Author: Zhuang, Weidong
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Chapter 2

Height estimates for solutions of S-unit equations

Let $|\cdot|_\infty$ denote the ordinary absolute value on $\mathbb{Q}$ and for a prime p, denote by $|\cdot|_p$ the p-adic absolute value, normalized such that $|p|_p = p^{-1}$. Let K be a number field and M_K its collection of places (equivalence classes of absolute values). For every $\nu \in M_K$, choose $|\cdot|_\nu$ from ν such that if ν lies above $p \in \{\infty\} \cup \{\text{primes}\}$. Then $|x|_\nu = |x|_p^{[K_\nu:\mathbb{Q}_p]}$ for $x \in \mathbb{Q}$.

We recall the Subspace Theorem, due to Schmidt and Schlickewei.

For $\mathfrak{X} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(K)$, define $|\mathfrak{X}|_\nu := \max(|x_1|_\nu, \ldots, |x_n|_\nu)$ for $\nu \in M_K$ and $H_K(\mathfrak{X}) = \prod_{\nu \in M_K} |\mathfrak{X}|_\nu$.

Subspace Theorem. *Let $n \geqslant 1$, and let S be a finite set of places of K. For $\nu \in S$, let $L_{0\nu}, \ldots, L_{n\nu}$ be linearly independent linear forms with coefficients in K. Further, let $C > 0, \delta > 0$. Then the set of solutions of the inequality*

$$\prod_{\nu \in S} \frac{|L_{0\nu}(\mathfrak{X}) \ldots L_{n\nu}(\mathfrak{X})|_\nu}{|\mathfrak{X}|_\nu^{n+1}} \leqslant C H_K(\mathfrak{X})^{-n-1-\delta}$$

in $\mathfrak{X} \in \mathbb{P}^n(K)$ is contained in a finite union of proper linear subspaces of $\mathbb{P}^n(K)$.

This was proved by Schmidt in [20], [21] in the case that S contains only archimedean places, and by Schlickewei [19] in full generality.

As a consequence, in [9] Evertse derived the following result.

Let S be a finite set of places of K containing all archimedean places. Define the ring of S-integers $\mathcal{O}_S = \{x \in K : |x|_\nu \leqslant 1 \text{ for } \nu \notin S\}$. Define

$$|x|_S := \prod_{\nu \in S} |x|_\nu \text{ for } x \in \mathcal{O}_S,$$

$$H_S(x_1, \dots, x_n) := \prod_{\nu \in S} \max(|x_1|_\nu, \dots, |x_n|_\nu) \text{ for } x_1, \dots, x_n \in \mathcal{O}_S.$$

Theorem (Evertse)**.** *Let K be an algebraic number field and S a finite set of valuations of K containing those archimedean ones. Assume $x_1, \dots, x_n \in \mathcal{O}_S$ such that $\sum_{i=1}^{n} x_i = 0$ but no non-empty proper subsum vanishes. Then for every $\varepsilon > 0$ we have*

$$H_S(x_1, \dots, x_n) < C(n, \varepsilon, S) |\prod_{i=1}^{n} x_i|_S^{1+\varepsilon}.$$

Here $C(n, \varepsilon, S)$ is an ineffective constant. In this chapter, we are going to prove a much stronger analogue of this result over function fields.

2.1 Height estimates

Let $K = k(t)$, L a finite extension of K. For $x_1, \dots, x_n \in L$, define

$$H_L(x_1, \dots, x_n) = \prod_{\omega \in M_L} \max(|x_1|_\omega, \dots, |x_n|_\omega),$$

$$H_L^*(x_1, \dots, x_n) = \prod_{\omega \in M_L} \max(1, |x_1|_\omega, \dots, |x_n|_\omega).$$

$$H(x_1, \dots, x_n) = \prod_{\omega \in M_L} \max(|x_1|_\omega, \dots, |x_n|_\omega)^{1/[L:K]},$$

$$H^*(x_1, \dots, x_n) = \prod_{\omega \in M_L} \max(1, |x_1|_\omega, \dots, |x_n|_\omega)^{1/[L:K]}.$$

For a finite set $T \subset M_L$, define

$$H_T(x_1, \dots, x_n) = \prod_{\omega \in T} \max(|x_1|_\omega, \dots, |x_n|_\omega).$$

Lemma 2.1.1 (Mason)**.** *Let L be a finite extension of $K = k(t)$, and T a finite set of valuations of L. Let $\gamma_1, \gamma_2, \gamma_3$ be non-zero elements of L satisfying $\gamma_1 + \gamma_2 + \gamma_3 = 0$ and $\nu(\gamma_1) = \nu(\gamma_2) = \nu(\gamma_3)$ for every valuation $\nu \notin T$. Then either $\frac{\gamma_1}{\gamma_2} \in k$, which means $H^*(\frac{\gamma_1}{\gamma_2}) = 1$, or $H^*(\frac{\gamma_1}{\gamma_2}) \leqslant e^{(\#T+2g_L-2)/[L:K]}$.*

Proof. See Chapter I, Lemma 2 of [17]. □

Corollary 2.1.2. *With the above notation, we have in both the cases $\frac{\gamma_1}{\gamma_2} \in k, \frac{\gamma_1}{\gamma_2} \notin k$ that*

$$H^*(\frac{\gamma_1}{\gamma_2}) \leqslant e^{(\#T+2g_L-1)/[L:K]}.$$

Proof. This follows directly from the facts that $g_L \geqslant 0$ and $\#T \geqslant 1$. □

Recall

$$\mathcal{O}_T := \{x \in L : |x|_\omega \leqslant 1 \text{ for } \omega \notin T\},$$

$$\mathcal{O}_T^\times := \{x \in L : |x|_\omega = 1 \text{ for } \omega \notin T\}.$$

Note that by the product formula, we have

$$\begin{aligned} H^*(\frac{\gamma_1}{\gamma_2}) &= \Big(\prod_{\omega \in M_L} \max(1, \left|\frac{\gamma_1}{\gamma_2}\right|_\omega)\Big)^{1/[L:K]} \\ &= \Big(\prod_{\omega \in M_L} \max(|\gamma_1|_\omega, |\gamma_2|_\omega)\Big)^{1/[L:K]} = H(\gamma_1, \gamma_2), \end{aligned}$$

and if $\gamma_1, \gamma_2 \in \mathcal{O}_T$, then $H(\gamma_1, \gamma_2)^{[L:K]} \leqslant H_T(\gamma_1, \gamma_2)$.

Brownawell and Masser obtained the following generalization:

Theorem 2.1.3. *Let L be a finite extension of $K = k(t)$, and T a finite set of valuations of L. Put $g' = \max(0, 2g-2)$. Let $u_1, \ldots, u_n$ be T-units in L satisfying $u_1 + \cdots + u_n = 0$ but $\sum_{i \in I} u_i \neq 0$ for every non-empty proper subset I of $\{1, \ldots, n\}$. Then*

$$H(u_1, \ldots, u_n) \leqslant e^{\frac{1}{2}(n-1)(n-2)(\#T+g')/[L:K]}.$$

Proof. See [6]. □

We deduce the following result, which will be improved in the next section.

