Author: Zhuang, Weidong
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Chapter 3

Geometry of numbers over function fields

Minkowski's results on successive minima of convex bodies have analogues over function fields. These are discussed in this chapter. Our main reference is Thunder [24].

3.1 Successive minima

Recall $K = k(t)$ is the rational function field over an algebraically closed field k of characteristic 0 and for $\nu \in M_K$, $R_\nu = \{x \in K_\nu : |x|_\nu \leqslant 1\}$. A subset $\mathcal{C}_\nu$ of K_ν^n is called a ν-adic convex symmetric body if it has the following properties:

- $\mathcal{C}_\nu$ is closed and bounded in the topology of K_ν^n induced by $|\cdot|_\nu$ and has $\mathbf{0}$ as an interior point;
- for every $\mathbf{x} \in \mathcal{C}_\nu, \alpha \in K_\nu$ with $|\alpha|_\nu \leqslant 1$, we have $\alpha\mathbf{x} \in \mathcal{C}_\nu$;
- for every $\mathbf{x}, \mathbf{y} \in \mathcal{C}_\nu$, we have $\mathbf{x} + \mathbf{y} \in \mathcal{C}_\nu$.

Remark 3.1.1. *These properties imply that $\mathcal{C}_\nu$ is an R_ν-module.*

Let S be a finite subset of M_K containing ν_∞ and $\mathbb{A}_S = \prod_{\nu \in S} K_\nu$ the ring of S-adeles. Consider K as a subring of $\mathbb{A}_S$ by identifying $x \in K$ with the adele $(x_\nu)_{\nu \in S}$ with $x_\nu = x$ for all $\nu \in S$. A subset $\mathcal{C}$ of $\mathbb{A}_S^n$ is called convex symmetric if $\mathcal{C} = \prod_{\nu \in S} \mathcal{C}_\nu$ with $\mathcal{C}_\nu$ ν-adic convex symmetric for $\nu \in S$. We need two lemmas.

Lemma 3.1.2. *Let $\nu \in M_K$ and $\mathcal{C}_\nu \subset K_\nu^n$. Then $\mathcal{C}_\nu$ is a ν-adic symmetric convex body if and only if $\mathcal{C}_\nu = \{\mathbf{x} \in K_\nu^n : \|A_\nu \mathbf{x}\|_\nu \leqslant 1\}$ for some $A_\nu \in GL_n(K_\nu)$.*

Proof. First, notice that from the definition, $\mathcal{C}_\nu$ is an R_ν-module. It is also bounded, hence there is a constant $C > 0$ such that $\|\mathbf{x}\|_\nu \leqslant C$ for every $\mathbf{x} \in \mathcal{C}_\nu$.

It is easy to see that $\mathcal{C}_\nu$ contains a basis of K_ν^n, since $\mathbf{0}$ is an interior point. Hence there exists a free R_ν-module $\mathcal{M}_1$ of rank n such that $\mathcal{M}_1 \subset \mathcal{C}_\nu$. Take $\alpha \in K_\nu$ with $|\alpha|_\nu \geqslant C$. Then $\mathcal{C}_\nu \subset \{\mathbf{x} \in K_\nu^n : \|\mathbf{x}\|_\nu \leqslant |\alpha|_\nu\}$. Hence $\mathcal{C}_\nu$ is contained in the free R_ν-module $\mathcal{M}_2 = \alpha R_\nu^n$ of rank n.

As is well-known, R_ν is a principal ideal domain, hence by Chapter III, Theorem 7.1 of [16], we know that $\mathcal{C}_\nu$ is also a free R_ν-module of rank n. Take an R_ν-basis of $\mathcal{C}_\nu$, let B_ν be the matrix whose columns consists of this basis, and let $A_\nu = B_\nu^{-1}$. Then $\mathcal{C}_\nu = \{B_\nu \mathbf{y} : \mathbf{y} \in R_\nu^n\} = \{\mathbf{x} \in K_\nu^n : \|A_\nu \mathbf{x}\|_\nu \leqslant 1\}$. □

