u_6)^2 \\
&\leqslant c_2^4|R(F,G)|_{S_1}^{-1}|u_1u_2u_3|_{S_1}^2\left(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant 3}(|u_i|_\nu)\right)^4\times \\
&\qquad \times|u_4u_5u_6|_{S_1}^2\left(\prod_{\nu\notin S_1}\max_{4\leqslant i\leqslant 6}(|u_i|_\nu)\right)^4 \\
&\leqslant c_2^4|R(F,G)|_{S_1}^3|FG|_{S_1}^{-8}. \qquad (7.2.9)
\end{aligned}$$

This leads to

$$\prod_{\substack{1\leqslant p<q\leqslant 6\\ p\not\equiv q\,(\mathrm{mod}\,2)}}H_{S_1}(u_p,u_q)\leqslant c_2^{16}|R(F,G)|_{S_1}^{15}|FG|_{S_1}^{-36}.$$

Finally, for case (iv), using the same idea we deduce that

$(u_1^3, u_2^3) = \frac{u_1 u_2}{R(F,G)}(-u_1^2 u_4 u_6, u_2^2 u_3 u_5)$ and so

$$\begin{aligned}
& H_{S_1}(u_1, u_2)^3 = H_{S_1}(u_1^3, u_2^3) \\
\leqslant\ & |u_1 u_2 R(F,G)^{-1}|_{S_1} H_{S_1}(u_1, u_3) H_{S_1}(u_1, u_5) H_{S_1}(u_4, u_2) H_{S_1}(u_6, u_2) \\
\leqslant\ & |u_1 u_2 R(F,G)^{-1}|_{S_1} H_{S_1}(u_1, u_3, u_5)^2 H_{S_1}(u_2, u_4, u_6)^2 \\
\leqslant\ & c_2^4 |R(F,G)|_{S_1}^{-1} |u_1 u_2|_{S_1} |u_1 u_3 u_5|_{S_1}^2 \left(\prod_{\nu \notin S_1} \max(|u_1|_\nu, |u_3|_\nu, |u_5|_\nu) \right)^4 \times \\
& \qquad \times |u_2 u_4 u_6|_{S_1}^2 \left(\prod_{\nu \notin S_1} \max(|u_2|_\nu, |u_4|_\nu, |u_6|_\nu) \right)^4 \\
\leqslant\ & c_2^4 |u_1 u_2 R(F,G)^3|_{S_1} |FG|_{S_1}^{-8}. \qquad (7.2.10)
\end{aligned}$$

Similar inequalities hold true for the other pairs (p, q) under consideration. Combining with (7.2.2), we have

$$\prod_{\substack{1 \leqslant p < q \leqslant 6 \\ p \not\equiv q (\bmod 2)}} H_{S_1}(u_p, u_q) \leqslant c_2^{12} |R(F,G)|_{S_1}^{11} |FG|_{S_1}^{-24}.$$

This finishes our proof. □

Proof of Proposition 7.2.1. This is a combination of (7.2.3) and Lemma 7.2.2, applying the ultra-metric inequality for non-archimedean valuations. □

Remark 7.2.3. *In this section we assumed only $F, G \in K_1[X, Y]$. If we require $F, G \in \mathcal{O}_{S_1}[X, Y]$, then $|FG|_{S_1} \geqslant 1$ and so Proposition 7.2.1 gives*

$$|R(F,G)|_{S_1} \geqslant e^{-\frac{90}{17}(2g_{K_1} - 1 + \#S_1)} |D(F)D(G)|_{S_1}^{\frac{3}{34}}.$$

7.3 Binary forms of arbitrary degree

Again, recall that K_1 is a finite extension of $K = k(t)$ with genus g_{K_1}, and S_1 a finite set of valuations on K_1.