Corollary 2.1.4. *Let L be a finite extension of $K = k(t)$, and T a finite set of valuations of L. Put $g' = \max(0, 2g-2)$. Let $u_1, \ldots, u_n$ be elements of $\mathcal{O}_T$ satisfying $u_1 + \cdots + u_n = 0$ but $\sum_{i \in I} u_i \neq 0$ for every non-empty proper subset I of $\{1, \ldots, n\}$. Then*

$$H_T(u_1, \ldots, u_n) \leqslant e^{\frac{1}{2}(n-1)(n-2)(\#T+g')} \Big|\prod_{i=1}^{n} u_i\Big|_T^{\frac{(n-1)(n-2)}{2}}.$$

Proof. Let U be the collection of $\omega \in M_L \backslash T$ such that $\omega(u_i)$, $i = 1, \ldots, n$, are not all equal. Then clearly $\#U < \infty$.

Now consider the complement of $T \cup U$. For every $\omega \notin T \cup U$, we have $\omega(u_1) = \cdots = \omega(u_n)$. Since $u_i \in \mathcal{O}_T$, there are two cases: either $\omega(u_i) = 0$, which is the case for almost all valuations, or $\omega(u_i) > 0$. Let $V = \{\omega \notin T \cup U : \omega(u_1) = \cdots = \omega(u_n) > 0\}$.

If $V = \emptyset$, then by Theorem 2.1.3, we have $H(u_1, \ldots, u_n) \leqslant e^{\frac{(n-1)(n-2)(\#T+\#U+g')}{2[L:K]}}$.

If $V \neq \emptyset$, then $\frac{u_1}{u_n} + \cdots + \frac{u_{n-1}}{u_n} + 1 = 0$ and each nontrivial partial sum is non-zero by assumption. As $\frac{u_i}{u_n}, i = 1, \ldots, n-1$, and 1 are all elements of $\mathcal{O}^*_{T \cup U}$, and the height function H is projective, we obtain by Theorem 2.1.3

$$H(u_1, \ldots, u_n) = H\Big(\frac{u_1}{u_n}, \ldots, \frac{u_{n-1}}{u_n}, 1\Big) \leqslant e^{\frac{(n-1)(n-2)(\#T+\#U+g')}{2[L:K]}}. \tag{2.1.1}$$

On the other hand, since $u_i \in \mathcal{O}_T$ for $i = 1, \ldots, n$, we have $\max_{1 \leqslant i \leqslant n} |u_i|_\omega \leqslant 1$, hence $\min_{1 \leqslant i \leqslant n} |u_i|_\omega \leqslant e^{-1}$ for $\omega \in U$, and therefore

$$e^{\#U} \leqslant \prod_{\omega \notin T} \frac{1}{\min_{1 \leqslant i \leqslant n} |u_i|_\omega}. \tag{2.1.2}$$

Combining (2.1.1) with (2.1.2) we derive that

$$\begin{aligned} H_T(u_1, \ldots, u_n) &\leqslant e^{\frac{1}{2}(n-1)(n-2)(\#T+g')} \Big(\prod_{\omega \notin T} \frac{1}{\min_{1 \leqslant i \leqslant n} |u_i|_\omega}\Big)^{\frac{(n-1)(n-2)}{2}} \prod_{\omega \notin T} \frac{1}{\max_{1 \leqslant i \leqslant n} |u_i|_\omega} \\ &\leqslant e^{\frac{1}{2}(n-1)(n-2)(\#T+g')} \prod_{\omega \notin T} \prod_{i=1}^{n} |u_i|_\omega^{-\frac{(n-1)(n-2)}{2}} \\ &= e^{\frac{1}{2}(n-1)(n-2)(\#T+g')} \Big|\prod_{i=1}^{n} u_i\Big|_T^{\frac{(n-1)(n-2)}{2}}, \end{aligned}$$

as claimed. □

2.2 S-unit equations and heights

Actually, from an effective version of the subspace theorem over function fields, we can deduce better results.

The following theorem is originally stated in terms of additive heights and over function fields K_1 associated to arbitrary nonsingular varieties. We restate it in our notation in the special case for curves, i.e., for function fields of transcendence degree 1. For $n \in \mathbb{Z}_{\geqslant 1}$, put

$$C(n) = e^{\binom{n}{2}(2g_{K_1}-2+\#S_1)}, C'(n) = e^{\binom{n}{2}\max(0,2g_{K_1}-2+\#S_1)}.$$

Theorem 2.2.1. *Let K_1 be a finite extension of $K = k(t)$ and $L_1, \ldots, L_q$ hyperplanes in $\mathbb{P}^N(K_1)$ defined by linear forms with coefficients in k. Let $S_1 \subset M_{K_1}$ be a finite set of valuations. If the coordinates of $\mathfrak{X} = [x_0 : \cdots : x_N] \in \mathbb{P}^N(K_1)$ are linearly independent over k, then*

$$\prod_{\nu \in S_1} \min_J \prod_{j \in J} \frac{|L_j(\mathfrak{X})|_\nu}{|\mathfrak{X}|_\nu} \geqslant C(N+1)^{-1} H(\mathfrak{X})^{-(N+1)[K_1:k(t)]},$$

where the minimum is taken over all subsets J of $\{1, \ldots, q\}$ such that the linear forms L_j $(j \in J)$ are linearly independent.

Proof. See Theorem 1 of [25]. □

Corollary 2.2.2. *Assume $x_1, \ldots, x_n \in \mathcal{O}_{S_1}$ are k-linearly independent. Then*

$$\left(\prod_{i=1}^{n} |x_i|_{S_1}\right) |x_1 + \cdots + x_n|_{S_1} \geqslant C(n)^{-1} H_{S_1}(x_1, \ldots, x_n).$$

Proof. We apply Theorem 2.2.1 with $N = n-1, \mathfrak{X} = [x_1 : \cdots : x_n], L_i = x_i\,(i = 1, \ldots, n), L_{n+1} = x_1 + \cdots + x_n$.

For each $\nu \in S_1$, choose $t(\nu) \in \{1, \ldots, n\}$ such that $|x_{t(\nu)}|_\nu = \max_{1 \leqslant i \leqslant n}(|x_i|_\nu)$, and take $J(\nu) = \{1, \ldots, n+1\} \backslash \{t(\nu)\}$. Then

$$\prod_{\nu \in S_1} \prod_{j \in J(\nu)} \frac{|L_j(\mathfrak{X})|_\nu}{|\mathfrak{X}|_\nu} \geqslant C(n)^{-1} H(\mathfrak{X})^{-n[K_1:k(t)]},$$

hence as $x_i \in \mathcal{O}_{S_1}$,

$$\frac{\left(\prod_{i=1}^{n} |x_i|_{S_1}|\right) |x_1 + \cdots + x_n|_{S_1}}{H_{S_1}(\mathfrak{X})^{n+1}} \geqslant C(n)^{-1} H_{S_1}(\mathfrak{X})^{-n}.$$

This completes the proof. □

Actually, the condition that $x_1, \ldots, x_n$ be k-linearly independent can be relaxed to the condition that $x_1 + \cdots + x_n$ have no vanishing subsum.