Remark 3.1.3. *For $\mathcal{C}_\nu = R_\nu$ we will choose A_ν to be I_n, the $n \times n$ identity matrix. This does not change $\mathcal{C}_\nu$.*

Example 3.1.4. *Take ν be the valuation corresponding to 0. Then $K_\nu = k((t)), R_\nu = \mathbb{C}[[t]]$. Let $\mathcal{C}_\nu = \{\mathbf{x} \in k((t))^n : \|\mathbf{x}\|_\nu < 1\}$. Then $\mathcal{C}_\nu = \{(x_1, \ldots, x_n) \in \mathcal{C}((t))^n : x_i \in tk[[t]], i = 1, \ldots, n\}$. We may take $A_\nu = diag(\frac{1}{t}, \ldots, \frac{1}{t})$ and this gives $\mathcal{C}_\nu = \{\mathbf{x} \in k((t))^n : \|A_\nu \mathbf{x}\|_\nu \leqslant 1\}$.*

Lemma 3.1.5. *For $\mathbf{x} \in K^n \backslash \{\mathbf{0}\}$ there exists $f \in K$ such that $\|A_\infty(f\mathbf{x})\|_\infty = H_A(\mathbf{x})$, $\|A_\nu(f\mathbf{x})\|_\nu = 1$ for $\nu \in S, \nu \neq \nu_\infty$ and $\|f\mathbf{x}\|_\nu = 1$ for $\nu \notin S$.*

Proof. For consistency put $A_\nu = I_n$ the $n \times n$ identity matrix for $\nu \notin S$. Let $\{\nu_1, \dots, \nu_m\} \subset M_K \backslash \{\nu_\infty\}$ be the finite set of valuations such that $\|A_\nu \mathbf{x}\|_\nu \neq 1$, with corresponding uniformizers $t - p_1, \dots, t - p_m \in K$. Let $n_i = -\nu_i(A_{\nu_i}\mathbf{x})$ for $1 \leqslant i \leqslant m$ and $f = \prod_{i=1}^{m}(t - p_i)^{n_i}$. Then $\|A_\infty(f\mathbf{x})\|_\infty = H_A(\mathbf{x})$, $\|A_{\nu_i}(f\mathbf{x})\|_{\nu_i} = 1$ for $i = 1, \dots, m$ and $\|A_{\nu_i}(f\mathbf{x})\|_{\nu_i} = 1$ for $\nu \notin \{\nu_1, \dots, \nu_m, \nu_\infty\}$, as claimed. □

Let $\mathcal{C} \subset \mathbb{A}_S^n$ be an S-convex symmetric body and $\lambda \in e^{\mathbb{Z}}$. By Lemma 3.1.2, there exists $A \in \mathrm{GL}_n(\mathbb{A}_S)$, such that $\mathcal{C}_\nu = \{\mathbf{x} \in K_\nu^n : \|A_\nu \mathbf{x}\|_\nu \leqslant 1\}$ for each $\nu \in S$. We view $\mathcal{O}_S^n$ as a subset of $\prod_{\nu \in S} K_\nu^n$ via the diagonal embedding.

For $\lambda \in e^{\mathbb{Z}}$, define

$$\lambda\mathcal{C}_\infty := \{a\mathbf{x} : \mathbf{x} \in \mathcal{C}_\infty, a \in K_\infty, |a|_\infty \leqslant \lambda\}, \tag{3.1.1}$$

$$\lambda\mathcal{C} := (\lambda\mathcal{C}_\infty) \times \prod_{\nu \in S, \nu \neq \nu_\infty} \mathcal{C}_\nu. \tag{3.1.2}$$

Then

$$\lambda\mathcal{C}_\infty = \{\mathbf{x} \in K_\infty^n : \|A_\infty \mathbf{x}\|_\infty \leqslant \lambda\},$$

and

$$\lambda\mathcal{C} \cap \mathcal{O}_S^n := \{\mathbf{x} \in \mathcal{O}_S^n : H_A(x) \leqslant \lambda\}.$$

Remark that by Lemma 3.1.5, for every $\mathbf{x} \in \mathcal{O}_S^n$ with $H_A(\mathbf{x}) \leqslant \lambda$, there exists $f \in K$ such that $\|f\mathbf{x}\|_\nu = 1$ for $\nu \neq \nu_\infty$ and $\|A_\infty(f\mathbf{x})\|_\infty = H_A(\mathbf{x}) \leqslant \lambda$. In particular, $f\mathbf{x} \in k[t]^n$.

