The handle http://hdl.handle.net/1887/36589 holds various files of this Leiden University dissertation.

Author: Zhuang, Weidong
Title: Symmetric diophantine approximation over function fields
Issue Date: 2015-12-03
Chapter 1

Preliminaries

In this chapter we collect some results related to discriminants, resultants, valuations, heights and twisted heights.

Unless otherwise stated, throughout this dissertation, $k$ will be an algebraically closed field of characteristic 0 and $K = k(t)$ the rational function field in the variable $t$. By a function field, we always mean a finite extension of $K$.

1.1 Discriminants and resultants

Let $L$ be an arbitrary field. Let

$$F(X, Y) = a_0X^n + a_1X^{n-1}Y + \cdots + a_nY^n \in L[X, Y]$$

be a binary form of degree $n \geq 2$.

We have a factorization $F(X, Y) = \prod_{i=1}^{n}(\alpha_iX + \beta_iY)$ over an algebraic closure $\overline{L}$ of $L$. As usual, we define the discriminant of $F$ to be

$$D(F) := \prod_{i<j}(\alpha_i\beta_j - \alpha_j\beta_i)^2.$$

This is a homogeneous polynomial of degree $2n - 2$ in $\mathbb{Z}[a_0, \ldots, a_n]$. In particular, for a linear form, we define its discriminant to be 1.
It is easy to show that for $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{L})$ and $\lambda \in \mathbb{L}^*$, we have

\[
D(\lambda F) = \lambda^{2n-2}D(F), \\
D(F_U) = (\text{det } U)^{n(n-1)}D(F),
\]

where $F_U(X, Y) = F(aX + bY, cX + dY)$.

Let $F(X, Y) = a_0X^m + a_1X^{m-1}Y + \cdots + a_mY^m$ and $G(X, Y) = b_0X^n + b_1X^{n-1}Y + \cdots + b_nY^n$ be two binary forms with coefficients in $L$. The resultant $R(F, G)$ of $F, G$ is defined by the determinant

\[
R(F, G) := \begin{vmatrix} a_0 & a_1 & \cdots & a_m \\ \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n \\ b_0 & b_1 & \cdots & b_s \\ \vdots & \vdots & \ddots & \vdots \\ b_0 & b_1 & \cdots & b_n \end{vmatrix}, \tag{1.1.1}
\]

where the first $n$ rows consist of coefficients of $F$ and the last $m$ rows of coefficients of $G$.

Over the algebraic closure $\overline{L}$ of $L$, suppose that we have factorizations

\[
F(X, Y) = \prod_{i=1}^{m}(\alpha_iX + \beta_iY), G(X, Y) = \prod_{j=1}^{n}(\gamma_jX + \delta_jY).
\]

Then

\[
R(F, G) = \prod_{i=1}^{m}\prod_{j=1}^{n}(\alpha_i\delta_j - \beta_i\gamma_j). \tag{1.1.2}
\]

Hence $R(F, G) = 0$ holds exactly when $F, G$ have a common factor.

The resultant has the following properties:

\[
R(\lambda F, \mu G) = \lambda^n\mu^mR(F, G),
\]

\[
R(F_1F_2, G) = R(F_1, G)R(F_2, G),
\]
\[ R(G, F) = (-1)^{mn} R(F, G), \]
\[ R(F, G + HF) = R(F, G), \]
where \( \lambda, \mu \in L, F, G, F_1, F_2 \) are binary forms and \( H \) is a binary form of degree \( n - m \) if \( n \geq m \).

For an invertible matrix \( U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), define \( F_U(X, Y) := F(aX + bY, cX + dY) \).

Then \( R(F_U, G_U) = (\det U)^{mn} R(F, G) \).

### 1.2 Valuations on function fields

Recall \( K = k(t) \). Denote by \( M_K \) the collection of normalized discrete valuations on \( K \) that are trivial on \( k \). This set is described as follows. For \( f \in k[t] \setminus \{0\} \), define \( \nu_p(f)(p \in k \cup \{\infty\}) \) by \( f = (t - p)^{\nu_p(f)} g \) where \( g \in k[t] \) and \( g(p) \neq 0 \) if \( p \in k \); further, define \( \nu_{\infty}(f) = -\deg f \). We extend this to \( k(t) \) by setting \( \nu_p(0) := \infty \) and \( \nu_p(fg) = \nu_p(f) - \nu_p(g) \) for \( f, g \in k[t], g \neq 0 \).