Theorem 7.3.1. *Assume $F, G \in K_1[X, Y]$ are two binary forms such that $\deg F = m \geqslant 3, \deg G = n \geqslant 3$, FG is square-free and has splitting field L*

over K_1. *Then*

$$|R(F,G)|_{S_1} \geqslant e^{-\frac{10mn}{17}(\frac{2g_L-1}{[L:K_1]}+\#S_1)}|D(F)|_{S_1}^{\frac{n}{17(m-1)}}|D(G)|_{S_1}^{\frac{m}{17(n-1)}}\left(|F|_{S_1}^{\frac{15n}{17}}|G|_{S_1}^{\frac{15m}{17}}\right).$$

In particular, if F, G *are irreducible, let* L' *be the field generated by one root of* $F(X,1)$ *and* $G(Y,1)$, *Then*

$$|R(F,G)|_{S_1} \geqslant e^{-\frac{10mn}{17}(\frac{6g_{L'}+4m^2n^2}{[L':K_1]}+\#S_1)}\times$$
$$\times|D(F)|_{S_1}^{\frac{n}{17(m-1)}}|D(G)|_{S_1}^{\frac{m}{17(n-1)}}\left(|F|_{S_1}^{\frac{15n}{17}}|G|_{S_1}^{\frac{15m}{17}}\right).$$

Lemma 7.3.2 (Castelnuovo's Inequality). *Let* F *be a function field of transcendence degree* 1 *over* k. *Let* F_1, F_2 *be two finite extensions of* $k(t)$ *and* F *their compositum. Suppose that*

(i) $F = F_1F_2$ *is the compositum of* F_1 *and* F_2,

(ii) $[F : F_i] = n_i$ *and* F_i *has genus* g_i *(i=1,2).*

Then the genus g *of* F *is bounded by*

$$g \leqslant n_1g_1 + n_2g_2 + (n_1-1)(n_2-1).$$

Proof. See Theorem 3.11.3 of [23]. □

Proof of Theorem 7.3.1. Let T be the set of valuations in L above those in S_1. Assume $F(X,Y) = \prod_{i=1}^{m}(\alpha_iX - \beta_iY), G(X,Y) = \prod_{j=1}^{n}(\gamma_jX - \delta_jY)$. We make a reduction to the case of cubic binary forms. Let

$$F_{pqr}(X,Y) = (\alpha_pX-\beta_pY)(\alpha_qX-\beta_qY)(\alpha_rX-\beta_rY) \text{ for } 1 \leqslant p < q < r \leqslant m,$$

$$G_{ijh}(X,Y) = (\gamma_iX-\delta_iY)(\gamma_jX-\delta_jY)(\gamma_hX-\delta_hY) \text{ for } 1 \leqslant i < j < h \leqslant n.$$

By Proposition 7.2.1, we have

$$|R(F_{pqr}, G_{ijh})|_T \geqslant e^{-\frac{90}{17}(2g_L-1+\#T)}|D(F_{pqr})D(G_{ijh})|_T^{\frac{3}{34}}|F_{pqr}G_{ijh}|_T^{\frac{45}{17}}.$$

Observ that

$$\prod_{1\leqslant p<q<r\leqslant m} F_{pqr} = F^{\binom{m-1}{2}},$$

$$\prod_{1\leqslant i<j<h\leqslant n} G_{ijh} = G^{\binom{n-1}{2}},$$

$$\prod_{1\leqslant p<q<r\leqslant m}\ \prod_{1\leqslant i<j<h\leqslant n} R(F_{pqr}, G_{ijh}) = R(F,G)^{\frac{(m-1)(n-1)(m-2)(n-2)}{4}},$$

$$\prod_{1\leqslant p<q<r\leqslant m} D(F_{pqr}) = D(F)^{m-2},$$

$$\prod_{1\leqslant i<j<h\leqslant n} D(G_{ijh}) = D(G)^{n-2}.$$

Hence, by taking the products, we deduce that

$$|R(F,G)|_T \geqslant e^{-\frac{10mn}{17}(2g_L-1+\#T)}|D(F)|_T^{\frac{n}{17(m-1)}}|D(G)|_T^{\frac{m}{17(n-1)}}\left(|F|_T^{\frac{15n}{17}}|G|_T^{\frac{15m}{17}}\right).$$

As $\#T \leqslant [L:K_1]\#S_1$ and $|x|_T = |x|_{S_1}^{[L:K_1]}$ for $x\in K_1$, we conclude that

$$|R(F,G)|_{S_1} \geqslant e^{-\frac{10mn}{17}(\frac{2g_L-1}{[L:K_1]}+\#S_1)}|D(F)|_{S_1}^{\frac{n}{17(m-1)}}|D(G)|_{S_1}^{\frac{m}{17(n-1)}}\left(|F|_{S_1}^{\frac{15n}{17}}|G|_{S_1}^{\frac{15m}{17}}\right).$$

If F,G are irreducible, write $F = a\prod_{i=1}^{m}(X-\gamma_i Y), G = b\prod_{j=1}^{n}(X-\delta_j Y)$, then all fields $K_1(\gamma_i,\delta_j)$ are isomorphic. Without loss of generality, assume $L' = K_1(\gamma_1,\delta_1)$.