Definition 3.1.6. *The i-th successive minimum λ_i of $\mathcal{C}$ is the minimum of all $\lambda \in e^{\mathbb{Z}}$ such that $\lambda\mathcal{C} \cap \mathcal{O}_S^n$ contains at least i K-linearly independent points.*

Clearly, given $\lambda \in e^{\mathbb{Z}}$ and $\mathbf{x} \in \lambda\mathcal{C}$, we have $H_A(\mathbf{x}) \leqslant \lambda$.

Theorem 3.1.7. *The successive minima exist and $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n < \infty$. Moreover, there exists a basis $\mathbf{x}_1, \dots, \mathbf{x}_n$ of K^n such that $\mathbf{x}_i \in \lambda_i\mathcal{C} \cap \mathcal{O}_S^n$, and $\|A_\infty(\mathbf{x}_i)\|_\infty = H_A(\mathbf{x}_i) = \lambda_i$ for all $i = 1, \dots, n$.*

Proof. For every $\mathbf{x} \in K^n\backslash\{0\}$, we have $H_A(\mathbf{x}) \in e^{\mathbb{Z}}$ and also $H_A(\mathbf{x}) \geqslant c_2 > 0$ unless $\mathbf{x} = \mathbf{0}$ by Lemma 1.5.2. Hence there is $\mathbf{x}_1 \in K^n\backslash\{0\}$ such that $H_A(\mathbf{x}_1)$ is minimal. Further, by Lemma 3.1.5 we may choose $\mathbf{x}_1$ such that $\|A_\infty(\mathbf{x}_1)\|_\infty = H_A(\mathbf{x}_1)$. Then automatically, $\lambda_1 = H_A(\mathbf{x}_1)$ is the first successive minimum. Successively, for $j = 1, \ldots, n-1$, we take $\mathbf{x}_{j+1}$ to be a point $\mathbf{x} \in K^n$ such that $\mathbf{x}$ is K-linearly independent of $\mathbf{x}_1, \ldots, \mathbf{x}_j$ and $H_A(\mathbf{x}_{j+1})$ is minimal with this property, and we may also assume that $\|A_\nu(\mathbf{x}_{j+1})\|_\nu = 1$ for $\nu \neq \nu_\infty$ by Lemma 3.1.5. With this choice, $H_A(\mathbf{x}_1) \leqslant \cdots \leqslant H_A(\mathbf{x}_n)$ and $\mathbf{x}_i \in H_A(\mathbf{x}_i)\mathcal{C}$ for $i = 1, \ldots, n$, hence λ_i exists and $\lambda_i \leqslant H_A(\mathbf{x}_i)$ for $i = 1, \ldots, n$. We claim that $\lambda_i = H_A(\mathbf{x}_i)$ and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are as required. Assume the contrary, let i be the smallest index such that $H_A(\mathbf{x}_i) > \lambda_i$. There are K-linearly independent points $\mathbf{y}_1, \ldots, \mathbf{y}_i$ in $K^n \cap \lambda_i\mathcal{C}$. Clearly, we have $H_A(\mathbf{x}_i) > \lambda_i \geqslant H_A(\mathbf{y}_j)$ for $j = 1, \ldots, i$. So by our choice of $\mathbf{x}_i$, we know that each $\mathbf{y}_j$ is K-linearly dependent of $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$, which contradicts the fact that $\mathbf{y}_1, \ldots, \mathbf{y}_i$ are K-linearly independent. This completes the proof of the claim. □

Theorem 3.1.8. *Let $\mathcal{C}$ be an S-convex body. Then there is an $\mathcal{O}_S$-module basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of $\mathcal{O}_S^n$ such that $\|A_\infty(\mathbf{a}_i)\|_\infty = H_A(\mathbf{a}_i) = \lambda_i$ for $i = 1, \ldots, n$. Also, we have $\prod_{i=1}^{n} \lambda_i = |\det A|$.*

Proof. By Lemma 1.5.4, we can choose a K-basis of column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of K^n such that $\prod_{i=1}^{n} H_A(\mathbf{a}_i) = H_A(K^n)$.