Corollary 2.2.3. *Let $x_1, \ldots, x_n \in \mathcal{O}_{S_1}$ such that $\sum_{i \in I} x_i \neq 0$ for any non-empty subset $I \subset \{1, \ldots, n\}$. Let S_1 be a finite subset of M_{K_1}, T a subset of S_1. Then*

$$\prod_{i=1}^{n} |x_i|_{S_1} |x_1 + \cdots + x_n|_T \geqslant C'(n)^{-1} H_T(x_1, \ldots, x_n).$$

Proof. We proceed by induction on n. For $n = 1$ the assertion is trivial since $x_1 \in \mathcal{O}_{S_1}$. Let $N \geqslant 2$ and assume the assertion is true for $n < N$. We now consider the case $n = N$. Since each $\nu \in S_1$ is non-archimedean, i.e., $|x_1 + \cdots + x_n|_\nu \leqslant \max_{1 \leqslant i \leqslant n} |x_i|_\nu$, it suffices to deal with the special case $T = S_1$.

First suppose that $x_1, \ldots, x_N$ are k-linearly independent. Then the assertion is true by Corollary 2.2.2. Next assume that $\operatorname{rank}_k\{x_1, \ldots, x_N\} < N$. Then, possibly after rearranging the indices, we may assume that $x_1 + \cdots + x_N = a_1 x_1 + \cdots + a_u x_u$, where $1 \leqslant u < N$, $a_1, \ldots, a_u \in k^*$ and u is minimal with this property. Then $x_1, \ldots, x_u$ are k-linearly independent and no subsum of the right-hand side is 0. Partition S_1 into two subsets

$$S^{(1)} = \{\nu \in S_1 : \max_{1 \leqslant i \leqslant N}(|x_i|_\nu) = \max_{1 \leqslant i \leqslant u}(|x_i|_\nu)\},$$

$$S^{(2)} = \{\nu \in S_1 : \max_{1 \leqslant i \leqslant N}(|x_i|_\nu) > \max_{1 \leqslant i \leqslant u}(|x_i|_\nu)\}.$$

Then we have $x_{u+1} + \cdots + x_N = (a_1 - 1)x_1 + \cdots + (a_u - 1)x_u$ and hence $|x_{u+1} + \cdots + x_N|_\nu \leqslant \max_{1 \leqslant i \leqslant u}(|x_i|_\nu)$ for $\nu \in S^{(2)}$. Now we have

$$\Big(\prod_{i=1}^{N} |x_i|_{S_1}\Big) |x_1 + \cdots + x_N|_{S_1} = \Big(\prod_{i=1}^{u} |a_i x_i|_{S_1} |a_1 x_1 + \cdots + a_u x_u|_{S_1}\Big) \prod_{i=u+1}^{N} |x_i|_{S_1}.$$

Combining this with the induction hypothesis, we derive that

$$
\begin{aligned}
&\Big(\prod_{i=1}^{N}|x_i|_{S_1}\Big)|x_1+\cdots+x_N|_{S_1}\\
\geqslant\ & C(u)^{-1}H_{S_1}(a_1x_1,\ldots,a_ux_u)\prod_{i=u+1}^{N}|x_i|_{S_1}\\
=\ & C(u)^{-1}H_{S^{(1)}}(x_1,\ldots,x_u)H_{S^{(2)}}(x_1,\ldots,x_u)\prod_{i=u+1}^{N}|x_i|_{S_1}\\
\geqslant\ & C(u)^{-1}H_{S^{(1)}}(x_1,\ldots,x_N)|x_{u+1}+\cdots+x_N|_{S^{(2)}}\prod_{i=u+1}^{N}|x_i|_{S_1}\\
\geqslant\ & C(u)^{-1}H_{S^{(1)}}(x_1,\ldots,x_N)C'(N-u)^{-1}H_{S^{(2)}}(x_{u+1},\ldots,u_N)\\
\geqslant\ & C'(N)^{-1}H_{S_1}(x_1,\ldots,x_N),
\end{aligned}
$$

which completes the induction step. □

With the help of two lemmas stated below and a similar idea as in the proof of Theorem 1, [26], we obtain a generalization of Theorem 1 of [26]. The following two lemmas are from [25], which deals with a more general case. We restate and prove the lemmas in our specific case. Recall that for every $z \in K_1\backslash k$ we have a derivation d/dz. For each valuation $\nu \in M_{K_1}$, we choose a local parameter $\xi = \xi_\nu$ with $\nu(\xi) = 1$. Then we have another corresponding derivation $d/d\xi$.

Let $f_1, \ldots, f_n \in K_1$ be k-linearly independent. Define the Wronskian related to z as $W = W_z(f_1, \ldots, f_n) := \det\big((d/dz)^{j-1}f_i\big)_{1\leqslant i,j\leqslant n}$. Then it is well-known that $W \neq 0$.

Lemma 2.2.4. *(i) For $h \in K_1$, we have*

$$W_z(hf_1,\ldots,hf_n) = h^n W_z(f_1,\ldots,f_n).$$

(ii) For any $\xi \in K_1\backslash k$ we have

$$W_z(f_1,\ldots,f_n) = \det\Big((\frac{d\xi}{dz})^{j-1}\cdot(d/d\xi)^{j-1}f_i\Big)_{1\leqslant i,j\leqslant n}.$$

Proof. For (i), assume $h \neq 0$, otherwise it is trivial. By the Leibniz rule for derivatives, we have for each $1 \leqslant i \leqslant n$ that

$$(d/dz)^m(hf_i) - h(d/dz)^m f_i = \sum_{l=0}^{m-1} \binom{m}{l} (d/dz)^{m-l}h \cdot (d/dz)^l f_i,$$

is a K_1-linear combination of $h(d/dz)^l f_i, 0 \leqslant l \leqslant m-1$. The determinant remains unchanged if we recursively replace the j-th column by $(h(d/dz)^{j-1}f_1, \ldots, h(d/dz)^{j-1}f_n)^{\mathrm{T}}$ for $j = 2, \ldots, n$. Then the assertion follows immediately.

