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Chapter 5

Height estimates in terms of the discriminant

We are going to prove a generalization of Theorem 1 in the introduction.

Main Theorem. *Let $K = k(t)$ and S a finite set of valuations of K containing ν_∞. Let $F \in \mathcal{O}_S[X, Y]$ be a binary form of degree $n \geqslant 4$ with non-zero discriminant. Then F is* $\mathrm{GL}\,(2, \mathcal{O}_S)$*-equivalent to a binary form F^* such that*

$$H^*(F^*) \leqslant e^{(n-1)\left(\#S(n+11)-5\right)} |D(F)|_S^{20+\frac{1}{n}}.$$

We mainly follow the arguments from [9].

5.1 Consequences of the Riemann-Hurwitz formula

First we deduce some consequences of the Riemann-Hurwitz formula. In this section, let K_1 be a finite extension of $k(t)$, unless otherwise stated. Here k is an algebraically closed field of characteristic 0, this implies that all residue degrees are 1. Let L be a finite extension of K_1. Then by the

Riemann-Hurwitz formula, we have

$$2g_L - 2 = [L : K_1](2g_{K_1} - 2) + \sum_{\nu \in M_{K_1}} \sum_{V|\nu} (e(V|\nu) - 1).$$

We denote by S_1 a finite set of valuations of K_1, and by T the set of valuations of L above S_1. Clearly, we have $\#T \leqslant [L : K_1]\#S_1$. For $\nu \in M_{K_1}$, we put $R_{L/K_1,\nu} = \sum_{V|\nu}(e(V|\nu) - 1)$, where the sum is taken over all valuations V of L lying above ν. Then $0 \leqslant R_{L/K_1,\nu} \leqslant [L : K_1] - 1$.

Lemma 5.1.1. $2g_L - 2 + \#T = [L : K_1](2g_{K_1} - 2 + \#S_1) + \sum_{\nu \notin S_1} R_{L/K_1,\nu}$.

Proof. By the Riemann-Hurwitz formula, we have

$$\begin{aligned} & 2g_L - 2 + \#T \\ = \ & [L : K_1](2g_{K_1} - 2) + \#T + \sum_{\nu \in S_1} R_{L/K_1,\nu} + \sum_{\nu \notin S_1} R_{L/K_1,\nu} \\ = \ & [L : K_1](2g_{K_1} - 2) + \#T + \sum_{\nu \in S_1} \Big(\sum_{V|\nu} e(V|\nu) \Big) - \#T + \sum_{\nu \notin S_1} R_{L/K_1,\nu} \\ = \ & [L : K_1](2g_{K_1} - 2) + \sum_{\nu \in S_1} [L : K_1] + \sum_{\nu \notin S_1} R_{L/K_1,\nu} \\ = \ & [L : K_1](2g_{K_1} - 2 + \#S_1) + \sum_{\nu \notin S_1} R_{L/K_1,\nu}, \end{aligned}$$

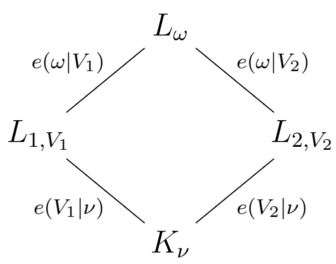
as claimed. □

Consider the compositum L of finite extensions $L_1, \dots, L_r$ of K_1.

Lemma 5.1.2. *Let* $\nu \in M_{K_1}$. *Then* $\frac{R_{L/K_1,\nu}}{[L:K_1]} \leqslant \sum_{i=1}^{r} \frac{R_{L_i/K_1,\nu}}{[L_i:K_1]}$.

Proof. It suffices to prove this in the case $r = 2$. Then the general statement follows easily by induction.

So assume $r = 2$. Let $\omega \in M_L$ with $\omega|\nu$ and let $V_1 \in M_{L_1}, V_2 \in M_{L_2}$ be such that $\omega|V_1, \omega|V_2$. We have the diagram



Since all residue degrees are equal to 1, every ramification index is equal to the extension degree. By general theory of field extensions, we know that $e(\omega|V_2) \leqslant e(V_1|\nu), e(\omega|V_1) \leqslant e(V_2|\nu)$ and $e(\omega|\nu) = e(\omega|V_1)e(V_1|\nu)$. On the other hand, since every ramification index is a positive integer, we deduce that

$$(e(\omega|V_2) - 1)(e(\omega|V_1) - 1) \geqslant 0,$$

hence

$$e(\omega|V_2)e(V_2|\nu) - e(\omega|V_1) - e(\omega|V_2) \geqslant -1,$$

and therefore

$$e(\omega|V_1)(e(V_1|\nu) - 1) + e(\omega|V_2)(e(V_2|\nu) - 1) \geqslant e(\omega|\nu) - 1.$$

By taking the sum we deduce that

$$\begin{aligned} \sum_{\substack{\omega \in M_L \\ \omega|\nu}} \Big(e(\omega|\nu) - 1\Big) \leqslant & \sum_{\substack{V_1 \in M_{L_1} \\ V_1|\nu}} \Big(\sum_{\substack{\omega \in M_L \\ \omega|V_1}} e(\omega|V_1)(e(V_1|\nu) - 1)\Big) \\ & + \sum_{\substack{V_2 \in M_{L_2} \\ V_2|\nu}} \Big(\sum_{\substack{\omega \in M_L \\ \omega|V_2}} e(\omega|V_2)(e(V_2|\nu) - 1)\Big). \end{aligned}$$

Noticing that $\sum\limits_{\omega|V_i} e(\omega|V_i) = [L : L_i]$ for $i = 1, 2$, this leads to

$$R_{L/K_1,\nu} \leqslant [L : L_1]R_{L_1/K_1,\nu} + [L : L_2]R_{L_2/K_1,\nu}.$$

which implies the desired result. □

We deduce some other genus estimates that will be needed later.

Recall $K = k(t)$. Assume that $F(1, 0) \neq 0$. Then we may assume that $F = a \prod\limits_{j=1}^{n} (X - \alpha_i Y), a \in K^*$ and that for every $\sigma \in \mathrm{Gal}(\overline{K}/K)$, there is a permutation of $(1, \ldots, n)$, also denoted by σ, such that for $j = 1, \ldots, n$ we have $\sigma(\alpha_j) = \alpha_{\sigma(j)}$. Suppose we have a factorization $F = \prod\limits_{i=1}^{d} F_i$ where $F_i \in \mathcal{O}_S[X, Y]$ is a primitive irreducible binary form of degree n_i. Let $\alpha_{i,j}, j = 1, \ldots, n_i$ be the zeros of $F_i(X, 1)$ among $\alpha_1, \ldots, \alpha_n$. Then all terms

$R_{K(\alpha_{i,j})/K,\nu}, 1 \leqslant j \leqslant n_i$ are equal. We put $L_{ij} = K(\alpha_{i,j})$ for $i = 1, \dots, d, j = 1, \dots, n_i$, and $I = \{1, \dots, n\}$. Then

$$\sum_{i=1}^{d} \sum_{j=1}^{n_i} \frac{R_{L_{ij}/K,\nu}}{[L_{ij} : K]} = \sum_{i=1}^{d} R_{L_{i1}/K,\nu}. \tag{5.1.1}$$

For such a field L_{ij} and a valuation ω of $\mathcal{O}_{L_{ij}}$, lying above the valuation ν of $k[t]$, we have by Lemma 1.2.3

$$\nu(D_{\mathcal{O}_{L_{ij}}/k[t]}) = \sum_{\omega|\nu} (e(\omega|\nu) - 1).$$

Further, by Lemma 4.1.1, we have $R_{L_{ij}/K,\nu} = \nu(D_{\mathcal{O}_{L_{ij}}/k[t]}) \leqslant \nu(D(F_i))$ for $\nu \notin S$, hence

$$\sum_{\nu \notin S} R_{L_{ij}/K,\nu} \leqslant \sum_{\nu \notin S} \nu(D(F_i)) = -\sum_{\nu \in S} \nu(D(F_i)). \tag{5.1.2}$$

For any set of indices $J = \{i_1, \dots, i_m\} \subset \{1, \dots, n\}$ we put $L_J = K(\alpha_{i_1}, \dots, \alpha_{i_m})$, and let T_J be the set of valuations of L_J above S. For each $i \in \{1, \dots, n\}$ we choose $\bar{i} \in \{1, \dots, d\}$ such that α_i is a root of $F_{\bar{i}}$. Recall that $F = F_1 \cdot \dots \cdot F_d$, where F_i is an irreducible factor of F in $\mathcal{O}_S[X, Y]$ with $F_i(\alpha_{\bar{i}}, 1) = 0$. Then by Lemma 5.1.1 with $g_K = 0$ and Lemma 5.1.2, we have

$$\begin{aligned}
\frac{2g_{L_J} - 2 + \#T_J}{[L_J : K]} &= (-2 + \#S) + \sum_{\nu \notin S} \frac{R_{L_J/K,\nu}}{[L_J : K]} \\
&\leqslant -2 + \#S + \sum_{\nu \notin S} \sum_{i \in J} \frac{R_{K(\alpha_i)/K,\nu}}{[K(\alpha_i) : K]} \\
&= -2 + \#S + \sum_{i \in J} \frac{1}{[K(\alpha_i) : K]} \sum_{\nu \notin S} R_{K(\alpha_i)/K,\nu} \\
&\leqslant -2 + \#S - \sum_{i \in J} \sum_{\nu \in S} \frac{\nu(D(F_{\bar{i}}))}{\deg F_{\bar{i}}}. \qquad (5.1.3)
\end{aligned}$$

Applying this to $J = I$ and combining this with (5.1.1) and (5.1.2), we obtain for the splitting field L of F over K,

$$\begin{aligned}\frac{2g_L - 2 + \#T}{[L:K]} &\leqslant (-2+\#S) - \sum_{\nu\in S}\sum_{i=1}^{n}\frac{\nu(D(F_{\bar{i}}))}{\deg F_{\bar{i}}} \\ &= (-2+\#S) - \sum_{\nu\in S}\nu(\prod_{i=1}^{d} D(F_i)) \\ &\leqslant (-2+\#S) - \sum_{\nu\in S}\nu(D(F)),\end{aligned}$$

where the last step follows from (4.1.3).