Let $\mu_i = H_A(\mathbf{a}_i)$, and assume that $\mu_1 \leqslant \cdots \leqslant \mu_n$ without loss of generality. By Lemma 3.1.5 we may also assume that $\|A_\infty(\mathbf{a}_i)\|_\infty = H_A(\mathbf{a}_i) = \mu_i$, whence $\mathbf{a}_i \in \mathcal{O}_S^n$ for $i = 1, \ldots, n$. Then $H_A(K^n) = \prod_{i=1}^{n} \mu_i$ and $\mathbf{a}_1, \ldots, \mathbf{a}_i \in \mu_i\mathcal{C}$. By the definition of successive minima, we have $\lambda_i \leqslant \mu_i$ for all i, hence $\prod_{i=1}^{n} \lambda_i \leqslant H_A(K^n)$.

On the other hand, by Theorem 3.1.7, we may take a K-basis $\mathbf{x}_1, \ldots, \mathbf{x}_n$ with $\mathbf{x}_i \in \lambda_i\mathcal{C}, H_A(\mathbf{x}_i) = \lambda_i$ for all i. By lemma 1.5.4, we have $\prod_{i=1}^{n} \lambda_i \geqslant H_A(K^n)$. Therefore, we get $\mu_i = \lambda_i$ and $\prod_{i=1}^{n} \lambda_i = H_A(K^n) = |\det A|$.

For $\nu \in S$ let $l_{i\nu}(\mathbf{x}) = \sum_j a_{ij\nu} x_j$ and let $A_\nu = (a_{ij\nu})_{i,j}$ be the $n \times n$ matrix with the coefficients of $l_{i\nu}$ on the i-th row. Let $\Delta_\nu = \det(l_{i\nu}(\mathbf{a}_j))$. Note that $\|A_\nu \mathbf{a}_j\|_\nu = \max_{1\leqslant i\leqslant n} |l_{i\nu}(\mathbf{a}_j)|_\nu$. By the rules of matrix multiplication, we have $A_\nu(\mathbf{a}_1, \ldots, \mathbf{a}_n) = (l_{i\nu}(\mathbf{a}_j))_{i,j} = \Delta_\nu$, where $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is the matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Then by taking determinants, we get

$$\begin{aligned}
\prod_{\nu\in S} |\det A_\nu|_\nu \prod_{\nu\in S} |\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)|_\nu &= \prod_{\nu\in S} |\Delta_\nu|_\nu \leqslant \prod_{\nu\in S}\prod_{j=1}^{n} \max_{1\leqslant i\leqslant n} |l_{i\nu}(\mathbf{a}_j)|_\nu \\
&= \prod_{j=1}^{n}(\prod_{\nu\in S} \|A_\nu \mathbf{a}_j\|_\nu) = \prod_{j=1}^{n} H_A(\mathbf{a}_j) \\
&= |\det A| = \prod_{\nu\in S} |\det A_\nu|_\nu.
\end{aligned}$$

Thus we deduce that $\prod_{\nu\in S} |\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)|_\nu \leqslant 1$. Since $\mathbf{a}_i \in \mathcal{O}_S^n$, we have $|\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)|_\nu \leqslant 1$ for $\nu \notin S$. By the product formula we have $\prod_{\nu\in S} |\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)|_\nu = 1$, hence $|\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)|_\nu = 1$ for $\nu \notin S$. This implies that $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is an $\mathcal{O}_S$-module basis of $\mathcal{O}_S^n$. □