For (ii), we will prove by induction that $(d/dz)^m f_i - (\frac{d\xi}{dz})^m \cdot (d/d\xi)^m f_i$ is a K_1-linear combination, independent of i, of $(d/d\xi)f_i, \ldots, (d/d\xi)^{m-1}f_i$. Then the assertion is clear for the same reason as in (i). By the chain rule, we know

$$(d/dz)f_i = \frac{d\xi}{dz} \cdot (d/d\xi)f_i,$$

$$(d/dz)^2 f_i = (d/dz)(\frac{d\xi}{dz}) \cdot (d/d\xi)f_i + (\frac{d\xi}{dz})^2 \cdot (d/d\xi)^2 f_i.$$

Let $m \geqslant 3$ and assume our assertion is true for $m-1$, i.e.,

$$(d/dz)^{m-1}f_i - (\frac{d\xi}{dz})^{m-1}(d/d\xi)^{m-1}f_i = \sum_{j=1}^{m-2} g_j (d/d\xi)^j f_i,$$

with each $g_j \in K_1$. Put $g_0 = 0$. Then by the chain rule, we have

$$\begin{aligned}(d/dz)^m f_i - (\frac{d\xi}{dz})^m (d/d\xi)^m f_i &= \sum_{j=1}^{m-2} \big((d/dz)g_j + g_{j-1}\frac{d\xi}{dz}\big)(d/d\xi)^j f_i \\ &+ \big((d/dz)(\frac{d\xi}{dz})^{m-1} + g_{m-2}\frac{d\xi}{dz}\big)(d/d\xi)^{m-1} f_i.\end{aligned}$$

This completes the induction and hence the proof. □

Lemma 2.2.5. *For every $\nu \in M_{K_1}$, we have*

$$\nu(W) + \binom{n}{2}\nu(\frac{dz}{d\xi_\nu}) \geqslant n \min_{1 \leqslant i \leqslant n} \nu(f_i),$$

where ξ_ν is a local parameter of ν.

Proof. Let $\nu \in M_{K_1}$ and a local parameter ξ_ν of ν. For k-linearly dependent $f_1, \ldots, f_n$ we have $W = 0$ and the assertion is clear. Assume that $f_1, \ldots, f_n$ are k-linearly independent. Let $m = -\min\limits_{1\leqslant i\leqslant n} \nu(f_i), l = -\nu(\frac{d\xi_\nu}{dz})$, and put $g_i = f_i\xi_\nu^m$. By Lemma 2.2.4 we have

$$\begin{aligned} W &= W_z(f_1, \ldots, f_n) \\ &= \xi_\nu^{-nm} \cdot W_z(g_1, \ldots, g_n) \\ &= \xi_\nu^{-nm} \cdot \det\left((\frac{d\xi_\nu}{dz})^{j-1} \cdot (d/d\xi_\nu)^{j-1} g_i \right)_{1\leqslant i,j\leqslant n} \\ &= \xi_\nu^{-nm} \cdot \xi_\nu^{-l\binom{n}{2}} \cdot \det\left(\xi_\nu^{l(j-1)} (\frac{d\xi_\nu}{dz})^{j-1} \cdot (d/d\xi_\nu)^{j-1} g_i \right)_{1\leqslant i,j\leqslant n} . \end{aligned}$$

Since $\nu(\xi_\nu^{l(j-1)}(\frac{d\xi_\nu}{dz})^{j-1}) = 0$ and $\nu(g_i) = \nu(f_i)+m \geqslant 0$, we have $\nu((d/d\xi_\nu)^{j-1}g_i) \geqslant 0$, hence $\nu(W) \geqslant -nm - l\binom{n}{2}$, as claimed. □

Lemma 2.2.6. *Let $f_1, \ldots, f_n$ be k-linearly independent elements of K_1. Then for every $\nu \in M_{K_1}$, we have*

$$\nu(W) + \binom{n}{2}\nu(\frac{dz}{d\xi_\nu}) \geqslant -\binom{n}{2} + \sum_{i=1}^{n} \nu(f_i).$$

Proof. See [26] or [6]. □

Lemma 2.2.7. *Let $f_1, \ldots, f_n$ be k-linearly independent elements of K_1 and $b = \sum\limits_{i=1}^{n} f_i$. Then*

$$C(n)|b|_{S_1} \prod_{i=1}^{n} |f_i|_{S_1} \Big(\prod_{\nu\notin S_1} \max_i(|f_i|_\nu) \Big)^n \geqslant H_{S_1}(f_1, \ldots, f_n).$$

Proof. Let $\nu \in S_1$, choose $j(\nu) \in \{1, \ldots, n\}$ such that $\nu(f_{j(\nu)}) = \min\limits_i(\nu(f_i))$. Then $W_z(f_1, \ldots, f_n)$ does not change if we replace $f_{j(\nu)}$ by b. Applying Lemma 2.2.6, we get

$$\nu(W) + \binom{n}{2}\nu(\frac{dz}{d\xi_\nu}) + \binom{n}{2} \geqslant \Big(\sum_{i=1}^{n} \nu(f_i) \Big) + \nu(b) - \min_{1\leqslant i\leqslant n} (\nu(f_i)).$$

Now let $\nu \notin S_1$. Then by Lemma 2.2.5, we get

$$\nu(W) + \binom{n}{2}\nu(\frac{dz}{d\xi_\nu}) \geqslant n \min_{1\leqslant i\leqslant n} \nu(f_i).$$

Taking the sum over all $\nu \in M_{K_1}$, and noticing that $\sum_{\nu\in M_{K_1}} \nu(W) = 0$, $\sum_{\nu\in M_{K_1}} \nu(\frac{dz}{d\xi_\nu}) = 2g_{K_1}$, we deduce that

$$\begin{aligned}&\binom{n}{2}(2g_{K_1} - 2) + \binom{n}{2}\#S_1 \\ &\geqslant \sum_{\nu\in S_1}\sum_{i=1}^{n}\nu(f_i) + \sum_{\nu\in S_1}\nu(b) - \sum_{\nu\in S_1}\min_i \nu(f_i) + n\sum_{\nu\notin S_1}\min_i \nu(f_i).\end{aligned}$$

Hence

$$C(n) \geqslant \Big(|b|_{S_1}\prod_{i=1}^{n}|f_i|_{S_1}\Big)^{-1}\prod_{\nu\in S_1}\max_i(|f_i|_\nu)\Big(\prod_{\nu\notin S_1}\max_i(|f_i|_\nu)\Big)^{-n},$$

or equivalently,

$$C(n)|b|_{S_1}\prod_{i=1}^{n}|f_i|_{S_1}\Big(\prod_{\nu\notin S_1}\max_i(|f_i|_\nu)\Big)^{n} \geqslant H_{S_1}(f_1,\ldots,f_n).$$

□

Lemma 2.2.8. *Let S_1, T be as in Corollary 2.2.3. Let $x_1,\ldots,x_n \in K_1$ be such that for each non-empty subset $I \subset \{1,\ldots,n\}$, we have $\sum_{i\in I} x_i \neq 0$. Then*

$$\prod_{i=1}^{n}|x_i|_{S_1}|x_1+\cdots+x_n|_T\Big(\prod_{\nu\notin S_1}\max_i(|x_i|_\nu)\Big)^{n} \geqslant C'(n)^{-1}H_T(x_1,\ldots,x_n),$$

with $T \subset S_1$.

This is a slight generalization of Corollary 2.2.3.

Proof. Observe that $C'(h)C'(l) \leqslant C'(h+l)$ for $h, l \geqslant 0$.