Then \( M_K = \{ \nu_p : p \in k \cup \{\infty\} \} \). In this thesis we often work with absolute values. We define the absolute value \(|\cdot|_\nu\) by \( e^{-\nu(\cdot)} \) for \( \nu \in M_K \). These absolute values satisfy the product formula

\[
\prod_{\nu \in M_K} |x|_\nu = 1
\]

for every \( x \in K^* \). All valuations of \( K \) are non-archimedean, so for a binary form \( F \in K[X,Y] \) we have

\[
|D(F)|_\nu \leq \max_{0 \leq j \leq n} (|a_j|_\nu^{2n-2}) \tag{1.2.1}
\]

for every \( \nu \in M_K \). Let \( S \) be a finite set of valuations of \( K \), containing the 'infinite valuation' \( \nu_{\infty} \). Define the ring of \( S \)-integers and group of \( S \)-units by

\[
O_S = \{ x \in K : |x|_\nu \leq 1 \text{ for } \nu \not\in S \},
\]
\[
O_S^\times = \{ x \in K : |x|_\nu = 1 \text{ for } \nu \not\in S \}.
\]
We define the $S$-norm of $x \in K$ by
\[
|x|_S = \prod_{\nu \in S} |x|_\nu.
\]
It is clear that $|x|_S \geq 1$ for $x \in \mathcal{O}_S \setminus \{0\}$ and $|x|_S = 1$ for $x \in \mathcal{O}_S^\times$.

**Remark 1.2.1.** Let $K$ be a purely transcendental extension of $k$ of transcendence degree 1. Choose $t$ such that $K = k(t)$. The 'infinite valuation' $\nu_\infty$ is the one with $\nu_\infty(t) < 0$. The choice of the infinite valuation depends on the choice of a transcendental element $t$ generating $K$. In what follows, we make a distinction between the infinite valuation $\nu_\infty$ and the other valuations on $K$. But we should mention that in our arguments we could as well have chosen any other valuation to play the role of the infinite valuation.

Recall that $k$ is an algebraically closed field of characteristic 0, and $K = k(t)$. Let $L$ be a finite extension of $K$. We say a valuation $\omega$ is normalized if $\omega(L^*) = \mathbb{Z}$. Denote by $M_L$ the normalized valuations on $L$ that are trivial on $k$. For valuations $\nu \in M_K$, $\omega \in M_L$, we say that $\omega$ lies above $\nu$, and denote it by $\omega|\nu$, if the restriction of $\omega$ to $K$ is a positive multiple of $\nu$. Then for every $\nu \in M_K$, we have finitely many valuations $\omega \in M_L$ above $\nu$. For every $\omega \in M_L$, we define the corresponding absolute value $|x|_\omega := e^{-\omega(x)}$. Then we have $\omega(x) = e(\omega|\nu)\nu(x)$ for $\omega|\nu, x \in K$, where $e(\omega|\nu)$ is called the ramification index. Let $L_\omega$ denote the completion of $L$ at $\omega$. In our case, $k$ is algebraically closed with char $k = 0$ and the residue field of $\nu$ is $k$, hence the residue degree is 1, implying that $e(\omega|\nu) = [L_\omega : K_\nu]$. Thus our chosen absolute value is a prolongation of $| \cdot |_{[L_\omega : K_\nu]}$, rather than $| \cdot |_\nu$, to $L$, hence by Proposition 1.2.7 of [4], we have the relation $|x|_\omega = |N_{L_\omega/K_\nu}(x)|_\nu$ for every $x \in L$. By assumption, $K$ has characteristic 0, so the extension $L/K$ is separable. Hence
\[
N_{L/K}(x) = \prod_{\omega|\nu} N_{L_\omega/K_\nu}(x) \text{ for } x \in L.
\]
so we have
\[ \prod_{\omega \mid \nu} |x|_{\omega} = |N_{L/K}(x)|_{\nu} \text{ for } x \in L, \nu \in M_K \]
and
\[ \prod_{\omega \in M_L} |x|_{\omega} = 1 \text{ for } x \in L^*. \]

Similarly, we define the $T$-norm of $x \in L$ by
\[ |x|_T = \prod_{\omega \in M_L} |x|_{\omega}. \]

We recall some facts about Dedekind domains. For a non-zero fractional ideal $a$ of a Dedekind domain $A$ and a prime ideal $\wp$ of $A$, we denote by $\nu_\wp(a)$ the exponent of $\wp$ in the prime ideal factorization of $a$.

**Lemma 1.2.2.** There is a bijection between the non-zero prime ideals of $A$ and the discrete valuations of $F$ that are non-negative on $A$, given by $p \mapsto \nu_p$ such that $\nu_p(a)$ is the exponent of $p$ in the unique prime ideal factorization of the ideal generated by $a$.