3.2 A generalization

For an arbitrary field L, we denote by $L[X_1, \ldots, X_n]^{\text{lin}}$ the L-vector space of linear forms in n variables with coefficients in L. Recall that $K = k(t)$ and S a finite set of valuations of K containing ν_∞. For each $\nu \in S$, let $m_{1\nu}, \ldots, m_{n\nu}$ be linearly independent linear forms from $K_\nu[X_1, \ldots, X_n]^{\text{lin}}$ and define

$$\mathcal{C}_\nu = \{\mathbf{x} \in K_\nu^n : \max_{1\leqslant i\leqslant n} |m_{i\nu}(\mathbf{x})|_\nu \leqslant 1\}.$$

By Lemma 3.1.2, this is indeed a symmetric ν-adic convex body. Then for $\mathcal{C} = \prod_{\nu\in S} \mathcal{C}_\nu$, Theorem 3.1.8 gives the equality

$$\prod_{i=1}^{n} \lambda_i = \prod_{\nu\in S} |\det(m_{1\nu}, \ldots, m_{n\nu})|_\nu,$$

with $\lambda_i, i = 1, \ldots, n$ the successive minima of $\prod_{\nu \in S} \mathcal{C}_\nu$. We may generalize this result as follows.

Let S be a finite set of valuations of K containing the infinite valuation ∞. For every $\nu \in S$, $|\cdot|_\nu$ has a unique extension to the algebraic closure $\overline{K_\nu}$. Let $l_{1\nu}, \ldots, l_{m\nu}, m \geqslant n$ be a set of linear forms in $\overline{K_\nu}[X_1, \ldots, X_n]^{\mathrm{lin}}$, with rank n. Let $\mathcal{C}_\nu = \{\mathbf{x} \in K_\nu^n : |l_{i\nu}(\mathbf{x})|_\nu \leqslant 1\, (1 \leqslant i \leqslant m)\}$ and $\mathcal{C} = \prod_{\nu \in S} \mathcal{C}_\nu$. Since $\mathrm{rank}(l_{1\nu}, \ldots, l_{m\nu}) = n$, $\mathcal{C}$ is indeed a convex symmetric body. We say that $\{l_{1\nu}, \ldots, l_{m\nu}\}$ is $\mathrm{Gal}(\overline{K_\nu}/K_\nu)$*-symmetric*, if for every $\sigma \in \mathrm{Gal}(\overline{K_\nu}/K_\nu)$, the linear forms $\sigma(l_{1\nu}), \ldots, \sigma(l_{m\nu})$ are a permutation of $l_{1\nu}, \ldots, l_{m\nu}$. With this setting, we have the following result, which is a function field analogue of a result from the geometry of numbers over number fields by Evertse [9].

Theorem 3.2.1. *Let $\mathcal{L}_\nu = \{l_{1\nu}, \ldots, l_{m\nu}\} \subset \overline{K_\nu}[X_1, \ldots, X_n]$ be a $\mathrm{Gal}(\overline{K_\nu}/K_\nu)$-symmetric set of linear forms of rank n for each $\nu \in S$. Let $\lambda_1, \ldots, \lambda_n$ be the successive minima of $\mathcal{C}$. Then*

$$\prod_{i=1}^{n} \lambda_i \geqslant \prod_{\nu \in S} \max_{1 \leqslant i_1 < \cdots < i_n \leqslant m} |\det(l_{i_1\nu}, \ldots, l_{i_n\nu})|_\nu,$$

$$\prod_{i=1}^{n} \lambda_i \leqslant e^{(m-1)n\#S/2} \prod_{\nu \in S} \max_{1 \leqslant i_1 < \cdots < i_n \leqslant m} |\det(l_{i_1\nu}, \ldots, l_{i_n\nu})|_\nu.$$