We proceed by induction on n. For $n = 1$, the assertion follows trivially from the product formula. Let $N \geqslant 2$ and assume the assertion is true for

$n < N$. We prove the assertion for $n = N$. We may again assume that $T = S_1$ without loss of generality since each $\nu \in S_1$ is non-archimedean, i.e., $|x_1 + \cdots + x_n|_\nu \leqslant \max_{1\leqslant i\leqslant n} |x_i|_\nu$.

First suppose that $x_1, \ldots, x_n$ are k-linearly independent. Then the assertion is true by Lemma 2.2.7.

Now, possibly after rearranging the indices, suppose that $x_1 + \cdots + x_n = a_1x_1 + \cdots + a_ux_u$ with $1 \leqslant u < n$, $a_1, \ldots, a_u \in k^*$ where u with this property has been chosen minimally and no proper subsum of the right-hand side vanishes. Then $x_1, \ldots, x_u$ are k-linearly independent. Partition S_1 into two subsets

$$S^{(1)} = \{\nu \in S_1 : \max_{1\leqslant i\leqslant n}(|x_i|_\nu) = \max_{1\leqslant i\leqslant u}(|x_i|_\nu)\},$$
$$S^{(2)} = \{\nu \in S_1 : \max_{1\leqslant i\leqslant n}(|x_i|_\nu) > \max_{1\leqslant i\leqslant u}(|x_i|_\nu)\}.$$

Then we have $x_{u+1} + \cdots + x_n = (a_1 - 1)x_1 + \cdots + (a_u - 1)x_u$ and hence $|x_{u+1} + \cdots + x_n|_\nu \leqslant \max_{1\leqslant i\leqslant u}(|x_i|_\nu)$ for $\nu \in S^{(2)}$. So

$$\prod_{i=1}^{n} |x_i|_{S_1} |x_1 + \cdots + x_n|_{S_1} = \Big(\prod_{i=1}^{u} |a_ix_i|_{S_1} |a_1x_1 + \cdots + a_ux_u|_{S_1}\Big) \prod_{i=u+1}^{n} |x_i|_{S_1}.$$

Combining this with the induction hypothesis, we derive that

$$\begin{aligned} & \prod_{i=1}^{n} |x_i|_{S_1} |x_1 + \cdots + x_n|_{S_1} \\ = \; & \Big(\prod_{i=1}^{u} |a_ix_i|_{S_1} |a_1x_1 + \cdots + a_ux_u|_{S_1}\Big) \prod_{i=u+1}^{n} |x_i|_{S_1} \\ \geqslant \; & C'(u)^{-1} H_{S_1}(a_1x_1, \ldots, a_ux_u)\Big(\prod_{\nu\notin S_1} \max_{1\leqslant i\leqslant u}(|a_ix_i|_\nu)\Big)^{-u} \prod_{i=u+1}^{n} |x_i|_{S_1} \end{aligned}$$

From the partition of S_1, the last expression can be rewritten as

$$\begin{aligned} & C'(u)^{-1} H_{S_1}(a_1x_1, \ldots, a_ux_u)\Big(\prod_{\nu\notin S_1} \max_{1\leqslant i\leqslant u}(|a_ix_i|_\nu)\Big)^{-u} \prod_{i=u+1}^{n} |x_i|_{S_1} \\ = \; & C'(u)^{-1} H_{S^{(1)}}(x_1, \ldots, x_u) H_{S^{(2)}}(x_1, \ldots, x_u) \prod_{i=u+1}^{n} |x_i|_{S_1} \Big(\prod_{\nu\notin S_1} \max_{1\leqslant i\leqslant u}(|x_i|_\nu)\Big)^{-u}. \end{aligned}$$

Hence

$$
\begin{aligned}
&\prod_{i=1}^{n}|x_i|_{S_1}|x_1+\cdots+x_n|_{S_1}\\
\geqslant\ & C'(u)^{-1}H_{S^{(1)}}(x_1,\ldots,x_u)H_{S^{(2)}}(x_1,\ldots,x_u)\prod_{i=u+1}^{n}|x_i|_{S_1}\Big(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant u}(|x_i|_\nu)\Big)^{-u}\\
\geqslant\ & C'(u)^{-1}H_{S^{(1)}}(x_1,\ldots,x_n)\Big(|x_{u+1}+\cdots+x_n|_{S^{(2)}}\prod_{i=u+1}^{n}|x_i|_{S_1}\Big)\times\\
&\quad\times\Big(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant u}(|x_i|_\nu)\Big)^{-u}\\
\geqslant\ & C'(u)^{-1}H_{S^{(1)}}(x_1,\ldots,x_n)C'(n-u)^{-1}H_{S^{(2)}}(x_{u+1},\ldots,u_n)\times\\
&\quad\times\Big(\prod_{\nu\notin S_1}\max_{i>u}(|x_i|_\nu)\Big)^{u-n}\Big(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant u}(|x_i|_\nu)\Big)^{-u}\\
\geqslant\ & C'(n)^{-1}H_{S_1}(x_1,\ldots,x_n)\Big(\prod_{\nu\notin S_1}\max_{1\leqslant i\leqslant n}|x_i|_\nu\Big)^{-n},
\end{aligned}
$$

which completes the induction step. □

Theorem 2.2.9. *Let S_1, T be as above. Let $x_1,\ldots,x_n\in K_1$ be such that $\sum_{i\in I}x_i\neq 0$ for any non-empty subset $I\subset\{1,\ldots,n\}$. Then*

$$\prod_{i=1}^{n}|x_i|_{S_1}\cdot|x_1+\cdots+x_n|_T\cdot\Big(\prod_{\nu\notin S_1}\max_i(|x_i|_\nu)\Big)^n\geqslant C'(u)^{-1}H_T(x_1,\ldots,x_n),$$

where $u=\mathrm{rank}_k\{x_1,\ldots,x_n\}$.

This result improves Lemma 2.2.8 and is inspired by an idea of Zannier [26].

Proof. Recall that $C'(s)=e^{\frac{s(s-1)}{2}\max(2g_{K_1}-2+\#S_1,0)}$ for $s\in\mathbb{N}$. Then we have $C'(s)C'(t)\leqslant C'(s+t)$.

First notice that the special case $u=n$ is just Lemma 2.2.7.

For the general case we proceed by induction on n, the case $n=1$ being trivial. Let $N\geqslant 2$ and assume the assertion is true for all $n<N$, now consider the case $n=N$. Like in the proof of Lemma 2.2.8, we only have to

consider the special case $T = S_1$. Let $x_1, \dots, x_u$ be k-linearly independent with u maximal and assume, renumbering indices if necessary, $x_1 + \cdots + x_N = \sum_{i=1}^{v} a_i x_i$ with $a_1, \dots, a_v \in k^*$ and $1 \leqslant v \leqslant u$.