On the other hand, by Lemma 5.1.2, we have

$$\begin{aligned}\frac{2g_L - 2}{[L:K]} &= -2 + \sum_{\nu\in M_K}\frac{R_{L/K,\nu}}{[L:K]} \\ &\leqslant -2 + \sum_{\nu\in M_K}\sum_{i=1}^{d}\sum_{j=1}^{n_i}\frac{R_{L_{ij}/K,\nu}}{[L_{ij}:K]} \\ &= -2 + \sum_{\nu\in M_K}\sum_{i=1}^{d} R_{L_{i1}/K,\nu}.\end{aligned}$$

Let g_i be the genus of $L_{i1}, i = 1, \ldots, d$. Then

$$2g_i - 2 = -2[L_{i1}:K] + \sum_{\nu\in M_K} R_{L_{i1}/K,\nu},\ i = 1, \ldots, d,$$

so

$$\frac{2g_L - 2}{[L:K]} \leqslant -2 + \sum_{i=1}^{d}(2g_i - 2 + 2n_i), \tag{5.1.4}$$

or,

$$\frac{2g_L - 2}{[L:K]} \leqslant 2\deg F - 2 + \sum_{i=1}^{d}(2g_i - 2). \tag{5.1.5}$$

5.2 A few lemmas

In the proof of Theorem 5 we follow the idea of [9]. We are going to construct a special admissible tuple $\mathbb{A}$ as in definition 4.3.1 with some good properties.

Let $F \in \mathcal{O}_S[X,Y]$ be a binary form of degree $n \geqslant 4$. Assume we have a factorization into linear forms $F = al_1 \dots l_n$ with $l_i = X - \alpha_i Y$. Denote by Δ_{ij} the determinant $\det(l_i, l_j)$. Then

$$\Delta_{ij}\Delta_{hl} + \Delta_{jh}\Delta_{il} + \Delta_{hi}\Delta_{jl} = 0. \tag{5.2.1}$$

We will use this identity and apply Lemma 2.1.1 (Mason's Theorem) to it. Let L' be the splitting field L of F or the field $L_{ijhl} = K(\alpha_i, \alpha_j, \alpha_h, \alpha_l)$, and T' the set of valuations of L' lying above those in S. The case when $L' = L_{ijhl}$ is prepared for Theorem 1, whilst the case $L' = L$ is a variation on Theorem 1, which will be needed as well.

We introduce some auxiliary quantities that will be used in the proof of Theorem 1. For $i = 1, \dots, n$, let

$$\xi_{i\omega} = \max(|\alpha_i|_\omega, 1) \text{ for } \omega \notin T,$$

$$\xi_{i\omega} = (\prod_{\omega' \notin T} \xi_{i\omega'})^{-1/\#T} \text{ for } \omega \in T.$$

Then $\prod_{\omega \in M_L} \xi_{i\omega} = 1$ for $i = 1, \dots, n$. We also have for $\omega \notin T$,

$$\begin{aligned} \xi_{\sigma(i),\omega} &= \max(|\alpha_{\sigma(i)}|_\omega, 1) \\ &= \max(|\alpha_i|_{\omega_\sigma}, 1) \\ &= \xi_{i,\omega_\sigma}. \end{aligned}$$

Hence $\xi_{\sigma(i)\omega} = \xi_{i\omega_\sigma}$ for each $\omega \in T$ as well.

Next we put

$$\theta_{ij\omega} = \frac{|\det(l_i, l_j)|_\omega}{\xi_{i\omega}\xi_{j\omega}}, i \neq j.$$

We have $\theta_{ij\omega} \leqslant 1$ for $\omega \notin T$, $\theta_{\sigma(i),\sigma(j),\omega} = \theta_{ij\omega\circ\sigma}$ for $i, j \in \{1, \dots, n\}$ with $i \neq j$ and $\sigma \in \mathrm{Gal}(L/K)$, and

$$\prod_{\omega \in M_L} \theta_{ij\omega} = 1.$$

Further, let $U' \subset M_{L'} \backslash T'$ be the set of valuations ω such that $|\Delta_{ij}\Delta_{hl}|_\omega$, $|\Delta_{jh}\Delta_{il}|_\omega$, $|\Delta_{hi}\Delta_{jl}|_\omega$ are not all equal. Then clearly $\#U' < \infty$.

Note that for $\gamma_1, \gamma_2 \in L^*$, we have by the product formula

$$\begin{aligned} H^*(\frac{\gamma_1}{\gamma_2}) &= \Big(\prod_{\omega \in M_L} \max(1, |\frac{\gamma_1}{\gamma_2}|_\omega)\Big)^{1/[L:K]} \\ &= \Big(\prod_{\omega \in M_L} \max(|\gamma_1|_\omega, |\gamma_2|_\omega)\Big)^{1/[L:K]} = H(\gamma_1, \gamma_2). \end{aligned}$$

Hence by (5.2.1) and Lemma 2.1.1, we have

$$\begin{aligned} & H(\Delta_{ij}\Delta_{hl}, \Delta_{jh}\Delta_{il}, \Delta_{hi}\Delta_{jl}) = H(\Delta_{ij}\Delta_{hl}, \Delta_{jh}\Delta_{il}) \\ &\leqslant e^{\max(2g_{L'}-2+\#T'+\#U',0)/[L':K]} \\ &\leqslant e^{\max(2g_{L'}-2+\#T',0)/[L':K]} e^{\#U'/[L':K]}. \end{aligned} \tag{5.2.2}$$

Let $U \subset M_L$ be the set of valuations outside T such that $|\Delta_{ij}\Delta_{hl}|_\omega$, $|\Delta_{jh}\Delta_{il}|_\omega$, $|\Delta_{hi}\Delta_{jl}|_\omega$ are not all equal. Then $\#U \leqslant [L:L']\#U'$.

Put $\theta'_{ij\omega'} = \frac{|\Delta_{ij}|_{\omega'}}{\xi_{i\omega'}\xi_{j\omega'}}$ with $\xi_{i\omega'} = \max(|\alpha_i|_{\omega'}, 1)$ for $\omega' \notin T'$. Then $\theta'_{ij\omega'} \leqslant 1$ and

$$U' \subset \{\omega' \notin T' : \min(\theta'_{ij\omega'}\theta'_{hl\omega'}, \theta'_{jh\omega'}\theta'_{il\omega'}, \theta'_{hi\omega'}\theta'_{jl\omega'}) < 1\}.$$

For $\omega' \in U'$, the minimum is in fact at most e^{-1}, so we have

$$\begin{aligned} e^{\#U'} &\leqslant \prod_{\omega' \in U'} \frac{1}{\min(\theta'_{ij\omega'}\theta'_{hl\omega'}, \theta'_{jh\omega'}\theta'_{il\omega'}, \theta'_{hi\omega'}\theta'_{jl\omega'})} \\ &\leqslant \prod_{\omega' \notin T'} \frac{1}{\min(\theta'_{ij\omega'}\theta'_{hl\omega'}, \theta'_{jh\omega'}\theta'_{il\omega'}, \theta'_{hi\omega'}\theta'_{jl\omega'})}, \end{aligned}$$

hence

$$\begin{aligned} e^{\#U'/[L':K]} &\leqslant \prod_{\omega' \notin T'} \left(\frac{\prod_{t \in \{i,j,h,l\}} \max(|\alpha_t|_{\omega'}, 1)}{\min(|\Delta_{ij}\Delta_{hl}|_{\omega'}, |\Delta_{jh}\Delta_{il}|_{\omega'}, |\Delta_{hi}\Delta_{jl}|_{\omega'})} \right)^{1/[L':K]} \\ &= \prod_{\omega \notin T} \left(\frac{\prod_{t \in \{i,j,h,l\}} \max(|\alpha_t|_{\omega}, 1)}{\min(|\Delta_{ij}\Delta_{hl}|_{\omega}, |\Delta_{jh}\Delta_{il}|_{\omega}, |\Delta_{hi}\Delta_{jl}|_{\omega})} \right)^{1/[L:K]} \\ &= \prod_{\omega \notin T} \left(\frac{1}{\min(\theta_{ij\omega}\theta_{hl\omega}, \theta_{jh\omega}\theta_{il\omega}, \theta_{hi\omega}\theta_{jl\omega})} \right)^{1/[L:K]}, \end{aligned} \tag{5.2.3}$$

where the first equality comes from the fact $\prod_{\omega|\omega'} |x|_\omega = |x|_{\omega'}^{[L:L']}, x \in L'$.