Proof. Let ν in S and define $G_\nu := \mathrm{Gal}(\overline{K_\nu}/K_\nu)$. Since $\mathcal{L}_\nu = \{l_{1\nu}, \ldots, l_{m\nu}\}$ is $\mathrm{Gal}(\overline{K_\nu}/K_\nu)$-symmetric, we have an action of G_ν on $\mathcal{L}_\nu$. Consider the G_ν-orbits and without loss of generality, assume that $l_{1\nu}, \ldots, l_{r\nu}$ are representatives for the orbits. Let $K_{i\nu}$ be the field over K_ν generated by the coefficients of $l_{i\nu}$, and $\sigma_{i\nu}^{(1)}, \ldots, \sigma_{i\nu}^{(m_{i\nu})}$ the K_ν-isomorphic embeddings of $K_{i\nu}$ into $\overline{K_\nu}$, where $m_{i\nu} = [K_{i\nu} : K_\nu]$. Then it is clear that

$$\mathcal{L}_\nu = \bigcup_{i=1}^{r} \{\sigma_{i\nu}^{(1)}(l_{i\nu}), \ldots, \sigma_{i\nu}^{(m_{i\nu})}(l_{i\nu})\}. \tag{3.2.1}$$

This implies

$$\mathcal{C}_\nu = \{\mathbf{x} \in K_\nu^n : |l_{i\nu}(\mathbf{x})|_\nu \leqslant 1\, (1 \leqslant i \leqslant r)\}.$$

Let $\mathcal{O}_{i\nu}$ be the integral closure of R_ν in $K_{i\nu}$. Then it is a free R_ν-module of rank $[K_{i\nu} : K_\nu]$ (see [22], Chap. II, Prop. 3). Let $\omega_{i\nu}^{(1)}, \ldots, \omega_{i\nu}^{(m_{i\nu})}$ be an R_ν-basis of $\mathcal{O}_{i\nu}$. Then it is also a K_ν-basis of $K_{i\nu}$. Hence we may write

$$l_{i\nu} = \sum_{j=1}^{m_{i\nu}} \omega_{i\nu}^{(j)} M_{i\nu}^{(j)}, \tag{3.2.2}$$

where $M_{i\nu}^{(j)} \in K_\nu[X_1, \ldots, X_n]^{\mathrm{lin}}$. By the choice of our R_ν-basis, it is easy to see that for $y = \sum_{j=1}^{m_{i\nu}} \omega_{i\nu}^{(j)} x_j$, with $x_j \in K_\nu$, we have $|y|_\nu \leqslant 1$ if and only if $|x_j|_\nu \leqslant 1$ for $j = 1, \ldots, m_{i\nu}$. Hence $|l_{i\nu}(\mathbf{x})|_\nu \leqslant 1$ if and only if $|M_{i\nu}^{(j)}(\mathbf{x})|_\nu \leqslant 1$ for $j = 1, \ldots, m_{i\nu}$, and therefore

$$\mathcal{C}_\nu = \{\mathbf{x} \in K_\nu^n : |M_{i\nu}^{(j)}(\mathbf{x})|_\nu \leqslant 1 \, (1 \leqslant i \leqslant r, 1 \leqslant j \leqslant m_{i\nu})\}.$$

By (3.2.1), we have $\sum_{i=1}^{r} m_{i\nu} = m$. Let $\{M_{1\nu}, \ldots, M_{m\nu}\}$ be the linear forms $M_{i\nu}^{(j)} (1 \leqslant i \leqslant r, 1 \leqslant j \leqslant m_{i\nu})$ in some order. Then

$$\mathcal{C}_\nu = \{\mathbf{x} \in K_\nu^n : |M_{i\nu}(\mathbf{x})|_\nu \leqslant 1 \, (1 \leqslant i \leqslant m)\}.$$

Without loss of generality, we may assume that for each $\nu \in S$,

$$|\det(M_{1\nu}, \ldots, M_{n\nu})|_\nu = \max_{1 \leqslant i_1 < \cdots < i_n \leqslant m} |\det(M_{i_1\nu}, \ldots, M_{i_n\nu})|_\nu.$$