First assume $v = u$. Then each x_i is a k-linear combination of $x_1, \dots, x_u$, hence

$$\max_{1 \leqslant i \leqslant N}(|x_i|_\nu) = \max_{1 \leqslant i \leqslant u}(|x_i|_\nu). \tag{2.2.1}$$

Then by applying Lemma 2.2.8 to $a_1 x_1, \dots, a_u x_u$, $|\cdot|_\nu$ and using that $|\cdot|_\nu$ is trivial on k^* for $\nu \in M_{K_1}$, we get

$$\prod_{i=1}^{u} |x_i|_{S_1} |x_1 + \cdots + x_N|_{S_1} \Big(\prod_{\nu \notin S_1} \max_{1 \leqslant i \leqslant u}(|x_i|_\nu) \Big)^u \geqslant c(u)^{-1} H_{S_1}(x_1, \dots, x_u). \tag{2.2.2}$$

Clearly, $\max_{1 \leqslant i \leqslant N}(|x_i|_\nu) \geqslant |x_i|_\nu$ for $i > u, \nu \notin S_1$, so

$$\Big(\prod_{\nu \notin S_1} \max_{1 \leqslant i \leqslant N}(|x_i|_\nu) \Big)^{N-u} \geqslant \prod_{i>u} \prod_{\nu \notin S_1} |x_i|_\nu$$

and hence by the product formula,

$$\prod_{i>u} |x_i|_{S_1} \Big(\prod_{\nu \notin S_1} \max_{1 \leqslant i \leqslant n}(|x_i|_\nu) \Big)^{N-u} \geqslant 1. \tag{2.2.3}$$

Combining (2.2.2) with (2.2.3) we derive the assertion when $v = u$.

So we assume $1 \leqslant v < u < N$. Applying again Lemma 2.2.8 to $a_1 x_1, \dots, a_v x_v$ we get

$$\prod_{i=1}^{v} |x_i|_{S_1} |x_1 + \cdots + x_N|_{S_1} \Big(\prod_{\nu \notin S_1} \max_{1 \leqslant i \leqslant v}(|x_i|_\nu) \Big)^v \geqslant c(v)^{-1} H_{S_1}(x_1, \dots, x_v). \tag{2.2.4}$$

We claim that there exists $h < N$ such that there are two finite sequences $\{u_l\}, \{j_l\}$ of integers of the same length h satisfying the following:

(i) $u_0 = v, u_h = u, u_l > u_{l-1}$ for $1 \leqslant l \leqslant h$,

(ii) $j_0 = 0$, $u < j_l \leqslant N$ for $l \geqslant 1$, $j_s \neq j_t$ for $s \neq t$,

(iii) there is a renumbering of the indices $v+1,\dots,u$ such that for $l>0$ x_{j_l} is a k-linear combination of $x_1,\dots,x_{u_l}$ and for all $l\geqslant 0$

$$\prod_{i=0}^{l}|x_{j_i}|_{S_1}\prod_{i=1}^{u_l}|x_i|_{S_1}|x_1+\cdots+x_N|_{S_1}\Big(\prod_{\nu\notin S_1}\max_{i\in A_l}(|x_i|_\nu)\Big)^{u_l+l} \geqslant c(u_l)^{-1}H_{S_1}(x_1,\dots,x_{u_l}), \tag{2.2.5}$$

where we put $|x_0|_{S_1}=1$ and $A_l=\{1,\dots,u_l\}\cup\{j_1,\dots,j_l\}$.

Then this construction will finish the proof in the end.

We prove this claim by induction on l. The first step when $l=0$ is just (2.2.4). Let $r\geqslant 0$ and assume that $u_0,\dots,u_r,j_0,\dots,j_r$ have been constructed such that $u_r<u$ and (2.2.5) holds for $l=0,\dots,r$. We show the existence of u_{r+1},j_{r+1} such that (2.2.5) holds for $l=r+1$. For any index $0<j\leqslant N$ we have

$$x_j=\sum_{i=1}^{u}\lambda_{i,j}x_i=\sum_{i=1}^{u_r}\lambda_{i,j}x_i+\sum_{i=u_r+1}^{u}\lambda_{i,j}x_i:=T_{j,r}+U_{j,r},$$

with $\lambda_{i,j}\in k$ uniquely determined.

We claim that there is j such that both $T_{j,r}$ and $U_{j,r}$ are non-zero. Assume the contrary, then for each $j\in\{1,\dots,N\}$, either $U_{j,r}=0$ i.e., $x_j=T_j$, or $T_{j,r}=0$, that is, $x_j=U_{j,r}$. Since $x_1+\cdots+x_N=\sum_{i=1}^{v}a_ix_i$ and $v<u_r+1$, we derive that $\sum_{j=1}^{N}U_{j,r}=0$, or equivalently, $\sum_{U_{j,r}\neq 0}x_j=0$. But $u_r<u$, so we have $U_{u,r}\neq 0$, and hence $\{j:U_{j,r}\neq 0\}\neq\emptyset$. This gives a vanishing subsum, which contradicts the assumption.

Let j be the smallest index with $U_{j,r}\neq 0, T_{j,r}\neq 0$ and put $j_{r+1}=j$. Then clearly $j>u$ because $x_1,\dots,x_u$ are k-linearly independent with u maximal. Renumbering the indices $u_r+1,\dots,u$, we can write

$$U_{j_{r+1},r}=\sum_{i=u_r+1}^{u_{r+1}}\lambda_{i,j_{r+1}}x_i, \tag{2.2.6}$$

where $\lambda_{i,j_{r+1}}\neq 0$ for $u_r+1\leqslant i\leqslant u_{r+1}$. This defines u_{r+1} satisfying $u_r<u_{r+1}\leqslant u$ and gives $x_{j_{r+1}}$ a linear combination of $x_1,\dots,x_{u_{r+1}}$. Since for

$l < r+1$, $u_l \leqslant u_r < u_{r+1}$ and x_{j_l} is a linear combination of $x_1, \dots, x_{u_l}$, we infer that $j_{r+1} \neq j_l$.

Put $B_l = A_{l+1} \backslash A_l = \{j_{l+1}\} \cup \{u_l + 1, \dots, u_{l+1}\}$. The assumption in (iii) for u_r, j_r gives

$$\left(\prod_{i=0}^{r} |x_{j_i}|_{S_1}\right)\left(\prod_{i=1}^{u_r} |x_i|_{S_1}\right) |x_1 + \cdots + x_N|_{S_1} \Big(\prod_{\nu \notin S_1} \max_{i \in A_r}(|x_i|_\nu)\Big)^{u_r + r} \geqslant c(u_r)^{-1} H_{S_1}(x_i : i \in A_r). \tag{2.2.7}$$

Notice that $T_{j_{r+1}} = \sum_{i=1}^{u_r} \lambda_{i,j_{r+1}} x_i = x_{j_{r+1}} - \sum_{i=u_r+1}^{u_{r+1}} \lambda_{i,j_{r+1}} x_i$ as a sum of $x_{j_{r+1}}$ and $-\lambda_{i,j_{r+1}} x_i, u_r + 1 \leqslant i \leqslant u_{r+1}$, the assumption of Lemma 2.2.8 is satisfied and the components are indeed k-linearly independent, since $T_{j_{r+1}} \neq 0$, $x_1, \dots, x_u$ are k-linearly independent with u maximal and $\lambda_{i,j_{r+1}} \neq 0$ for $u_r + 1 \leqslant i \leqslant u_{r+1}$. By Lemma 2.2.8, we obtain

$$\prod_{i \in B_r} |x_i|_{S_1} |T_{j_{r+1}}|_{S_1} \Big(\prod_{\nu \notin S_1} \max_{i \in B_r}(|x_i|_\nu)\Big)^{u_{r+1} - u_r + 1} \geqslant c(u_{r+1} - u_r + 1)^{-1} H_{S_1}(x_i : i \in B_r). \tag{2.2.8}$$