Lemma 5.2.1. *We have*

(i) $\theta_{il\omega}\theta_{jh\omega} \leqslant \max(\theta_{ij\omega}\theta_{hl\omega}, \theta_{ih\omega}\theta_{jl\omega})$ *for all* $\omega \in T$*;*

(ii) $\prod_{\omega\in T} \max(\theta_{ij\omega}\theta_{hl\omega}, \theta_{ih\omega}\theta_{jl\omega}, \theta_{il\omega}\theta_{jh\omega})$
$\leqslant e^{\max(2g_{L'}-2+\#T',0)[L:L']} \cdot \Big(\prod_{\omega\in T} \theta_{ij\omega}\theta_{hl\omega}\theta_{ih\omega}\theta_{jl\omega}\theta_{il\omega}\theta_{jh\omega}\Big)$*;*

(iii) $(\prod_{\omega\in T}\prod_{1\leqslant i<j\leqslant n} \theta_{ij\omega})^{1/[L:K]} = |D(F)|_S^{1/2}$ *if* F *is primitive.*

Proof. (i) It is easy to see that (i) is a direct consequence of (5.2.1).

(ii) We have

$$
\begin{aligned}
&\left(\prod_{\omega\in T}\max(\theta_{ij\omega}\theta_{hl\omega}, \theta_{ih\omega}\theta_{jl\omega}, \theta_{il\omega}\theta_{jh\omega})\right)^{1/[L:K]}\\
= &\left(\frac{\prod_{\omega\in M_L}\max(\theta_{ij\omega}\theta_{hl\omega},\theta_{ih\omega}\theta_{jl\omega},\theta_{il\omega}\theta_{jh\omega})}{\prod_{\omega\notin T}\max(\theta_{ij\omega}\theta_{hl\omega},\theta_{ih\omega}\theta_{jl\omega},\theta_{il\omega}\theta_{jh\omega})}\right)^{1/[L:K]}\\
= &\; H(\Delta_{ij}\Delta_{hl}, \Delta_{jh}\Delta_{il}, \Delta_{hi}\Delta_{jl})\Big(\prod_{\omega\notin T}\frac{1}{\max(\theta_{ij\omega}\theta_{hl\omega},\theta_{ih\omega}\theta_{jl\omega},\theta_{il\omega}\theta_{jh\omega})}\Big)^{1/[L:K]}\\
\overset{(1)}{\leqslant} &\; e^{\max(2g_{L'}-2+\#T',0)/[L':K]} \times\\
\times &\prod_{\omega\notin T}\Big(\frac{1}{\min(\theta_{ij\omega}\theta_{hl\omega},\theta_{jh\omega}\theta_{il\omega},\theta_{ih\omega}\theta_{jl\omega})}\frac{1}{\max(\theta_{ij\omega}\theta_{hl\omega},\theta_{ih\omega}\theta_{jl\omega},\theta_{il\omega}\theta_{jh\omega})}\Big)^{1/[L:K]}\\
\overset{(2)}{\leqslant} &\; e^{\max(2g_{L'}-2+\#T',0)/[L':K]}\left(\prod_{\omega\in T}\theta_{ij\omega}\theta_{hl\omega}\theta_{ih\omega}\theta_{jl\omega}\theta_{il\omega}\theta_{jh\omega}\right)^{1/[L:K]},
\end{aligned}
$$

where (1) follows from (5.2.2) and (5.2.3), and (2) is deduced from the product formula and the simple fact that if $a, b, c \leqslant 1$, then
$abc \leqslant \max(a, b, c)\min(a, b, c)$. This gives (ii).

(iii) If F is primitive, we have $|a|_\omega \prod_{i=1}^{n} \xi_{i\omega} = 1$ for $\omega \notin T$. So

$$\prod_{\omega\notin T}(|a|_\omega \prod_{i=1}^{n}\xi_{i\omega}) = 1,$$

which implies that

$$\prod_{\omega\in T}(|a|_\omega \prod_{i=1}^{n}\xi_{i\omega}) = 1. \tag{5.2.4}$$

Notice that

$$D(F) = a^{2n-2}\prod_{1\leqslant i<j\leqslant n}\det(l_i,l_j)^2,$$

$$|D(F)|_T = |D(F)|_S^{[L:K]}$$

and

$$\prod_{\omega\in T}\prod_{1\leqslant i<j\leqslant n}\xi_{i\omega}\xi_{j\omega} = \prod_{\omega\in T}\prod_{i=1}^{n}(\xi_{i\omega})^{n-1}.$$

Hence

$$\begin{aligned}\prod_{\omega\in T}\prod_{1\leqslant i<j\leqslant n}\theta_{ij\omega} &= \prod_{\omega\in T}\prod_{1\leqslant i<j\leqslant n}\frac{|\det(l_i,l_j)|_\omega}{\xi_{i\omega}\xi_{j\omega}}\\ &= \frac{\prod_{\omega\in T}|D(F)/a^{2n-2}|_\omega^{\frac{1}{2}}}{\prod_{\omega\in T}\prod_{i=1}^{n}\xi_{i\omega}^{n-1}}\\ &= |D(F)|_T^{\frac{1}{2}}.\end{aligned}$$

This completes the proof. □

Let

$$M = \prod_{\omega}\prod_{i=1}^{n}A_{i\omega},$$

$$R = \prod_{\omega\in T}\max_{1\leqslant i<j\leqslant n}\frac{|\det(l_i,l_j)|_\omega}{A_{i\omega}A_{j\omega}},$$

$$A'_{i\omega} = \frac{A_{i\omega}}{\xi_{i\omega}},$$

$$M' = \prod_{\omega\in T}\prod_{i=1}^{n}A'_{i\omega}.$$

Let $F = al_1\cdots l_n, n \geqslant 3$, where $l_i = X - \alpha_i Y$. By Theorem 4.3.3, F is equivalent to a binary form F^* such that

$$H^*(F^*) \leqslant (e^{n(n+1)\#S}|a|_S^2 R^{n/[L:K]}M^{2/[L:K]})^{(n-1)/(n-2)}. \tag{5.2.5}$$

We now state our important proposition.

Proposition 5.2.2. *Suppose F is primitive. Then there is an admissible tuple $\mathbb{A}$ such that*

(i) $\prod_{\omega\in T}\max_{1\leqslant i<j\leqslant n}\frac{|\det(l_i,l_j)|_\omega}{A_{i\omega}A_{j\omega}}\leqslant 1$;

(ii) $(\prod_{i=1}^{n}\prod_{\omega\in T}A_{i\omega})^{\frac{2(n-1)}{n-2}}\leqslant |a|_T^{-\frac{2(n-1)}{n-2}}e^{[L:K](\#S-1)5(n-1)}|D(F)|_T^{20+\frac{1}{n}}$.

Before prove this proposition, we sketch the rough idea behind it.

Without loss of generality, let us assume for the moment that we have to deal with only one absolute value, simply denoted by $|\cdot|$. We are aiming at minimizing $M^2R^n=(A_1\cdots A_n)^2\big(\max_{1\leqslant i<j\leqslant n}\frac{|\det(l_i,l_j)|}{A_iA_j}\big)^n$. By replacing A_i by λA_i for $i=1,\ldots,n$, we may assume $\max_{1\leqslant i<j\leqslant n}\frac{|\det(l_i,l_j)|}{A_iA_j}=1$. So we aim at minimizing $A_1\cdots A_n$ subject to $\max_{1\leqslant i<j\leqslant n}\frac{|\det(l_i,l_j)|}{A_iA_j}=1$.

By taking logarithms this translates into a linear programming problem. Let $x_i=\log A_i,\delta_{ij}=\log|\Delta_{ij}|$. We want to minimize $x_1+\cdots+x_n$ subject to $\max_{1\leqslant i<j\leqslant n}(\delta_{ij}-x_i-x_j)\leqslant 0$, which is to say, $x_i+x_j\geqslant\delta_{ij}$ for all i,j with $1\leqslant i<j\leqslant n$.

We also have some conditions

$$\delta_{pq}+\delta_{ij}\leqslant\max(\delta_{pi}+\delta_{qj},\delta_{pj}+\delta_{qi})$$

for all distinct i,j,p,q by (5.2.1), and by Mason's result,

$$\max(\delta_{ij}+\delta_{pq},\delta_{iq}+\delta_{jp})\leqslant\delta_{ij}+\delta_{pq}+\delta_{iq}+\delta_{jp}+\delta_{ip}+\delta_{jq}+C$$

for all distinct i,j,p,q.

We want to estimate $x_1+\cdots+x_n$ in terms of $\sum_{1\leqslant i<j\leqslant n}\delta_{ij}$ because $D(F)=\prod_{1\leqslant i<j\leqslant n}|\Delta_{ij}|^2=\exp\big(2\sum_{1\leqslant i<j\leqslant n}\delta_{ij}\big)$.

Our idea is as follows. Fix p,q and let $x_p^{(pq)}=\frac{1}{2}\delta_{pq}+z^{(pq)},x_q^{(pq)}=\frac{1}{2}\delta_{pq}-z^{(pq)}$, where $z^{(pq)}$ will be determined later. Then $x_p^{(pq)}+x_q^{(pq)}=\delta_{pq}$.

We need $x_i^{(pq)} + x_p^{(pq)} \geqslant \delta_{ip}, x_i^{(pq)} + x_q^{(pq)} \geqslant \delta_{iq}$

So we take $x_i^{(pq)} = \max(\delta_{ip} - x_p^{(pq)}, \delta_{iq} - x_q^{(pq)}) = \max(\delta_{ip} - \frac{1}{2}\delta_{pq} - z^{(pq)}, \delta_{iq} - \frac{1}{2}\delta_{pq} + z^{(pq)})$ for $i = 1, \ldots, n, i \neq p, q$.

Thus $x_i^{(pq)} + x_j^{(pq)} \geqslant \delta_{ip} - \frac{1}{2}\delta_{pq} - z^{(pq)} + \delta_{jq} - \frac{1}{2}\delta_{pq} + z^{(pq)} = \delta_{ip} + \delta_{jq} - \delta_{pq} \geqslant \delta_{ij}$.