By Lemma 3.1.2, we must have $|\det(M_{1\nu}, \ldots, M_{n\nu})|_\nu > 0$, so we may write $M_{j\nu} = \sum_{h=1}^{n} \xi_{jh} M_{h\nu}$, with $\xi_{jh} \in K_\nu$ for all $j = 1, \ldots, m$. By Cramer's rule, we have $\xi_{jh} = \frac{\det(M_{1\nu}, \ldots, M_{j\nu}, \ldots, M_{n\nu})}{\det(M_{1\nu}, \ldots, M_{h\nu}, \ldots, M_{n\nu})}$ and hence $|\xi_{jh}|_\nu \leqslant 1$. By the ultrametric inequality, we have $|M_{j\nu}(\mathbf{x})|_\nu \leqslant \max_{1 \leqslant i \leqslant n} |M_{i\nu}(\mathbf{x})|_\nu$ for every $\mathbf{x} \in K_\nu^n$. Therefore, we have

$$\mathcal{C}_\nu = \{\mathbf{x} \in K_\nu^n : |M_{i\nu}(\mathbf{x})|_\nu \leqslant 1 \, (1 \leqslant i \leqslant n)\}.$$

By Theorem 3.1.8, we have for the successive minima of $\mathcal{C}$

$$\begin{aligned} \prod_{i=1}^{n} \lambda_i &= \prod_{\nu \in S} |\det(M_{1\nu}, \ldots, M_{n\nu})|_\nu \\ &= \prod_{\nu \in S} \max_{1 \leqslant i_1 < \cdots < i_n \leqslant m} |\det(M_{i_1\nu}, \ldots, M_{i_n\nu})|_\nu. \end{aligned} \tag{3.2.3}$$

By applying $\sigma_{i\nu}^{(1)}, \ldots, \sigma_{i\nu}^{(m_{i\nu})} : K_{i\nu} \hookrightarrow \overline{K_\nu}$ to (3.2.2), we infer that

$$\sigma_{i\nu}^{(h)}(l_{i\nu}) = \sum_{j=1}^{m_{i\nu}} \sigma_{i\nu}^{(h)}(\omega_{i\nu}^{(j)}) M_{i\nu}^{(j)}, 1 \leqslant h \leqslant m_{i\nu}, 1 \leqslant i \leqslant r.$$

In view of (3.2.1) we may write this in a matrix form

$$\begin{pmatrix} l_{1\nu} \\ \vdots \\ l_{m\nu} \end{pmatrix} = \Omega_\nu \begin{pmatrix} M_{1\nu}^{(1)} \\ \vdots \\ M_{r\nu}^{(m_{r\nu})} \end{pmatrix},$$

or simply $\mathcal{L}_\nu = \Omega_\nu \mathcal{M}_\nu$, where $\Omega_\nu = \mathrm{diag}(B_{1\nu}, \ldots, B_{r\nu})$ is a block matrix with $B_{i\nu} = (\sigma_{i\nu}^{(h)}(\omega_{i\nu}^{(j)}))_{h,j}$. Since $\omega_{i\nu}^{(1)}, \ldots, \omega_{i\nu}^{(m_{i\nu})}$ is an R_ν-basis of $\mathcal{O}_{i\nu}$, and integral over R_ν, we know that their conjugates $\sigma_{i\nu}^{(h)}(\omega_{i\nu}^{(j)})$ are also integral over R_ν, and moreover that every matrix $B_{i\nu}$ is invertible. Further, every entry of Ω_ν^{-1} is of the form $\frac{\mu}{\det \Omega_\nu}$ with $|\mu|_\nu \leqslant 1$.

Now we have $|\det \Omega_\nu|_\nu = \prod_{i=1}^{r} |\det B_{i\nu}|_\nu$ and as is well known, $(\det B_{i\nu})^2$ generates the ideal $D_{K_{i\nu}/K_\nu}$, where $D_{K_{i\nu}/K_\nu}$ is the local discriminant of $K_{i\nu}/K_\nu$. Recall that K_ν is complete, hence there is exactly one valuation $V_{i\nu}$ on $K_{i\nu}$ above ν, with ramification index $e_{i\nu} = m_{i\nu}$. By Lemma 1.2.3,

$$2 \cdot \nu(\det B_{i\nu}) = e_{i\nu} - 1.$$

We deduce that

$$\prod_{i=1}^{r} |\det B_{i\nu}|_\nu = e^{-\sum_{i=1}^{r}(e_{i\nu}-1)/2} = e^{\frac{-m+r}{2}} \geqslant e^{\frac{-m+1}{2}}.$$