Combining this with (2.2.7), we get

$$\begin{aligned} &\prod_{i=0}^{r+1} |x_{j_i}|_{S_1} \prod_{i=1}^{u_{r+1}} |x_i|_{S_1} |T_{j_{r+1}}|_{S_1} |x_1 + \cdots + x_N|_{S_1} \times \\ &\quad \times \Big(\prod_{\nu \notin S_1} \max_{i \in A_r}(|x_i|_\nu)\Big)^{u_r + r} \Big(\prod_{i \in B_r} \max_{i \in B_r}(|x_i|_\nu)\Big)^{u_{r+1} - u_r + 1} \\ &\geqslant C'(u_r)^{-1} C'(u_{h+1} - u_r + 1)^{-1} H_{S_1}(x_i : i \in A_r) H_{S_1}(x_i : i \in B_r). \end{aligned} \tag{2.2.9}$$

Noticing that for any $\nu \in M_{K_1}$,

$$|T_{j_{r+1}}|_\nu \leqslant \min(\max(|x_i|_\nu : i \in A_r), \max(|x_i|_\nu : i \in B_r)),$$

$$\max(|x_i|_\nu : i = 1, \dots, u_{r+1}) \leqslant \max(\max(|x_i|_\nu : i \in A_r), \max(|x_i|_\nu : i \in B_r)),$$

we deduce that for $\nu \in M_{K_1}$

$$|T_{j_{r+1}}|_\nu \max(|x_i|_\nu : i = 1, \dots, u_{r+1}) \leqslant \max(|x_i|_\nu : i \in A_r) \max(|x_i|_\nu : i \in B_r)).$$

Taking the product over $\nu \in S_1$, and inserting (2.2.9), we obtain

$$\prod_{i=0}^{r+1} |x_{j_i}|_{S_1} \prod_{i=1}^{u_{r+1}} |x_i|_{S_1} \cdot |x_1 + \cdots + x_N|_{S_1} \times \\ \times \Big(\prod_{\nu \notin S_1} \max_{i \in A_r}(|x_i|_\nu) \Big)^{u_r + r} \Big(\prod_{i \in B_r} \max_{i \in B_r}(|x_i|_\nu) \Big)^{u_{r+1} - u_r + 1} \\ \geqslant C'(u_r)^{-1} C'(u_{r+1} - u_r + 1)^{-1} H_{S_1}(x_1, \ldots, x_{u_{r+1}}).$$

Observing that $C'(x)C'(y+1) \leqslant C'(x+y)$ for $x, y \geqslant 1$ and $A_l \cup B_l = A_{l+1}$, we get

$$\prod_{i=0}^{r+1} |x_{j_i}|_{S_1} \prod_{i=1}^{u_{r+1}} |x_i|_{S_1} |x_1 + \cdots + x_N|_{S_1} \Big(\prod_{\nu \notin S_1} \max_{i \in A_{r+1}} (|x_i|_\nu) \Big)^{u_{r+1} + r + 1} \\ \geqslant C'(u_{r+1})^{-1} H_{S_1}(x_1, \ldots, x_{u_{r+1}}).$$

This verifies (iii) for $r+1$ in place of r (in case $u_r < u$), and completes the proof of the claim.

Now for u_h, j_h, since $x_1, \ldots, x_u$ are k-linearly independent with u maximal, we have

$$\prod_{i=0}^{h} |x_{j_i}|_{S_1} \prod_{i=1}^{N} |x_i|_{S_1} |x_1 + \cdots + x_N|_{S_1} \Big(\prod_{\nu \notin S_1} \max_{1 \leqslant i \leqslant u} (|x_i|_\nu) \Big)^{u+h} \\ \geqslant C'(u)^{-1} H_{S_1}(x_1, \ldots, x_N). \tag{2.2.10}$$

Clearly we have

$$\prod_{\nu \notin S_1} \max_{1 \leqslant i \leqslant N} (|x_i|_\nu) \geqslant \prod_{\nu \notin S_1} |x_i|_\nu \text{ for } i \notin C := \{1, \ldots, u, j_1, \ldots, j_h\},$$

so

$$\Big(\prod_{\nu \notin S_1} \max_{1 \leqslant i \leqslant N} (|x_i|_\nu) \Big)^{N-u-h} \geqslant \prod_{\nu \notin S_1} \prod_{i \in C} |x_i|_\nu = \prod_{i \in C} |x_i|_{S_1}^{-1}. \tag{2.2.11}$$

Combining (2.2.10) and (2.2.11) completes the proof. □

Example 2.2.10. *Let x_1, x_2, x_3, x_4, x_5 be k-linearly independent, $x_6 = -x_4 - x_5 + 2x_3$. Then $n = 6, u = 5, v = 3, j_1 = 6, u_1 = u$.*

Let x_1, x_2, x_3, x_4 be k-linearly independent, $x_5 = 2x_3 - x_4, x_6 = -3x_3 + x_2$. Then $n = 6, u = 4, v = 2, j_1 = 6, u_1 = 3, j_2 = 5, u_2 = u$.

Let $x_1, \ldots, x_5$ be k-linearly independent, $x_6 = -x_3 + x_2 - x_4, x_7 = -x_5 + x_1$. Then $n = 7, u = 5, v = 2$, and we get $j_1 = 6, u_1 = 4, j_2 = 7, u_2 = u$, or we reorder x_3, x_4, x_5 by x'_5, x'_4, x'_3, then we get $j_1 = 7, u_1 = 3, j_2 = 6, u_2 = u$.

Corollary 2.2.11. *Let $n \geqslant 3$. Assume $x_1, \ldots, x_n \in K_1$ and $\sum_{i=1}^{n} x_i = 0$ but no non-empty proper subsum vanishes. Then*

$$H_{S_1}(x_1, \ldots, x_n) \leqslant e^{\binom{n-1}{2} \max(2g_{K_1} - 2 + \#S_1, 0)} \left(\prod_{i=1}^{n} |x_i|_{S_1} \right) \left(\prod_{\nu \notin S_1} \max_i (|x_i|_\nu) \right)^{n-1}.$$

If $x_1, \ldots, x_n$ are k-linearly independent, then we can replace $\max(2g_{K_1} - 2 + \#S_1, 0)$ *by* $2g_{K_1} - 2 + \#S_1$.