Let

$$F_{pqr}(X,Y) = (X-\gamma_p Y)(X-\gamma_q Y)(X-\gamma_r Y) \text{ for } 1\leqslant p<q<r\leqslant m,$$

$$G_{ijh}(X,Y) = (X-\delta_i Y)(X-\delta_j Y)(X-\delta_h Y) \text{ for } 1\leqslant i<j<h\leqslant n.$$

Let $M = K_1(\gamma_p,\gamma_q,\gamma_r,\delta_i,\delta_j,\delta_h)$ and T be the set of valuations of M above those in S_1. Here we omit the subscript because all such field are isomorphic and all T have the same cardinality. By Proposition 7.2.1,

$$|R(F_{pqr},G_{ijh})|_T \geqslant e^{-\frac{90}{17}(2g_M-1+\#T)}|D(F_{pqr})D(G_{ijh})|_T^{\frac{3}{34}}|F_{pqr}G_{ijh}|_T^{\frac{45}{17}}.$$

Applying Lemma 7.3.2 to $L_1 = K_1(\gamma_p,\delta_p,\gamma_q,\delta_q)$ and its subfields $K_1(\gamma_p,\delta_p)$ and $K_1(\gamma_q,\delta_q)$ we obtain

$$g_{L_1} \leqslant 2dg_{L'} + (d-1)^2,$$

where

$$\begin{aligned} d &= [L_1 : K_1(\gamma_q, \delta_q)] \\ &= [L_1 : K_1(\gamma_p, \delta_p)] \\ &\leqslant [L_1 : K_1(\gamma_p, \delta_p, \gamma_q)][K_1(\gamma_p, \delta_p, \gamma_q) : K_1(\gamma_p, \delta_p)] \\ &\leqslant [K_1(\delta_p, \delta_q) : K_1(\delta_p)][K_1(\gamma_p, \gamma_q) : K_1(\gamma_p)] \\ &\leqslant (m-1)(n-1) \\ &< mn. \end{aligned}$$

Observing that $[M : L_1] \leqslant d$ and $[M : K_1(\gamma_r, \delta_r)] \leqslant d^2$, and applying Lemma 7.3.2 to M and its subfields $L_1, K_1(\gamma_r, \delta_r)$ we obtain

$$g_M \leqslant [M : K_1(\gamma_r, \delta_r)]g_{L'} + [M : L_1]g_{L_1} + (d-1)(d^2-1).$$

Hence

$$\begin{aligned} g_M &\leqslant 3[M : K_1(\gamma_p, \delta_p)]g_{L'} + (2d+1)(d-1)^2 \\ &< \frac{3[M : K_1]}{[L' : K_1]}g_{L'} + 2m^2n^2\frac{[M : K_1]}{[L' : K_1]} \end{aligned}$$

and

$$|R(F_{pqr}, G_{ijh})|_{S_1} \geqslant e^{-\frac{90}{17}(\frac{6g_{L'}+4m^2n^2}{[L':K_1]}+\#S_1)}|D(F_{pqr})D(G_{ijh})|_{S_1}^{\frac{3}{34}}|F_{pqr}G_{ijh}|_{S_1}^{\frac{45}{17}}.$$

By taking the products over all triple (p, q, r) and (i, j, h) we deduce that

$$|R(F, G)|_{S_1} \geqslant e^{-\frac{10mn}{17}(\frac{6g_{L'}+4m^2n^2}{[L':K_1]}+\#S_1)}|D(F)|_{S_1}^{\frac{n}{17(m-1)}}|D(G)|_{S_1}^{\frac{m}{17(n-1)}}\left(|F|_{S_1}^{\frac{15n}{17}}|G|_{S_1}^{\frac{15m}{17}}\right).$$