**Proof.** See [1].

**Lemma 1.2.3.** Let $A$ be a Dedekind domain with fraction field $K_1$. Let $L$ be a finite separable extension of $K_1$, and $B$ the integral closure of $A$ in $L$. Assume that $L/K_1$ is tamely ramified. Denote by $D_{B/A}$ the discriminant ideal and $\mathfrak{D}_{B/A}$ the different ideal of $B$ over $A$. Let $p$ be a prime ideal of $A$, let $\wp_1, \ldots, \wp_r$ be the prime ideals of $B$ above $p$, and $\nu$ the valuation corresponding to $p$, and $\omega_i$ corresponding to $\wp_i$ for $i = 1, \ldots, r$. Then
\[ N_{L/K_1}(\mathfrak{D}_{B/A}) = D_{B/A}. \]

Further
\[ \nu(D_{B/A}) = \sum_{i=1}^{r} \left( e(\omega_i|\nu) - 1 \right). \]
Chapter 1. Preliminaries

Proof. For the first part, see Proposition 6, §3, Chapter III of [22].

Since the extension \( L/K \) is tamely ramified with residue degree \( f(\omega_i|\nu) = 1 \), we get by Proposition 13, §6, Chapter III of [22],

\[ \omega_i(\mathfrak{O}_{B/A}) = e(\omega_i|\nu) - 1 \text{ for } i = 1, \ldots, r, \]

hence

\[ \nu(D_{B/A}) = \nu(\mathfrak{N}_{L/K}(\mathfrak{O}_{B/A})) = \sum_{i=1}^{r} (e(\omega_i|\nu) - 1), \]

which gives the claim. \( \square \)

Later we will apply this lemma frequently to the case \( K_1 = k(t), A = k[t] \) and \( K_1 = K_\nu \), the completion of \( K \) at \( \nu \) and \( A = R_\nu := \{ x \in K_\nu : \nu(x) \geq 0 \} \) for \( \nu \in M_K \).

1.3 Polynomials and heights

Recall \( K = k(t) \). For \( \nu \in M_K \), denote by \( K_\nu \) the completion of \( K \) at the valuation \( \nu \). Then \( \nu \) has a unique extension to \( K_\nu \). Define

\[ R_\nu = \{ x \in K_\nu : \nu(x) \geq 0 \} \]

to be the local ring of \( K_\nu \). Then its group of units is

\[ R_\nu^\times = \{ x \in K_\nu : \nu(x) = 0 \}. \]

For \( x = (x_1, x_2, \ldots, x_n) \in K_\nu^n \), define

\[ \nu(x) = \min_{1 \leq i \leq n} \nu(x_i), \]
\[ \|x\|_\nu = e^{-\nu(x)} = \max_{1 \leq i \leq n} |x_i|_\nu, \]

and for \( x \in K^n \), define the homogeneous height and \( S \)-height

\[ H_K(x) = \prod_{\nu \in M_K} \|x\|_\nu, \]
1.4. Galois theory of valuations

\[ H_S(x) = \prod_{\nu \in S} \|x\|_\nu. \]

Clearly, the product is well-defined and \( H_K(x) \geq 1 \) for every \( x \neq 0 \) because of the product formula. Also, \( H_K(\lambda x) = H_K(x) \).

For a polynomial \( P \in K[x_1, \ldots, x_n] \) or \( P \in K_\nu[x_1, \ldots, x_n] \) we define \( |P|_\nu \) to be the maximum of the \( |\cdot|_\nu \)-values of its coefficients.

**Lemma 1.3.1** (Gauss’ lemma). Let \( K \) be a field, \( |\cdot|_\nu \) a non-archimedean absolute value on \( K \), and \( P = \prod_{i=1}^{t} P_i \) with \( P_i \in K[x_1, \ldots, x_n] \) for \( i = 1, \ldots, t \). Then

\[ |P|_\nu = \prod_{i=1}^{t} |P_i|_\nu. \]

**Proof.** See [14].

As a direct consequence, we have

**Corollary 1.3.2.** Let \( F = \prod_{i=1}^{n} (\alpha_i X + \beta_i Y) \) with \( \alpha_i, \beta_i \in K \) for \( i = 1, \ldots, n \). Then \( |F|_\nu = \prod_{i=1}^{n} \max(|\alpha_i|_\nu, |\beta_i|_\nu) \) for every \( \nu \in M_K \).

For \( L \) a finite extension of \( K \) and a polynomial \( P \in L[x_1, \ldots, x_m] \), we define

\[ N_{L/K}(P) = \prod_{i=1}^{[L:K]} \sigma_i(P), \]

where \( \sigma_1, \ldots, \sigma_{[L:K]} \) are the \( K \)-embeddings of \( L \) into \( \overline{K} \), and \( \sigma_i(P) \) is obtained by the action of \( \sigma_i \) on the coefficients of \( P \).

### 1.4 Galois theory of valuations

In this section, we give a brief sketch of some aspects of Galois theory of valuations that will be needed later.
Lemma 1.4.1. Let $K$ be a field with a non-trivial absolute value $|\cdot|_\nu$, and $L$ a finite Galois extension of $K$ with Galois group $G = \text{Gal}(L/K)$. Then for every two absolute values $|\cdot|_\omega, |\cdot|_{\omega'}$ on $L$ prolonging $|\cdot|_\nu$, there is $\sigma \in G$ such that $|x|_\omega = |\sigma(x)|_{\omega'}$ for $x \in L$.