Proof. Simply apply Theorem 2.2.9 for $x_1, \ldots, x_{n-1}$. □

Corollary 2.2.12. *Let $n \geqslant 3$. Assume $x_1, \ldots, x_n \in \mathcal{O}_{S_1}$ and $\sum_{i=1}^{n} x_i = 0$ but no non-empty proper subsum vanishes. Then*

$$H_{S_1}(x_1, \ldots, x_n) \leqslant e^{\binom{n-1}{2} \max(2g_{K_1} - 2 + \#S_1, 0)} \prod_{i=1}^{n} |x_i|_{S_1}.$$

Proof. This is a direct consequence since for $x \in \mathcal{O}_{S_1}, |x|_{S_1} \leqslant 1$. □

For $n = 4$, the constant $\frac{(n-1)(n-2)}{2}$ is best possible, as is shown by the following example from [5].

Example 2.2.13. *Let $K = k(t)$, and $x_1 = (t^r + 1)^3, x_2 = -t^{3r}, x_3 = -3t^r(t^r + 1), x_4 = -1$ where r is a positive integer. Take S to be the set of valuations corresponding to ∞ and the prime factors of $t(t^r + 1)$. Then $\#S = r + 2$, $x_i \, (i = 1, 2, 3, 4)$ are S-units, $H_S(x_1, x_2, x_3, x_4) = e^{3r} = e^{3(2g_K - 2 + \#S)}$. This implies that for $n = 4$ the constant $\frac{(n-1)(n-2)}{2}$ is best possible.*

Remark 2.2.14. *Corollary 2.2.12 is much stronger than its analogue over number fields, i.e., Lemma 2. Lemma 2 first involves an exponent $1 + \varepsilon$ on*

$\prod_{i=1}^{n} |x_i|_{S_1}$ *and second an ineffective constant* $C(n, S_1, \varepsilon)$*, which is caused by the ineffectivity of the Subspace Theorem. We also notice the improvement in comparison with the result of Corollary 2.1.4, with a much sharper exponent* 1 *instead of* $\frac{(n-1)(n-2)}{2}$.

Theorem 2.2.9 and its Corollary 2.2.11 imply the following results:

Lemma 2.2.15 (Mason, Stothers)**.** *Let* L *be a finite extension of* $K = k(t)$*, and* T *a finite set of valuations of* L*. Let* $\gamma_1, \gamma_2, \gamma_3$ *be non-zero elements of* L *satisfying* $\gamma_1 + \gamma_2 + \gamma_3 = 0$ *and* $\nu(\gamma_1) = \nu(\gamma_2) = \nu(\gamma_3)$ *for every valuation* $\nu \notin T$*. Then either* $\frac{\gamma_1}{\gamma_2} \in k$*, which means* $H^*(\frac{\gamma_1}{\gamma_2}) = 1$*, or*

$$H^*(\frac{\gamma_1}{\gamma_2}) \leqslant e^{(\#T + 2g_L - 2)/[L:K]}.$$

In particular, let $a(t), b(t), c(t)$ *be coprime polynomials over* k *such that* $a(t) + b(t) = c(t)$ *and not all of them are constants. Then*

$$\max(\deg a(t), \deg b(t), \deg c(t)) \leqslant \deg(\mathrm{rad}(abc)) - 1,$$

where rad(f) *denotes the product of the distinct prime factors of* f*.*

Proof. Assume that γ_1, γ_2 are k-linearly independent. Apply Corollary 2.2.11 with $n = 3$, $K_1 = L$, $S_1 = T$ and $x_i = \gamma_i$ $(i = 1, 2, 3)$ and apply the product formula. For the particular case that $a(t), b(t), c(t)$ are polynomials from $k[t]$ without a common factor, let S_1 be the set of valuations consisting of ν_∞ and those corresponding to the zeros of abc. Then $\#S_1 = \deg(\mathrm{rad}(abc)) - 1$ and thus our assertion follows directly from Corollary 2.2.11. □

Theorem 2.2.16 (Brownawell, Masser)**.** *Assume* $u_1, \ldots, u_n$ *are* S_1*-units satisfying* $u_1 + \cdots + u_n = 0$ *but no non-empty proper subsum vanishes. Then*

$$H_{K_1}(u_1, \ldots, u_n) \leqslant \exp\left(\binom{n-1}{2} \max(\#S + 2g_{K_1} - 2, 0)\right).$$

This is mentioned after Theorem B of [6].

Proof. Apply Corollary 2.2.11 by taking $T = S_1$ and noticing that for an S_1-unit x, we have $|x|_{S_1} = 1$ and $|x|_\nu = 1$ for every $\nu \notin S_1$. □

Theorem 2.2.17 (Zannier). *Let $a_1, \ldots, a_n \in K_1$ be S_1-units such that $\sum_{i\in\Gamma} a_i \neq 0$ for every nonempty $\Gamma \subset \{1, \ldots, n\}$. Put $b = a_1 + \cdots + a_n$. Then*

$$\sum_{\nu\in S_1} (\nu(b) - \min \nu(a_i)) \leqslant \binom{\mu}{2} \max(\#S_1 + 2g_{K_1} - 2, 0)$$

where $\mu = \operatorname{rank}\{a_1, \ldots, a_n\}$.

This is Theorem 1 of [26], except that there the result was stated $\#S_1 + 2g_{K_1} - 2$ instead of $\max(\#S_1 + 2g_{K_1} - 2, 0)$. However, the proof in [26] gives only the inequality with the maximum with 0.

Proof. This follows directly by taking $T = S_1, x_i = a_i\,(i = 1, \ldots, n)$ in Theorem 2.2.9 and using the fact that $a_1, \ldots, a_n$ are S_1-units. □

Theorem 2.2.18 (Davenport). *If $f(t), g(t)$ are nonzero polynomials over k such that $g(t)^2 \neq f(t)^3$, then*

$$\deg(g(t)^2 - f(t)^3) \geqslant \frac{1}{2} \deg f(t) + 1.$$

Proof. This is an analogue of Hall's conjecture over the function fields. It is first proved by Davenport in [8].

In Corollary 2.2.11, let $T = S_1$ be the set S consisting of ν_∞ and the valuations corresponding to the zeros of fg, and $x_1 = f(t)^3, x_2 = -g(t)^2, x_3 = g(t)^2 - f(t)^3$. Then

$$H_S(f(t)^3, g(t)^2) \leqslant e^{\#S-2} |g(t)^2 - f(t)^3|_S.$$

In particular, when $f(t), g(t)$ are coprime, we deduce that

$$\begin{aligned} \frac{1}{2}(3 \deg f(t) + 2 \deg g(t)) &\leqslant \max(\deg f(t)^3, \deg g(t)^2) \\ &\leqslant \deg(\operatorname{rad}(fg)) - 1 + \deg(g(t)^2 - f(t)^3) \\ &\leqslant \deg f(t) + \deg g(t) + \deg(g(t)^2 - f(t)^3) - 1. \end{aligned}$$

Hence

$$1 + \frac{1}{2} \deg f(t) \leqslant \deg(g(t)^2 - f(t)^3).$$

The case when $f(t), g(t)$ are not coprime is a direct consequence of the above.

□