This completes the proof. □

Corollary 7.3.3. *Let* $F, G \in \mathcal{O}_{S_1}[X, Y]$ *be two binary forms such that* $\deg F = m \geqslant 3, \deg G = n \geqslant 3$, *$FG$ is square-free and has splitting field L over K_1. Then*

$$|R(F, G)|_{S_1} \geqslant e^{-\frac{10mn}{17}(\frac{2g_L-1}{[L:K_1]}+\#S_1)}|D(F)|_{S_1}^{\frac{n}{17(m-1)}}|D(G)|_{S_1}^{\frac{m}{17(n-1)}}.$$

In particular, if F, G are irreducible, let L' be the field generated by one root of $F(X, 1)$ and $G(Y, 1)$ instead, then

$$|R(F, G)|_{S_1} \geqslant e^{-\frac{10mn}{17}(\frac{6g_{L'}+4m^2n^2}{[L':K_1]}+\#S_1)}|D(F)|_{S_1}^{\frac{n}{17(m-1)}}|D(G)|_{S_1}^{\frac{m}{17(n-1)}},$$

Proof. Since $F, G \in \mathcal{O}_{S_1}[X, Y]$, $|F|_{S_1} \geqslant 1, |G|_{S_1} \geqslant 1$. Then apply Theorem 7.3.1. In particular, if F, G are irreducible, then $F(1,0), G(1,0) \in \mathcal{O}_{S_1}$ and the rest is clear. □

Remark 7.3.4. *Theorem 7.3.1 and Corollary 7.3.3 do not hold if* $m = 2$ *or* $n = 2$*. For instance, if* $m = 2, n > 2$*, take* $F = X^2 - (t^2-1)Y^2, G = \prod_{i=1}^{n}(a_iX - b_iY)$ *where* $a_i, b_i\,(i = 1, \ldots, n)$ *satisfy* $a_i^2 - (t^2-1)b_i^2 = 1$*. Say,* $u_j, v_j \in k[t]$ *are the unique solution of* $u_j + v_j\sqrt{t^2-1} = (t+\sqrt{t^2-1})^j\ (j \in \mathbb{N})$ *and* $a_i = u_{l_i}, b_i = v_{l_i}\,(i = 1, \ldots, n)$ *with* $l_1 < \cdots < l_n$*. Then* $R(F, G) = 1, D(F) = 4(t^2-1)$ *and*

$$\begin{aligned} D(G) &= \prod_{1 \leqslant i < j \leqslant n} (a_ib_j - a_jb_i)^2 \\ &= \frac{1}{4(t^2-1)} \prod_{1 \leqslant i < j \leqslant n} \Big((t+\sqrt{t^2-1})^{l_i - l_j} - (t-\sqrt{t^2-1})^{l_i-l_j}\Big)^2 \\ &= \frac{1}{(t^2-1)} \prod_{1 \leqslant i < j \leqslant n} u_{l_i - l_j}^2. \end{aligned}$$

It is easy to check $\deg(u_j) = j$*, hence* $|D(G)|_\infty \to \infty$ *while* $\max_{1 \leqslant i < j \leqslant n}(l_i - l_j) \to \infty$*. This gives a counter-example.*

7.4 A result on Thue-Mahler equations

The idea of the following sections comes from [10]. We work out an analogue for function fields. Let L be a finite extension of $K = k(t)$, and $T \subset M_L$ a finite set of valuations. As in K, for a binary form F with coefficients $a_0, \ldots, a_n$, put

$$H_T(F) = \prod_{\omega \in T} \max(|a_0|_\omega, \ldots, |a_n|_\omega),$$

$$H_L(F) = \prod_{\omega \in L} \max(|a_0|_\omega, \ldots, |a_n|_\omega),$$

$$H(F) = H_L(F)^{1/[L:K]}.$$

Lemma 7.4.1. *Let $F(X,Y) \in L[X,Y]$ be a binary form of degree $m \geqslant 3$ with $D(F) \neq 0$. Let $A \geqslant 1$ and suppose F splits in L. Then every solution $(x,y) \in L^2$ of the Thue-Mahler equation*

$$|F(x,y)|_T = A \tag{7.4.1}$$

satisfies

$$H_T(x,y) \leqslant e^{2g_L - 1 + \#T} \prod_{\nu \notin T} \max(|x|_\nu, |y|_\nu)^2 \left(A \frac{H_L^2(F)}{H_T(F)} \right)^{3/m}.$$

In particular, if $F(X,Y) \in \mathcal{O}_T[X,Y]$ and $(x,y) \in \mathcal{O}_T^2$, then

$$H_T(x,y) \leqslant e^{2g_L - 1 + \#T} \left(A \cdot H_T(F) \right)^{3/m}.$$

Proof. Suppose we have a factorization $F(X,Y) = \prod_{i=1}^{m} (\alpha_i X + \beta_i Y)$ in L.