Now appropriate choices of $z^{(pq)}$ $(1 \leqslant p < q \leqslant n)$ and $x_i = \frac{2}{n(n-1)} \sum_{1 \leqslant p < q \leqslant n} x_i^{(pq)}$ $(i = 1, \ldots, n)$ will give a nearby solution.

Lemma 5.2.3. *If F is primitive, then $M'^{1/[L:K]} = |a|_S M^{1/[L:K]}$.*

Proof. We deduce that

$$\Big(\frac{M'}{M}\Big)^{1/[L:K]} = \Big(\prod_{i=1}^{n} \prod_{\omega} \xi_{i\omega}\Big)^{-1/[L:K]} = \Big(\prod_{i=1}^{n} \prod_{\omega \notin T} \xi_{i\omega}\Big)^{1/[L:K]}.$$

By Gauss' Lemma, we have

$$1 = |F|_\omega = |a|_\omega \prod_{i=1}^{n} \max(|\alpha_i|_\omega, |\beta|_\omega) = |a|_\omega \prod_{i=1}^{n} \xi_{i\omega}$$

for $\omega \notin T$, hence

$$\Big(\frac{M'}{M}\Big)^{1/[L:K]} = \Big(\prod_{\omega \notin T} \frac{1}{|a|_\omega}\Big)^{1/[L:K]} = \Big(\prod_{\omega \in T} |a|_\omega\Big)^{1/[L:K]} = |a|_S.$$

□

Hence we have $H^*(F^*) \leqslant (e^{n(n+1)\#S}(M'^2 R^n)^{1/[L:K]})^{(n-1)/(n-2)}$.

We can rewrite R as $R = \prod_{\omega \in T} \max_{1 \leqslant i < j \leqslant n} \frac{\theta_{ij\omega}}{A'_{i\omega} A'_{j\omega}}$. Now it is clear that Proposition 5.2.2 is equivalent to the following:

Proposition 5.2.4. *Suppose F is primitive. Then there is an admissible tuple $\mathbb{A}' = (A'_{i\omega} : \omega \in T, i = 1, \ldots, n)$ such that*

(i) $\prod_{\omega \in T} \max_{1 \leqslant i < j \leqslant n} \frac{\theta_{ij\omega}}{A'_{i\omega} A'_{j\omega}} \leqslant 1$;

(ii) $\Big(\prod_{i=1}^{n} \prod_{\omega \in T} A'_{i\omega}\Big)^{\frac{2(n-1)}{n-2}} \leqslant e^{[L:K](\#S-1)5(n-1)} |D(F)|_T^{20+\frac{1}{n}}$.

We prove (i) first. For the proof of (ii), we need some further preparations.

Proof of (i). Fix two distinct elements $p, q \in \{1, \ldots, n\}$ and $\omega \in T$.

Define

$$\Phi_{pq\omega}(x) = \prod_{h \neq p,q} \max(\theta_{ph\omega} e^{-x}, \theta_{qh\omega} e^{x}).$$

This function is continuous over the reals $\mathbb{R}$ and goes to infinity as x tends to $\pm\infty$. Hence, $\Phi_{pq\omega}$ assumes a minimum; let $x_{pq\omega}$ be the smallest real number at which $\Phi_{pq\omega}(x)$ assumes this minimum.

Put

$$A'^{(pq)}_{p\omega} = \theta^{\frac{1}{2}}_{pq\omega} e^{x_{pq\omega}},$$
$$A'^{(pq)}_{q\omega} = \theta^{\frac{1}{2}}_{pq\omega} e^{-x_{pq\omega}},$$
$$A'^{(pq)}_{h\omega} = \theta^{-\frac{1}{2}}_{pq\omega} \max(\theta_{ph\omega} e^{-x_{pq\omega}}, \theta_{qh\omega} e^{x_{pq\omega}}) \text{ for } h \neq p, q.$$

Next, put $A'_{i\omega} = \{\prod_{p \neq q} A'^{(pq)}_{i\omega}\}^{\frac{1}{n(n-1)}}$ for $i = 1, \ldots, n$, where the product is taken over all distinct pairs (p, q) with $1 \leqslant p, q \leqslant n$ and $p \neq q$. Finally, put $\mathbb{A}' = (A'_{i\omega} : i = 1, \ldots, n, \omega \in T)$.

We claim that this $\mathbb{A}'$ is admissible.

For every $\sigma \in \mathrm{Gal}(L/K), \omega \in T$ and $p, q \in \{1, \ldots, n\}, p \neq q$, we have

$$0 < \theta_{\sigma(i),\sigma(j),\omega} = \frac{|\Delta_{\sigma(i),\sigma(j)}|_\omega}{\xi_{\sigma(i)\omega}\xi_{\sigma(j)\omega}} = \frac{|\sigma(\Delta_{ij})|_\omega}{\xi_{\sigma(i)\omega}\xi_{\sigma(j)\omega}} = \frac{\Delta_{ij\omega_\sigma}}{\xi_{i\omega_\sigma}\xi_{j\omega_\sigma}} = \theta_{ij\omega_\sigma}.$$

From this, we deduce that

$$\begin{aligned}
\Phi_{\sigma(p),\sigma(q),\omega}(x) &= \prod_{h \neq p,q} \max(\theta_{\sigma(p),\sigma(h),\omega} e^{-x}, \theta_{\sigma(q),\sigma(h),\omega} e^{x}) \\
&= \prod_{h \neq p,q} \max(\theta_{ph\omega_\sigma} e^{-x}, \theta_{qh\omega_\sigma} e^{x}) \\
&= \Phi_{pq\omega_\sigma}(x).
\end{aligned}$$

Therefore, $x_{\sigma(p),\sigma(q),\omega} = x_{pq\omega_\sigma}$.

So we get $A'^{(\sigma(p)\sigma(q))}_{\sigma(h)\omega} = A'^{(pq)}_{h\omega_\sigma}$ for $h = 1, \ldots, n, p, q \in \{1, \ldots, n\}, p \neq q$ and hence $A'_{\sigma(i)\omega} = A'_{i\omega_\sigma}$. This shows that $\mathbb{A}'$ is admissible. Notice that

$$\theta_{pq\omega} = A'^{(pq)}_{p\omega} A'^{(pq)}_{q\omega},$$
$$\theta_{ph\omega} \leqslant A'^{(pq)}_{p\omega} A'^{(pq)}_{h\omega}, \theta_{qh\omega} \leqslant A'^{(pq)}_{q\omega} A'^{(pq)}_{h\omega} \text{ for } h \neq p, q.$$

Further, by $\Delta_{pq}\Delta_{ij} = \Delta_{pi}\Delta_{qj} - \Delta_{pj}\Delta_{qi}$, we have

$$\theta_{pq\omega}\theta_{ij\omega} \leqslant A'^{(pq)}_{p\omega} A'^{(pq)}_{q\omega} A'^{(pq)}_{i\omega} A'^{(pq)}_{j\omega},$$

hence

$$\theta_{ij\omega} \leqslant A'^{(pq)}_{i\omega} A'^{(pq)}_{j\omega}.$$

By taking the geometric means over all pairs p, q we get $\theta_{ij\omega} \leqslant A'_{i\omega} A'_{j\omega}$ for $\omega \in T$. This proves (i). □

We proceed to prove (ii). This will be much more involved, and requires some extra results.

For $p, q \in \{1, \ldots, n\}, p \neq q, \omega \in T$, set

$$\phi_{pq\omega} = \Phi_{pq\omega}(x_{pq\omega}),\ \phi_{pq} = \prod_{\omega \in T} \phi_{pq\omega}.$$

We have

$$\begin{aligned} \prod_{h=1}^{n} A'^{(pq)}_{h\omega} &= \theta_{pq\omega} \prod_{h \neq p,q} \theta_{pq\omega}^{-\frac{1}{2}} \max(\theta_{ph\omega} e^{-x_{pq\omega}}, \theta_{qh\omega} e^{x_{pq\omega}}) \\ &= \theta_{pq\omega}^{-\frac{n}{2}+2} \phi_{pq\omega} \\ &= \theta_{pq\omega}^{-\frac{n}{2}+2} \phi_{pq\omega}, \end{aligned}$$

and

$$\prod_{p \neq q} \prod_{\omega \in T} \theta_{pq\omega} = |D(F)|_T,$$

since F is primitive by Lemma 5.2.1 (iii).

Hence

$$\begin{aligned} M' &= \prod_{i=1}^{n}\prod_{\omega\in T} A'_{i\omega} \\ &= \left(\prod_{p\neq q}\prod_{\omega\in T}\prod_{h=1}^{n} A'^{(pq)}_{h\omega}\right)^{\frac{1}{n(n-1)}} \\ &= \left(\prod_{p\neq q}\prod_{\omega\in T}\theta_{pq\omega}^{-\frac{n}{2}+2}\phi_{pq\omega}\right)^{\frac{1}{n(n-1)}} \\ &= \left(|D(F)|_T^{-\frac{n}{2}+2}\prod_{p\neq q}\phi_{pq}\right)^{\frac{1}{n(n-1)}}. \end{aligned} \tag{5.2.6}$$

We estimate $\prod_{p\neq q}\phi_{pq}$. To this end, we need the following notation and a lemma.

For a fixed pair $\{p,q\}$ with $p,q\in\{1,\dots,n\}, p\neq q$, put $W_{pq} = \{1,\dots,n\}\backslash\{p,q\}$. For $J\subset W_{pq}, \omega\in T$, define the quantities $M_\omega(J)$ as follows.