Hence $\Omega_\nu^{-1} = (\omega_\nu^{ij})_{i,j}$ with $|\omega_\nu^{ij}|_\nu \leqslant e^{\frac{m-1}{2}}$. From $\mathcal{M}_\nu = \Omega_\nu^{-1}\mathcal{L}_\nu$ we know that each $M_{i\nu}$ is a linear combination of the linear forms $l_{i\nu}$ with coefficients whose $|\cdot|_\nu$-value is at most $e^{(m-1)/2}$. Combining this with (3.2.3), and applying the Cauchy-Binet formula from linear algebra which is valid over any field, we conclude that

$$\prod_{i=1}^{n} \lambda_i \leqslant e^{(m-1)n\#S/2} \prod_{\nu \in S} \max_{1 \leqslant i_1 < \cdots < i_n \leqslant m} |\det(l_{i_1\nu}, \ldots, l_{i_n\nu})|_\nu.$$

On the other hand, each entry of $\mathcal{M}_\nu$ has $|\cdot|_\nu$-value no more than 1, hence similarly as above, we have

$$\max_{1\leqslant i_1<\cdots<i_n\leqslant m} |\det(l_{i_1\nu},\ldots,l_{i_n\nu})|_\nu \leqslant \max_{1\leqslant i_1<\cdots<i_n\leqslant m} |\det(M_{i_1\nu},\ldots,M_{i_n\nu})|_\nu,$$

which combined with (3.2.3) gives

$$\prod_{i=1}^n \lambda_i \geqslant \prod_{\nu\in S} \max_{1\leqslant i_1<\cdots<i_n\leqslant m} |\det(l_{i_1\nu},\ldots,l_{i_n\nu})|_\nu.$$

□

Remark 3.2.2. *The proof of Theorem 3.2.1 remains valid if for $\nu \in S$, we take sets of linear forms $\{l_{1\nu},\ldots,l_{m(\nu),\nu}\}$ of different cardinalities $m(\nu) \geqslant n$, and different numbers $r(\nu)$ of G_ν-orbits. In that case, our estimate for $\prod_{i=1}^n \lambda_i$ becomes*

$$\prod_{i=1}^n \lambda_i \leqslant \prod_{\nu\in S} e^{n(m(\nu)-r(\nu))/2} \max_{i_1,\ldots,i_n} |\det(l_{i_1\nu},\ldots,l_{i_n\nu})|_\nu.$$

Now let L be a finite extension of K of degree m and of genus g_L.

Lemma 3.2.3. *Let $c = e^{2g_L}$. Then for every tuple $(\alpha_\omega : \omega \in M_L)$ such that*

$$\alpha_\omega \in e^{\mathbb{Z}} \text{ for } \omega \in M_L, \alpha_\omega = 1 \text{ for almost all } \omega, \prod_{\omega\in M_L} \alpha_\omega \geqslant c,$$

there is an $x \in L^*$ such that $|x|_\omega \leqslant \alpha_\omega$ for all $\omega \in M_L$.

Proof. Let $\alpha_\omega = e^{r_\omega}$ for $\omega \in M_L$ with $r_\omega \in \mathbb{Z}$ and $r_\omega = 0$ for almost all ω. Consider the divisor $D = \sum_{\omega\in M_L} r_\omega \omega$. By the Riemann-Roch theorem, if $\deg D = \sum_{\omega\in M_L} r_\omega \geqslant 2g_L$, then the dimension $\dim_k\{x \in L : \omega(x) \geqslant -r_\omega \text{ for } \omega \in M_L \text{ or } x = 0\}$ is positive, hence there exists $x \in L^*$ such that $\omega(x) \geqslant -r_\omega$, i.e., $|x|_\omega \leqslant \alpha_\omega$ for all $\omega \in M_L$. □