Put $\Delta_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$ for $i, j = 1, \ldots, m$. Then

$$|\Delta_{ij}|_\omega \leqslant \max(|\alpha_i|_\omega, |\beta_i|_\omega) \max(|\alpha_j|_\omega, |\beta_j|_\omega) \text{ for } \omega \in M_L.$$

Fix an arbitrary triple $\{r,s,t\} \subset \{1, \ldots, m\}$ and $(x,y) \in L^2$, and put

$$A_r = \Delta_{st}(\alpha_r X + \beta_r Y),\ a_r = A_r(x,y)$$

and similarly for A_s, A_t, a_s, a_t. Then observe that

$$\begin{aligned} A_r + A_s + A_t &= 0, \\ a_r + a_s + a_t &= 0. \end{aligned} \tag{7.4.2}$$

Applying Corollary 2.2.11 to (7.4.2) , we obtain

$$H_T(a_r, a_s, a_t) \leqslant e^{\max(2g_L - 2 + \#T, 0)} |a_r a_s a_t|_T \left(\prod_{\omega \notin T} \max(|a_r|_\omega, |a_s|_\omega, |a_t|_\omega) \right)^2, \tag{7.4.3}$$

where

$$|a_r a_s a_t|_T = |\Delta_{rs} \Delta_{st} \Delta_{tr}|_T \prod_{i \in \{r,s,t\}} |\alpha_i x + \beta_i y|_T,$$

$$\prod_{\omega\notin T}\max(|a_r|_\omega,|a_s|_\omega,|a_t|_\omega)\leqslant\prod_{\omega\notin T}\Big(\max(|x|_\omega,|y|_\omega)\prod_{i\in\{r,s,t\}}\max(|\alpha_i|_\omega,|\beta_i|_\omega)\Big).$$

Also,

$$\begin{aligned}\Delta_{rs}\Delta_{st}\Delta_{tr}X&=\Delta_{tr}\beta_sA_r-\Delta_{st}\beta_rA_s,\\ \Delta_{rs}\Delta_{st}\Delta_{tr}Y&=-\Delta_{tr}\alpha_sA_r+\Delta_{st}\alpha_rA_s.\end{aligned}\tag{7.4.4}$$

Then for each solution (x,y) of (7.4.1) and each $\omega\in T$,

$$|\Delta_{rs}\Delta_{st}\Delta_{tr}|_\omega\max(|x|_\omega,|y|_\omega)\leqslant\left(\prod_{i\in\{r,s,t\}}\max(|\alpha_i|_\omega,|\beta_i|_\omega)\right)\max(|a_r|_\omega,|a_s|_\omega).$$

Hence

$$|\Delta_{rs}\Delta_{st}\Delta_{tr}|_TH_T(x,y)\leqslant\Big(\prod_{i\in\{r,s,t\}}H_T(\alpha_i,\beta_i)\Big)H_T(a_r,a_s).\tag{7.4.5}$$

Noticing that $H_T(a_r,a_s)=H_T(a_r,a_s,a_t)$, combining (7.4.5) with (7.4.3), we deduce that

$$\begin{aligned}H_T(x,y)\ &\leqslant\ e^{2g_L-1+\#T}\prod_{\omega\notin T}\max(|x|_\omega,|y|_\omega)^2\\ &\times\prod_{i\in\{r,s,t\}}\Big(|\alpha_ix+\beta_iy|_T\prod_{\omega\in T}\max(|\alpha_i|_\omega,|\beta_i|_\omega)\prod_{\omega\notin T}\max(|\alpha_i|_\omega,|\beta_i|_\omega)^2\Big).\end{aligned}$$