If $\#J=0, J=\emptyset$, put $M_\omega(J)=1$;

If $\#J=1, J=\{j\}$, put $M_\omega(J)=\sqrt{\theta_{pj\omega}\theta_{qj\omega}}$;

If $\#J\geqslant 2$, put

$$M_\omega(J) = \max\Big\{\prod_{h\in I}\theta_{ph\omega}\prod_{h\in J\backslash I}\theta_{qh\omega} : I\subset J, \#I=\frac{1}{2}\#J\Big\} \text{ if } \#J \text{ is even,}$$

$$M_\omega(J) = \sqrt{M_{1\omega}(J)M_{2\omega}(J)} \text{ if } \#J \text{ is odd,}$$

where

$$M_{1\omega(J)} = \max\Big\{\prod_{h\in I}\theta_{ph\omega}\prod_{h\in J\backslash I}\theta_{qh\omega} : I\subset J, \#I=\tfrac{1}{2}(\#J+1)\Big\},$$

$$M_{2\omega(J)} = \max\Big\{\prod_{h\in I}\theta_{ph\omega}\prod_{h\in J\backslash I}\theta_{qh\omega} : I\subset J, \#I=\tfrac{1}{2}(\#J-1)\Big\}.$$

Finally, put $M(J)=\prod_{\omega\in T}M_\omega(J)$.

Lemma 5.2.5. $\phi_{pq}=M(W_{pq})$.

Proof. This is taken from [9], which deals with number fields. But over function fields, the argument is the same. We repeat it again here.

It suffices to prove $\phi_{pq\omega} = M_\omega(W_{pq})$ for every $\omega \in T$.

Take $f(x) = \log \Phi_{pq\omega}(x) = \sum_{h \in W_{pq}} \max(f_{ph} - x, f_{qh} + x)$ where $f_{ph} = \log \theta_{ph\omega}, f_{qh} = \log \theta_{qh\omega}$.

We can express $f(x)$ as

$$f(x) = \max\{C_0 - (n-2)x, C_1 - (n-2)x, \cdots, C_{n-3} + (n-4)x, C_{n-2} + (n-2)x\},$$

where

$$\begin{aligned} C_s &= \max\Big\{\sum_{h\in I} f_{ph} + \sum_{h \in W_{pq}\setminus I} f_{qh} : I \subset W_{pq}, \#I = n-2-s\Big\} \\ &= \log\max\Big\{\prod_{h \in I}\theta_{ph\omega} \prod_{h \in W_{pq}\setminus I} \theta_{qh\omega} : I \subset W_{pq}, \#I = n-2-s\Big\}, \end{aligned}$$

for $s = 0, \ldots, n-2$.

Let

$$I_s = \{x \in \mathbb{R} : f(x) = C_s - (n-2-2s)x\}\ (s = 0, \ldots, n-2).$$

We first show that I_s is nonempty.

Clearly, $I_0 = \{x \in \mathbb{R} : f(x) = C_0 - (n-2)x\} \neq \emptyset$, and similarly $I_{n-2} \neq \emptyset$.

We show that $I_s \neq \emptyset$ for $s \in \{1, \ldots, n-3\}$. Choose $I \subset W_{pq}$, with $\#I = n-2-s$ such that

$$C_s = \sum_{h\in I} f_{ph} + \sum_{h \in W_{pq}\setminus I} f_{qh}.$$

Take $i \in I, j \in W_{pq}\setminus I$ and consider the same sum but with $I' = \{j\} \cup I\setminus\{i\}$ instead of I. This sum is at most C_s, and so $f_{pi} + f_{qj} \geqslant f_{pj} + f_{qi}$ and hence $f_{pi} - f_{qi} \geqslant f_{pj} - f_{qj}$. So there exists $x \in \mathbb{R}$ such that

$$\max_{j \in I^c} \frac{1}{2}(f_{pj} - f_{qj}) \leqslant x \leqslant \min_{i \in I} \frac{1}{2}(f_{pi} - f_{qi}).$$

For this specific x, we have $f_{pi} - x \geqslant f_{qi} + x$ and $f_{pj} - x \leqslant f_{qj} + x$ for any $i \in I, j \in I^c$, and hence

$$f(x) = \sum_{i \in I} f_{pi} + \sum_{j \in I^c} f_{qj} - (n-2-2s)x = C_s - (n-s-2s)x.$$

So indeed, $I_s \neq \emptyset$.

Now we may use Lemma 12 of [9] to conclude that

$$\log \phi_{pq\omega} = \min\{f(x) : x \in \mathbb{R}\} = C_{\frac{1}{2}(n-2)} = \log M_\omega(W_{pq})$$

when n is even; and similarly

$$\log \phi_{pq\omega} = \frac{1}{2}(C_{\frac{1}{2}(n-1)} + C_{\frac{1}{2}(n-3)}) = \log M_\omega(W_{pq})$$

when n is odd. This completes the proof.

□

Next, we estimate $M(J)$ from above by induction on $\#J$, and eventually deduce an upper bound for $M(W_{pq}) = \phi_{pq}$.

Put $\Theta_p(J) = \Theta_q(J) = 1, D(J) = 1$ if $J = \emptyset$; $D(J) = 1$ if $\#J = 1$; and

$$\begin{cases} \Theta_p(J) = \prod_{\omega \in T} \prod_{h \in J} \theta_{ph\omega}, \\ \Theta_q(J) = \prod_{\omega \in T} \prod_{h \in J} \theta_{qh\omega}, \\ D(J) = \prod_{\omega \in T} \prod_{h \neq l \in J} \theta_{hl\omega} \end{cases} \tag{5.2.7}$$

if $\#J \geqslant 2$.

For $s \geqslant 2$, let

$$d(s) = \begin{cases} \frac{1}{s} & s \text{ even} \\ \frac{s}{s^2-1} & s \text{ odd,} \end{cases}$$

and

$$a(0) = 0, a(1) = 0, a(s) = a(s-2) + 1 + 4(s-2)d(s), s \geqslant 2;$$

$$b(0) = 0, b(1) = \frac{1}{2}, b(s) = \frac{s-2}{s} b(s-2) + \frac{2}{s} + \frac{4(s-2)}{s} d(s);$$

$$c(0) = 0, c(1) = 0, c(s) = \frac{(s-2)(s-3)}{s(s-1)} c(s-2) + \frac{1 + 4(s-2)d(s)}{s(s-1)}.$$

It is not difficult to show, by a straightforward computation

$$a(s) \leqslant 1 + \frac{5}{2}(s-2), b(s) \leqslant 3, c(s) < \frac{5}{2s-2}.$$

Take $c(s)$ as an example. We have

$$s(s-1)c(s) = (s-2)(s-3)c(s-2) + 1 + 4d(s)(s-2).$$

When s is even, we have

$$\begin{aligned} s(s-1)c(s) &= \sum_{h=1}^{s/2} 1 + 4(2h-2)d(2h) \\ &= \sum_{h=1}^{s/2} (5 - \frac{4}{h}) \\ &= \frac{5s}{2} - 4\sum_{h=1}^{s/2} \frac{1}{h} \\ &< \frac{5s}{2}, \end{aligned}$$

hence $c(s) < \frac{5}{2s-2}$.

When s is odd, we derive $c(s) < \frac{5}{2s-2}$ by a similar computation.

Lemma 5.2.6. *Put* $C_{pqij} = e^{\max(2g_{L_{pqij}} - 2 + \#T_{pqij}, 0)[L:L_{pqij}]}$. *We have*

$$M(J) \leqslant C(J)(\prod_{\omega \in T} \theta_{pq\omega})^{a(s)} (\Theta_p(J)\Theta_q(J))^{b(s)} D(J)^{c(s)},$$

where $J \subset W_{pq}, s = \#J$ *and*

$$C(J) = \begin{cases} 1 & \text{for } s = 0, 1; \\ \Big(\prod_{i \neq j \in J} C_{pqij}\Big)^{c(s)} & \text{for } s \geqslant 2. \end{cases} \tag{5.2.8}$$

where the notation $\prod_{i \neq j \in J}$ *means that the product is taken over all ordered pairs* (i, j) *with* $i, j \in J$ *and* $i \neq j$.

Proof. If $s = 0$, then $M(J) = 1$, and if $s = 1$, $J = \{j\}$, then $M(J) = \prod_{\omega \in T} \sqrt{\theta_{pj\omega}\theta_{qj\omega}}$. So in these cases, Lemma 5.2.6 is trivial.

Let $s \geqslant 2$ and assume the assertion is true for sets J of cardinality strictly smaller than s. Let $J \subset W_{pq}, \#J = s$. Fix $i, j \in J$ with $i \neq j$, let $J_{ij} = J \backslash \{i, j\}$ and fix any $\omega \in T$.

We distinguish the cases s even and s odd.

First suppose s is even.

Let $I \subset J, \#I = \frac{1}{2}s \geqslant 1$, $g(I) = \prod_{h\in I} \theta_{ph\omega} \cdot \prod_{h\in J\backslash I} \theta_{qh\omega}$.

If $i \in I, j \in J\backslash I$, then since

$$M_\omega(J_{ij}) \geqslant \prod_{h\in K\backslash\{i\}} \theta_{ph\omega} \prod_{h\in J\backslash\{I\cup\{j\}\}} \theta_{qh\omega},$$

we have

$$g(I) \leqslant \theta_{pi\omega}\theta_{qj\omega}M_\omega(J_{ij}).$$

Hence

$$g(I) \leqslant \max(\theta_{pi\omega}\theta_{qj\omega}, \theta_{pj\omega}\theta_{qi\omega})M_\omega(J_{ij}). \tag{5.2.9}$$

This is also true if $j \in I, i \in J\backslash I$.

If $i, j \in I$, then pick $l \in J\backslash I$ such that $\frac{\theta_{pj\omega}\theta_{ql\omega}}{\theta_{pl\omega}\theta_{qj\omega}}$ is minimal for all $l \in J\backslash I$. Since $J\backslash I \subset J_{ij}$, we have

$$\begin{aligned} \frac{\theta_{pj\omega}\theta_{ql\omega}}{\theta_{pl\omega}\theta_{qj\omega}} &\leqslant \left(\prod_{h\in J\backslash I} \max(1, \frac{\theta_{pj\omega}\theta_{qh\omega}}{\theta_{ph\omega}\theta_{qj\omega}})\right)^{2/s} \\ &\leqslant \left(\prod_{h\in J_{ij}} \max(1, \frac{\theta_{pj\omega}\theta_{qh\omega}}{\theta_{ph\omega}\theta_{qj\omega}})\right)^{2/s}. \end{aligned}$$

Take $I' = I \cup \{l\}\backslash\{j\}$. Then $\#I' = \frac{s}{2}, i \in I', j \in J\backslash I'$. From (5.2.9), we get

$$\begin{aligned} g(I) &= \frac{\theta_{pj\omega}\theta_{ql\omega}}{\theta_{pl\omega}\theta_{qj\omega}} g(I') \\ &\leqslant \max(\theta_{pi\omega}\theta_{qj\omega}, \theta_{pj\omega}\theta_{qi\omega})M_\omega(J_{ij}) \left(\prod_{h\in J_{ij}} \max(1, \frac{\theta_{pj\omega}\theta_{qh\omega}}{\theta_{ph\omega}\theta_{qj\omega}})\right)^{2/s}. \end{aligned}$$

Similarly, we have

$$g(I) \leqslant \max(\theta_{pi\omega}\theta_{qj\omega}, \theta_{pj\omega}\theta_{qi\omega})M_\omega(J_{ij}) \left(\prod_{h\in J_{ij}} \max(1, \frac{\theta_{pi\omega}\theta_{qh\omega}}{\theta_{ph\omega}\theta_{qi\omega}})\right)^{2/s}.$$

So we get

$$g(I) \leqslant \left(\prod_{h\in J_{ij}} \max(1, \frac{\theta_{pi\omega}\theta_{qh\omega}}{\theta_{ph\omega}\theta_{qi\omega}}) \max(1, \frac{\theta_{pj\omega}\theta_{qh\omega}}{\theta_{ph\omega}\theta_{qj\omega}})\right)^{1/s} \times$$
$$\times \max(\theta_{pi\omega}\theta_{qj\omega}, \theta_{pj\omega}\theta_{qi\omega}) M_\omega(J_{ij}).$$

If $i, j \in J\backslash I$, by interchanging $I, J\backslash I$ and also p, q, we have

$$g(I) \leqslant \left(\prod_{h\in J_{ij}} \max(1, \frac{\theta_{ph\omega}\theta_{qi\omega}}{\theta_{pi\omega}\theta_{qh\omega}}) \max(1, \frac{\theta_{ph\omega}\theta_{qj\omega}}{\theta_{pj\omega}\theta_{qh\omega}})\right)^{1/s} \times$$
$$\times \max(\theta_{pi\omega}\theta_{qj\omega}, \theta_{pj\omega}\theta_{qi\omega}) M_\omega(J_{ij}).$$

This gives altogether

$$M_\omega(J) = \max g(I) \leqslant H_\omega^{\frac{1}{s}} \max(\theta_{pi\omega}\theta_{qj\omega}, \theta_{pj\omega}\theta_{qi\omega}) M_\omega(J_{ij}),$$

where

$$H_\omega = \prod_{h\in J_{ij}} \max\left(1, \frac{\theta_{pi\omega}\theta_{qh\omega}}{\theta_{ph\omega}\theta_{qi\omega}}\right) \max\left(1, \frac{\theta_{pj\omega}\theta_{qh\omega}}{\theta_{ph\omega}\theta_{qj\omega}}\right)$$
$$\times \max\left(1, \frac{\theta_{ph\omega}\theta_{qi\omega}}{\theta_{pi\omega}\theta_{qh\omega}}\right) \max\left(1, \frac{\theta_{ph\omega}\theta_{qj\omega}}{\theta_{pj\omega}\theta_{qh\omega}}\right).$$

If $s \geqslant 3$ is odd, the argument is similar.

Finally, we have

$$M_\omega(J) \leqslant H_\omega^{d(s)} \max(\theta_{pi\omega}\theta_{qj\omega}, \theta_{pj\omega}\theta_{qi\omega}) M_\omega(J_{ij}).$$

Let $H = \prod_{\omega\in T} H_\omega$. Note that this quantity depends on J_{ij}. We have

$$M(J) \leqslant H^{d(s)} \prod_{\omega\in T} \max(\theta_{pi\omega}\theta_{qj\omega}, \theta_{pj\omega}\theta_{qi\omega}, \theta_{pq\omega}\theta_{ij\omega}) M(J_{ij}). \tag{5.2.10}$$

By Lemma 5.2.1, we have

$$\prod_{\omega\in T} \max(\theta_{pq\omega}\theta_{ij\omega}, \theta_{pi\omega}\theta_{qj\omega}, \theta_{pj\omega}\theta_{qi\omega}) \leqslant C_{pqij} \prod_{\omega\in T} \theta_{pq\omega}\theta_{ij\omega}\theta_{pi\omega}\theta_{qj\omega}\theta_{pj\omega}\theta_{qi\omega},$$

where $L_{pqij} = K(\alpha_i, \alpha_j, \alpha_p, \alpha_q)$ and T_{pqij} is the set of valuations in L_{pqij} above those in T.

Using Lemma 5.2.1 again, we obtain

$$\begin{aligned}
H &= \prod_{h\in J_{ij}} \prod_{\omega\in T} \left(\frac{\max(\theta_{pi\omega}\theta_{qh\omega}, \theta_{ph\omega}\theta_{qi\omega})^2 \max(\theta_{pj\omega}\theta_{qh\omega}, \theta_{ph\omega}\theta_{qj\omega})^2}{\theta^2_{ph\omega}\theta^2_{qh\omega}\theta_{qi\omega}\theta_{pi\omega}\theta_{qj\omega}\theta_{pj\omega}} \right) \\
&\leqslant \prod_{h\in J_{ij}} C^2_{pqih} C^2_{pqjh} \times \\
&\quad \times \prod_{h\in J_{ij}} \prod_{\omega\in T} \left(\frac{(\theta_{pi\omega}\theta_{qh\omega}\theta_{ph\omega}\theta_{qi\omega}\theta_{pq\omega}\theta_{ih\omega})^2 (\theta_{pj\omega}\theta_{qh\omega}\theta_{ph\omega}\theta_{qj\omega}\theta_{pq\omega}\theta_{jh\omega})^2}{\theta^2_{ph\omega}\theta^2_{qh\omega}\theta_{qi\omega}\theta_{pi\omega}\theta_{qj\omega}\theta_{pj\omega}} \right) \\
&= \prod_{h\in J_{ij}} \left(C^2_{pqih} C^2_{pqjh} \right) \cdot \prod_{h\in J_{ij}} \prod_{\omega\in T} \left(\theta^4_{pq\omega}\theta^2_{qh\omega}\theta^2_{ph\omega}\theta^2_{ih\omega}\theta^2_{jh\omega}\theta_{pi\omega}\theta_{qi\omega}\theta_{pj\omega}\theta_{qj\omega} \right).
\end{aligned}$$

By substituting these estimates into (5.2.10), we get

$$\begin{aligned}
M(J) &\leqslant \Big(\prod_{h\in J_{ij}} C^2_{pqih} C^2_{pqjh} \Big)^{d(s)} \times \\
&\quad \times \left(\prod_{h\in J_{ij}} \prod_{\omega\in T} \theta^4_{pq\omega}\theta^2_{qh\omega}\theta^2_{ph\omega}\theta^2_{ih\omega}\theta^2_{jh\omega}\theta_{pi\omega}\theta_{qi\omega}\theta_{pj\omega}\theta_{qj\omega} \right)^{d(s)} \times \\
&\quad \times C_{pqij} \cdot \left(\prod_{\omega\in T} \theta_{pq\omega}\theta_{ij\omega}\theta_{pi\omega}\theta_{qj\omega}\theta_{pj\omega}\theta_{qi\omega} \right) M(J_{ij}).
\end{aligned}$$

This inequality is valid for each pair $i, j \in J$ with $i \neq j$. By taking the geometric means over these pairs we get

$$\begin{aligned}
M(J) &\leqslant \left(\prod_{i\neq j\in J} \prod_{h\in J_{ij}} C^2_{pqih} C^2_{pqjh} \right)^{\frac{d(s)}{s(s-1)}} \times \\
&\quad \times \left(\prod_{i\neq j\in J} \prod_{h\in J_{ij}} \prod_{\omega\in T} \theta^4_{pq\omega}\theta^2_{qh\omega}\theta^2_{ph\omega}\theta^2_{ih\omega}\theta^2_{jh\omega}\theta_{pi\omega}\theta_{qi\omega}\theta_{pj\omega}\theta_{qj\omega} \right)^{\frac{d(s)}{s(s-1)}} \times \\
&\quad \times \left(\prod_{i\neq j\in J} C_{pqij} \Big(\prod_{\omega\in T} \theta_{pq\omega}\theta_{ij\omega}\theta_{pi\omega}\theta_{qj\omega}\theta_{pj\omega}\theta_{qi\omega} \Big) M(J_{ij}) \right)^{\frac{1}{s(s-1)}}.
\end{aligned}$$