However, by Gauss' lemma

$$\prod_{i=1}^{m}\max(|\alpha_i|_\omega,|\beta_i|_\omega)=|F|_\omega.$$

Then by taking the products over all triples $\{r,s,t\}\subset\{1,\ldots,m\}$ and (7.4.1), we deduce that

$$H_T(x,y)\leqslant e^{2g_L-1+\#T}\prod_{\omega\notin T}\max(|x|_\omega,|y|_\omega)^2\left(A\frac{H_L^2(F)}{H_T(F)}\right)^{3/m},\tag{7.4.6}$$

if $F(X,Y)\in\mathcal{O}_T[X,Y]$ and $(x,y)\in\mathcal{O}_T^2$, then we get $H_T(F)\geqslant1$ and $\max(|x|_\omega,|y|_\omega)\leqslant1$ for $\omega\notin T$, hence

$$H_T(x,y)\leqslant e^{2g_L-1+\#T}\left(A\cdot H_T(F)\right)^{3/m}.$$

□

7.5 Lower bounds for resultants in terms of heights

In this section we estimate the resultants from below in terms of heights. Again let $K = k(t)$, and let S be a finite set of valuations of M_K. Further, let $F, G \in K[X,Y]$ be two binary forms of degree m, n respectively. Recall that for $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\det U \neq 0$, define $F_U(X,Y) = F(aX + bY, cX + dY)$ and the same for G_U. Then

$$R(F_U, G_U) = (\det U)^{mn} R(F,G).$$

By the definition of resultant (1.1.1) and the non-archimedean property of the absolute values $|\cdot|_\nu$ on K, we have

$$|R(F,G)|_\nu \leqslant |F|_\nu^n |G|_\nu^m \text{ for } \nu \in M_K,$$

and hence

$$|R(F,G)|_S \leqslant |F|_S^n |G|_S^m.$$

Theorem 7.5.1. *Let $m, n > 2$ and let F, G be binary forms in $\mathcal{O}_S[X,Y]$ such that FG is square-free and with splitting field L over K. Then there exists $U \in GL_2(\mathcal{O}_S)$ such that*

$$|R(F,G)|_S \geqslant c(m,n,S,L)^{-1} H_S(G_U)^{\frac{m}{717}} H_S(F_U)^{\frac{n}{717}},$$

where

$$c(m,n,S,L) = \exp\left(\frac{422mn(2g_L - 1)}{717[L:K]} + mn(4m + 4n + 433)\frac{\#S}{717}\right).$$

Lemma 7.5.2. *Let $F \in \mathcal{O}_S[X,Y]$ be a binary form of degree m with non-zero discriminant. Then there exists $U \in GL_2(\mathcal{O}_S)$such that*

$$|D(F)|_S \geqslant e^{-\frac{m-1}{21}\left((m^2+6m-7)\#S + \frac{(5m-5)(2g_L-1)}{24[L:K]}\right)} H_S(F_U)^{\frac{m-1}{21}}.$$

Proof. This follows from Theorem 5.3.2, observing that $H_S(F') \leqslant H^*(F')$ for any $F' \in \mathcal{O}_S[X,Y]$. □

Proof of Theorem 7.5.1. Without loss of generality, assume that $|D(F)|_S^{\frac{n}{m-1}} \leqslant |D(G)|_S^{\frac{m}{n-1}}$.

By Lemma 7.5.2, there exists $U \in \mathrm{GL}_2(\mathcal{O}_S)$ such that

$$|D(F)|_S \geqslant e^{-\frac{m-1}{21}\left((m^2+6m-7)\#S+\frac{(5m-5)(2g_L-1)}{24[L:K]}\right)} H_S(F_U)^{\frac{m-1}{21}}.$$

Combining this with Corollary 7.3.3, we deduce that

$$\begin{aligned} |R(F,G)|_S &\geqslant e^{-\frac{10mn}{17}(\frac{2g_L-1}{[L:K]}+\#S)}|D(F)|_S^{\frac{n}{17(m-1)}}|D(G)|_S^{\frac{m}{17(n-1)}} \\ &\geqslant e^{-\frac{10mn}{17}(\frac{2g_L-1}{[L:K]}+\#S)}|D(F)|_S^{\frac{2n}{17(m-1)}} \\ &\geqslant C(L,S)H_S(F_U)^{\frac{2n}{357}}, \end{aligned} \tag{7.5.1}$$

where $C(L,S) = e^{-\frac{10mn}{17}(\frac{2g_L-1}{[L:K]}+\#S)-\frac{2n}{357}\left((m^2+6m-7)\#S+\frac{(5m-5)(2g_L-1)}{24[L:K]}\right)}$.