By inserting the upper bound for $M(J_{ij})$ following from the induction hypothesis, we get

$$
\begin{aligned}
M(J) \;\leqslant\; & \Bigg(\prod_{i\neq j\in J}\prod_{h\in J_{ij}} C^2_{pqih}C^2_{pqjh}\Bigg)^{\frac{d(s)}{s(s-1)}} \times \\
\times\; & \Bigg(\prod_{i\neq j\in J}\prod_{h\in J_{ij}}\prod_{\omega\in T}\theta^4_{pq\omega}\theta^2_{qh\omega}\theta^2_{ph\omega}\theta^2_{ih\omega}\theta^2_{jh\omega}\theta_{pi\omega}\theta_{qi\omega}\theta_{pj\omega}\theta_{qj\omega}\Bigg)^{\frac{d(s)}{s(s-1)}} \times \\
\times\; & \Bigg(\prod_{i\neq j\in J} C_{pqij}\cdot\Big(\prod_{\omega\in T}\theta_{pq\omega}\theta_{ij\omega}\theta_{pi\omega}\theta_{qj\omega}\theta_{pj\omega}\theta_{qi\omega}\Big)\Bigg)^{\frac{1}{s(s-1)}} \times \\
\times\; & \Bigg(\prod_{i\neq j\in J} C(J_{ij})(\prod_{\omega\in T}\theta_{pq\omega})^{a(s-2)}(\Theta_p(J_{ij})\Theta_q(J_{ij}))^{b(s-2)}D(J_{ij})^{c(s-2)}\Bigg)^{\frac{1}{s(s-1)}},
\end{aligned}
$$

where $C(J_{ij})$ is defined by (5.2.8) with J_{ij} replacing J.

Put

$$
C'(J) = \Big(\prod_{i\neq j\in J} C(J_{ij})\Big)^{\frac{1}{s(s-1)}}\Big(\prod_{i\neq j\in J} C_{pqij}\Big)^{\frac{1+4d(s)(s-2)}{s(s-1)}}.
$$

Then by the previous inequality, we get

$$
\begin{aligned}
M(J) \;\leqslant\; & C'(J)\Big(\prod_{\omega\in T}\theta_{pq\omega}\Big)^{a(s)}\Big(\prod_{i\neq j}\prod_{h\in J_{ij}}\prod_{\omega\in T}\theta_{qh\omega}\theta_{ph\omega}\theta_{ih\omega}\theta_{jh\omega}\Big)^{\frac{2d(s)}{s(s-1)}} \times \\
& \times\Big(\prod_{i\neq j\in J}\prod_{\omega\in T}\prod_{h\in J_{ij}}\theta_{ph\omega}\theta_{qh\omega}\Big)^{\frac{b(s-2)}{s(s-1)}}\Big(\prod_{i\neq j} D(J_{ij})\Big)^{\frac{c(s-2)}{s(s-1)}} \times \\
& \times\Big(\prod_{i\neq j\in J}\prod_{\omega\in T}\theta_{ij\omega}\Big)^{\frac{1}{s(s-1)}}\Big(\prod_{i\neq j}\prod_{\omega\in T}\theta_{pi\omega}\theta_{qi\omega}\theta_{pj\omega}\theta_{qj\omega}\Big)^{\frac{(s-2)d(s)+1}{s(s-1)}},
\end{aligned}
$$

hence

$$\begin{aligned} M(J) &= C'(J)\Big(\prod_{\omega\in T}\theta_{pq\omega}\Big)^{a(s)}(\Theta_p(J)\Theta_q(J))^{2d(s)}\times \\ &\times\Big(\prod_{i\neq j}\prod_{h\in J_{ij}}\prod_{\omega\in T}\theta_{ih\omega}\theta_{jh\omega}\Big)^{\frac{2d(s)}{s(s-1)}}\Big(\prod_{i\neq j}\prod_{\omega\in T}\prod_{h\in J_{ij}}\theta_{ph\omega}\theta_{qh\omega}\Big)^{\frac{b(s-2)}{s(s-1)}}\times \\ &\times\Big(\prod_{i\neq j}D(J_{ij})\Big)^{\frac{c(s-2)}{s(s-1)}}\Big(\prod_{i\neq j\in J}\prod_{\omega\in T}\theta_{ij\omega}\Big)^{\frac{1}{s(s-1)}}\times \\ &\times\Big(\prod_{i\neq j}\prod_{\omega\in T}\theta_{pi\omega}\theta_{qi\omega}\theta_{pj\omega}\theta_{qj\omega}\Big)^{\frac{(s-4)d(s)+1}{s(s-1)}}. \end{aligned}$$

Now by (5.2.8) with J_{ij} replacing J, we have

$$C'(J)=\Big(\prod_{i\neq j\in J}\prod_{h\neq l\in J_{ij}}C_{pqhl}\Big)^{\frac{c(s-2)}{s(s-1)}}\Big(\prod_{i\neq j\in J}C_{pqij}\Big)^{\frac{1+4d(s)(s-2)}{s(s-1)}},$$

hence

$$\begin{aligned} C'(J) &= \Big(\prod_{i\neq j\in J}C_{pqij}\Big)^{((s-2)(s-3)c(s-2)+1+4(s-2)d(s))/(s(s-1))} \\ &= \Big(\prod_{i\neq j\in J}C_{pqij}\Big)^{c(s)} \\ &= C(J). \end{aligned} \tag{5.2.11}$$

Now combining the just established upper bound for $M(J)$ with (5.2.7) and the obvious identities

$$\Big(\prod_{i\neq j}\theta_{pi\omega}\theta_{qi\omega}\theta_{pj\omega}\theta_{qj\omega}\Big)^{\frac{1}{s(s-1)}}=\Big(\prod_{h\in J}\prod_{\omega\in T}\theta_{ph\omega}\theta_{qh\omega}\Big)^{\frac{2}{s}},$$

$$\Big(\prod_{i\neq j}\prod_{\omega\in T}\prod_{h\in J_{ij}}\theta_{ph\omega}\theta_{qh\omega}\Big)^{\frac{1}{s(s-1)}}=\Big(\prod_{\omega\in T}\prod_{h\in J}\theta_{ph\omega}\theta_{qh\omega}\Big)^{\frac{s-2}{s}},$$

$$(\prod_{i\neq j}\prod_{h\in J_{ij}}\prod_{\omega\in T}\theta_{ih\omega}\theta_{jh\omega})^{\frac{1}{s(s-1)}}=(D(J))^{\frac{2(s-2)}{s(s-1)}},$$

$$(\prod_{i\neq j}\prod_{\omega\in T}\theta_{ij\omega})^{\frac{1}{s(s-1)}}=(D(J))^{\frac{1}{s(s-1)}},$$

$$(\prod_{i\neq j}D(J_{ij}))^{\frac{1}{s(s-1)}}=(D(J))^{\frac{(s-2)(s-3)}{s(s-1)}}.$$

we deduce that

$$M(J)\leqslant C(J)(\prod_{\omega\in T}\theta_{pq\omega})^{a(s)}(\Theta_p(J)\Theta_q(J))^{b(s)}D(J)^{c(s)},$$

which completes the induction step and the proof of Lemma 5.2.6. □

Proof of (ii) of Proposition 5.2.4. Now Lemma 5.2.6 with $J=W_{pq}$ and Lemma 5.2.5 give that

$$\begin{aligned}\phi_{pq}=M(W_{pq}) &\leqslant \Big(\prod_{i\neq j\in W_{pq}}C_{pqij}\Big)^{c(n-2)}\\ &\times \Big(\prod_{\omega\in T}\theta_{pq\omega}\Big)^{a(n-2)}\Big(\Theta_p(W_{pq})\Theta_q(W_{pq})\Big)^{b(n-2)}D(W_{pq})^{c(n-2)}.\end{aligned}$$

Notice that since F is primitive, it follows from (5.2.4) that $|a|_T=\prod_{i=1}^{n}\prod_{\omega\in T}\frac{1}{\xi_{i\omega}}$, so by an easy computation, we have

$$\prod_{p\neq q}\prod_{\omega\in T}\theta_{pq\omega}=|D(F)|_T,$$

$$\prod_{p\neq q}\Theta_p(W_{pq})\Theta_q(W_{pq})=|D(F)|_T^{2n-4},$$

$$\prod_{p\neq q}D(W_{pq})=|D(F)|_T^{(n-2)(n-3)}.$$

Thus, we deduce that

$$\prod_{\substack{p,q\in\{1,\dots,n\}\\ p\neq q}}\phi_{pq}\ \leqslant\ U\cdot\Big(\prod_{p\neq q}\prod_{i\neq j\in W_{pq}}C_{pqij}\Big)^{c(n-2)},\tag{5.2.12}$$

where $U = |D(F)|_T^{a(n-2)+(2n-4)b(n-2)+(n-2)(n-3)c(n-2)}$.

We need to estimate $\prod_{p\neq q}\prod_{i\neq j\in W_{pq}} C_{pqij}$ from above.