On the other hand, let $T \subset M_L$ be the set of valuations above those in S. Assume F_U, G_U factor in L as

$$F_U(X,Y) = \prod_{i=1}^{m}(\alpha_i X + \beta_i Y),$$

$$G_U(X,Y) = \prod_{j=1}^{n}(\gamma_j X + \delta_j Y).$$

Then

$$|R(F,G)|_S^{[L:K]} = |R(F_U,G_U)|_S^{[L:K]} = |R(F_U,G_U)|_T = \prod_{j=1}^{n}|F_U(\delta_j,-\gamma_j)|_T.$$

By Lemma 7.4.1, we obtain for $j=1,\ldots,n$ that

$$H_T(\delta_j,-\gamma_j) \leqslant e^{2g_L-1+\#T}\prod_{\omega\notin T}\max(|\delta_j|_\omega,|\gamma_j|_\omega)^2\Big(|F_U(\delta_j,-\gamma_j)|_T\frac{H_L(F_U)^2}{H_T(F_U)}\Big)^{3/m}.$$

Combining this with Gauss' lemma, we deduce that

$$\begin{aligned} H_T(G_U) &= \prod_{j=1}^{n} H_T(\gamma_j, \delta_j) \\ &\leqslant e^{n(2g_L-1+\#T)} \prod_{\omega\notin T}\prod_{j=1}^{n} \max(|\delta_j|_\omega, |\gamma_j|_\omega)^2 \Big(\prod_{j=1}^{n} |F_U(\delta_j, -\gamma_j)|_T \frac{H_L(F_U)^{2n}}{H_T(F_U)^n}\Big)^{3/m} \\ &= e^{n(2g_L-1+\#T)} \prod_{\omega\notin T} |G_U|_\omega^2 \Big(|R(F_U, G_U)|_T \frac{H_L(F_U)^{2n}}{H_T(F_U)^n}\Big)^{3/m}. \end{aligned}$$

Therefore

$$H_S(G_U) \leqslant e^{n(\frac{2g_L-1}{[L:K]}+\#S)} \prod_{\nu\notin S} |G_U|_\nu^2 \Big(|R(F,G)|_S \frac{H_K(F_U)^{2n}}{H_S(F_U)^n}\Big)^{3/m}.$$

Noticing that $F_U, G_U \in \mathcal{O}_S[X,Y]$, we obtain

$$H_S(G_U) \leqslant e^{n(\frac{2g_L-1}{[L:K]}+\#S)} \Big(|R(F,G)|_S H_S(F_U)^n\Big)^{3/m}. \tag{7.5.2}$$

Combining (7.5.1) with (7.5.2) we conclude that

$$\begin{aligned} H_S(G_U)^m H_S(F_U)^n &\leqslant e^{mn(\frac{2g_L-1}{[L:K]}+\#S)} |R(F,G)|_S^3 H_S(F_U)^{4n} \\ &\leqslant c|R(F,G)|_S^{717}, \end{aligned}$$

where

$$\begin{aligned} c &= \exp\Big(\big(421mn + \tfrac{5n(m-1)}{6}\big)\tfrac{2g_L-1}{[L:K]} + \big(421mn + 4n(m^2+6m-7)\big)\#S\Big) \\ &< \exp\Big(\tfrac{422mn(2g_L-1)}{[L:K]} + mn(4m+4n+433)\#S\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} &H_S(G_U)^m H_S(F_U)^n \\ &\quad < \exp\Big(\tfrac{422mn(2g_L-1)}{[L:K]} + mn(4m+4n+433)\#S\Big)|R(F,G)|_S^{717}. \end{aligned}$$

□

Corollary 7.5.3. *If F, G are irreducible over K, let L_1 be an extension of K generated by a root of $F(X,1)$ and L_2 an extension of K by a root of $G(X,1)$, and suppose that L_i has genus g_i for $i = 1,2$. Then*

$$|R(F,G)|_S \geqslant c(m,n,S,L_1,L_2)^{-1} H_S(G_U)^{\frac{m}{717}} H_S(F_U)^{\frac{n}{717}},$$

where

$$c(m,n,S,L_1,L_2) = \exp\left(\frac{422mn(m+n-5+2g_1+2g_2)}{717} + mn(4m+4n+433)\frac{\#S}{717}\right).$$

Proof. When F, G are irreducible, the claim is a combination of Theorem 7.5.1 and (5.1.5). □