Denote the field $K(\alpha_h)$ by L_h. By (5.1.3), we have

$$\frac{2g_{pqij}-1+\#T_{pqij}}{[L_{pqij}:K]} \leqslant -1+\#S-\sum_{h\in\{p,q,i,j\}}\sum_{\nu\in S}\frac{\nu(D(F_{\overline{h}}))}{\deg F_{\overline{h}}}.$$

so we get

$$(2g_{pqij}-1+\#T_{pqij})[L:L_{pqij}] \leqslant [L:K]\Big(-1+\#S-\sum_{h\in\{p,q,i,j\}}\sum_{\nu\in S}\frac{\nu(D(F_{\overline{h}}))}{\deg F_{\overline{h}}}\Big).$$

Hence

$$\begin{aligned}
&\sum_{p\neq q}\sum_{i\neq j\in W_{pq}}\max(2g_{pqij}-2+\#T_{pqij},0)[L:L_{pqij}]\\
\leqslant\;&\sum_{p\neq q}\sum_{i\neq j\in W_{pq}}(2g_{pqij}-1+\#T_{pqij})[L:L_{pqij}]\\
\leqslant\;&[L:K]\sum_{p\neq q}\sum_{\substack{i,j\neq p,q\\ i\neq j}}\Big(-1+\#S-\sum_{h\in\{p,q,i,j\}}\sum_{\nu\in S}\frac{\nu(D(F_{\overline{h}}))}{\deg F_{\overline{h}}}\Big)\\
=\;&[L:K]\Big(n(n-1)(n-2)(n-3)(\#S-1)\\
&\qquad -4(n-1)(n-2)(n-3)\sum_{i=1}^{d}\sum_{\nu\in S}\nu(D(F_i))\Big)\\
\leqslant\;&[L:K]\Big(n(n-1)(n-2)(n-3)(\#S-1)\\
&\qquad -4(n-1)(n-2)(n-3)\sum_{\nu\in S}\nu(D(F))\Big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\prod_{p\neq q}\phi_{pq} \;\leqslant\;& \Big(e^{[L:K](n-1)(n-2)(n-3)(n(\#S-1)-4\sum_{\nu\in S}\nu(D(F)))}\Big)^{c(n-2)}\times\\
&\times|D(F)|_T^{a(n-2)+(2n-4)b(n-2)+(n-2)(n-3)c(n-2)}\\
=\;& e^{[L:K]n(n-1)(n-2)(n-3)(\#S-1)c(n-2)}\times\\
&\times|D(F)|_T^{4(n-1)(n-2)(n-3)c(n-2)+a(n-2)+(2n-4)b(n-2)+(n-2)(n-3)c(n-2)}.
\end{aligned}$$

As $a(n-2) \leqslant \frac{5n}{2} - 9, b(n-2) \leqslant 3, c(n-2) < \frac{5}{2n-6}$ we conclude from (5.2.6) that

$$\begin{aligned} {M'}^{\frac{2(n-1)}{n-2}} &\leqslant \left(|D(F)|_T^{-\frac{n}{2}+2} \prod_{p \neq q} \phi_{pq} \right)^{\frac{2}{n(n-2)}} \\ &\leqslant e^{[L:K](\#S-1)5(n-1)} |D(F)|_T^{20+\frac{1}{n}}. \end{aligned}$$

This gives Proposition 5.2.4 (ii). □

5.3 Completion of the Proof of the Main Theorem

Main Theorem. *Let $F \in \mathcal{O}_S[X,Y]$ be a binary form of degree $n \geqslant 4$ with non-zero discriminant. Then F is $\mathrm{GL}\,(2, \mathcal{O}_S)$-equivalent to a binary form F^* such that*

$$H^*(F^*) \leqslant e^{(n-1)\left(\#S(n+11)-5\right)} |D(F)|_S^{20+\frac{1}{n}}.$$

Proof. When F is primitive, this follows directly from Proposition 5.2.4 and Theorem 4.3.3.

In the general case, write $F = a\tilde{F}$ such that $a \in \mathcal{O}_S$ and $\tilde{F}$ is primitive. Then there exists $\tilde{F}_1$ that is $\mathrm{GL}_n(2, \mathcal{O}_S)$-equivalent to $\tilde{F}$ such that

$$H^*(\tilde{F}_1) \leqslant e^{(n-1)\left(\#S(n+11)-5\right)} |D(\tilde{F})|_S^{20+\frac{1}{n}}.$$

Let $F_1 = a\tilde{F}_1$. Since $\tilde{F}_1$ is a binary form over $\mathcal{O}_S$, $H_S(\tilde{F}_1) \leqslant H^*(\tilde{F}_1)$. Noticing that $D(F) = a^{2n-2}D(\tilde{F})$, we deduce that

$$\begin{aligned} H_S(F_1) &= |a|_S H_S(\tilde{F}_1) \\ &\leqslant |a|_S e^{(n-1)\left(\#S(n+11)-5\right)} |D(\tilde{F})|_S^{20+\frac{1}{n}} \\ &= e^{(n-1)\left(\#S(n+11)-5\right)} |D(F)|_S^{20+\frac{1}{n}} |a|_S^{1-(2n-2)(20+1/n)} \\ &\leqslant e^{(n-1)\left(\#S(n+11)-5\right)} |D(F)|_S^{20+\frac{1}{n}}. \end{aligned}$$

By Lemma 4.2.1, there exists $u \in \mathcal{O}_S^*$ such that $H^*(uF_1) = H_S(F_1)$. Put $F^* = uF_1 = au\tilde{F}_1$. Then F^* is $\mathrm{GL}_n(2, \mathcal{O}_S)$-equivalent to F and

$$H^*(F^*) \leqslant e^{(n-1)\left(\#S(n+11)-5\right)}|D(F)|_S^{20+\frac{1}{n}},$$

as claimed. □

We need a variation of the Main Theorem. To get this, in the proof of Lemma 5.2.6 we repeat all computations but with all fields L_{pqij} replaced by L. Then we get Lemma 5.2.6 with C_{pqij} replaced by $e^{\max(2g_L-2+\#T,0)}$. This gives, instead of (5.2.12),

$$\begin{aligned}\prod_{p\neq q\in\{1,\dots,n\}} \phi_{pq} &< e^{\max(2g_L-2+\#T,0)\binom{n}{4}c(n-2)} \times \\ &\quad \times |D(F)|_T^{a(n-2)+(2n-4)b(n-2)+(n-2)(n-3)c(n-2)} \\ &\leqslant e^{\max(2g_L-2+\#T,0)\binom{n}{4}c(n-2)}|D(F)|_T^{11n-26}. \qquad (5.3.1)\end{aligned}$$

Similarly as before, this leads together with (5.2.6), to the following:

Proposition 5.3.1. *Suppose F is primitive with splitting field L. Then there is an admissible tuple $\mathbb{A}' = (A'_{i\omega} : \omega \in T, i = 1, \dots, n)$ such that*

(i) $\prod_{\omega\in T} \max_{1\leqslant i<j\leqslant n} \frac{\theta_{ij\omega}}{A'_{i\omega}A'_{j\omega}} \leqslant 1$;

(ii) $\left(\prod_{i=1}^{n}\prod_{\omega\in T} A'_{i\omega}\right)^{\frac{2(n-1)}{n-2}} \leqslant e^{\frac{5}{24}(n-1)\max(2g_L-2+\#T,0)}|D(F)|_T^{\frac{21}{n}}$.

Now with the same idea as in the proof of the Main Theorem, we have

Theorem 5.3.2. *Let $F \in \mathcal{O}_S[X,Y]$ be a binary form of degree $n \geqslant 4$ with non-zero discriminant. Then F is $\mathrm{GL}\,(2,\mathcal{O}_S)$-equivalent to a binary form F^* such that*

$$H^*(F^*) \leqslant \exp\left((n^2+6n-7)\#S + \frac{(5n-5)(2g_L-1)}{24[L:K]}\right)|D(F)|_S^{\frac{21}{n}}.$$

Proof. First, if F is primitive, this follows from Proposition 5.3.1 and Theorem 4.3.3 by a direct computation.

If F is not primitive, we assume that $F = a\tilde{F}$ with $a \in \mathcal{O}_S$ and $\tilde{F}$ primitive. Then there exists $\tilde{F}_1$ that is $\mathrm{GL}_n(2, \mathcal{O}_S)$-equivalent to $\tilde{F}$ such that

$$H^*(\tilde{F}_1) \leqslant \exp\left((n^2 + 6n - 7)\#S + \tfrac{(5n-5)(2g_L-1)}{24[L:K]}\right) |D(\tilde{F})|_S^{\frac{21}{n}}.$$

Let $F_1 = a\tilde{F}_1$. Then

$$\begin{aligned} H_S(F_1) &= |a|_S H_S(\tilde{F}_1) \\ &\leqslant |a|_S \exp\left((n^2 + 6n - 7)\#S + \tfrac{(5n-5)(2g_L-1)}{24[L:K]}\right) |D(\tilde{F})|_S^{\frac{21}{n}} \\ &= \exp\left((n^2 + 6n - 7)\#S + \tfrac{(5n-5)(2g_L-1)}{24[L:K]}\right) |D(F)|_S^{\frac{21}{n}} |a|_S^{1-42(n-1)/n} \\ &\leqslant \exp\left((n^2 + 6n - 7)\#S + \tfrac{(5n-5)(2g_L-1)}{24[L:K]}\right) |D(F)|_S^{\frac{21}{n}}. \end{aligned}$$

By Lemma 4.2.1, there is $u \in \mathcal{O}_S^*$ such that $H^*(uF_1) = H_S(F_1)$. Take $F^* = uF_1 = au\tilde{F}_1$, then it is $\mathrm{GL}_n(2, \mathcal{O}_S)$-equivalent to $F = a\tilde{F}$ and

$$H^*(F^*) \leqslant \exp\left((n^2 + 6n - 7)\#S + \tfrac{(5n-5)(2g_L-1)}{24[L:K]}\right) |D(F)|_S^{\frac{21}{n}}.$$

□

Remark 5.3.3. *This result is weaker than the Main Theorem in the sense that the constant depends on the splitting field L of F as well; however, it is apparently stronger because the exponent of $|D(F)|_S$ is much smaller and tends to zero when n goes to infinity.*