Proof. See Corollary 1.3.5 of [4].

For $\nu \in M_K$ and $L$ a Galois extension of $K$, denote by $A(\nu)$ the set of normalized valuations of $L$ above $\nu$. Fix $\omega_1 \in A(\nu)$. The completion $L_{\omega_1}$ of $L$ at $\omega_1$ is a Galois extension of $K_\nu$. We may view $L$ as a subfield of $L_{\omega_1}$. As mentioned before, the absolute values on $L$ defined above satisfy the relation $|x|_{\omega_1} = |N_{L_{\omega_1}/K_\nu}(x)|_\nu$ for $x \in L_{\omega_1}$. Without loss of generality, we may assume $K \subset K_\nu \subset L_{\omega_1} \subset \overline{K}_\nu$ and $K \subset L \subset L_{\omega_1} \subset \overline{K}_\nu$. Let $E(\omega_1|\nu)$ be the set $\{\sigma \in G : \omega_1 \circ \sigma = \omega_1\}$ equipped with composition. This is by definition the decomposition group of $\omega_1$ over $\nu$. By, for instance, §9, Chapter II of [18], we have an isomorphism
\[
\text{Gal}(L_{\omega_1}/K_\nu) \rightarrow E(\omega_1|\nu),
\sigma \mapsto \sigma|_L.
\]
Thus we may view $\text{Gal}(L_{\omega_1}/K_\nu)$ as a subgroup of $G$. Further, let
\[
E(\omega|\nu) = \{\sigma \in G : \omega = \omega_1 \circ \sigma\} \text{ for } \omega \in A(\nu).
\]
(1.4.1)
Since $G$ acts transitively on $A(\nu)$ (see §9, Chapter II, [18]), the sets $E(\omega|\nu)$ form a partition of $G$, and in fact they are the right cosets of $\text{Gal}(L_{\omega_1}/K_\nu)$ in $G$, so have the same cardinality:
\[
[L_\omega : K_\nu] = [L_{\omega'} : K_\nu]\text{ for } \omega, \omega' \text{ above } \nu.
\]
(1.4.2)
It is now reasonable to put $g_\nu := \#E(\omega|\nu) = [L_{\omega_1} : K_\nu]$. If we still denote by $|\cdot|_\nu$ the prolongation of $|\cdot|_\nu$ from $K$ to $\overline{K}_\nu$, and hence on $L_{\omega_1}$, then $|x|_\nu = |N_{L_{\omega_1}/K_\nu}(x)|_\nu^{1/[L_{\omega_1}:K_\nu]}$ for $x \in L_{\omega_1}$. It follows that for $x \in L, \omega \in A(\nu), \sigma \in E(\omega|\nu)$, we have
\[
|x|_\omega = |\sigma(x)|_{\omega_1} = |\sigma(x)|_\nu^{g_\nu}.
\]
(1.4.3)
Notice that $\sigma \in \text{Gal}(L/K)$, hence we may extend $\sigma \in \mathcal{E}(\omega|\nu)$ to a $K_\nu$-isomorphism from $L_\omega$ to $L_{\omega_1}$, by sending $\alpha = \lim_{n \to \infty} \alpha_n$ to $\sigma(\alpha) = \lim_{n \to \infty} \sigma(\alpha_n)$ where $\alpha \in L_\omega$ and $\alpha_n \in L$. Moreover, for every $x \in L_\omega$, we also have $|x|_\omega = |\sigma(x)|_{\omega_1} = |\sigma(x)|_\nu^\nu$.

### 1.5 Twisted heights

Let $S$ be a finite set of valuations of $K$. We define the ring of $S$-adeles

$$\mathbb{A}_S := \prod_{\nu \in S} K_\nu = \{ (x_\nu) | x_\nu \in K_\nu \text{ for every } \nu \in S \}$$

with componentwise addition and multiplication.

Further, let

$$\text{GL}_n(\mathbb{A}_S) = \{ (A_\nu) | A_\nu \in \text{GL}_n(K_\nu) \text{ for every } \nu \in S \},$$

where $\text{GL}_n(R_\nu)$ is the subgroup of $\text{GL}_n(K_\nu)$ of $n \times n$ matrices whose entries are in $R_\nu$ and whose determinant is in $R_\nu^\times$.

For $A = (A_\nu) \in \text{GL}_n(\mathbb{A}_S)$, define

$$|\det(A)|_S := \prod_{\nu \in S} |\det(A_\nu)|_\nu.$$

Also, we define the $\nu$-norm of $A_\nu$ as follows: if $A_\nu = (a_{ij})_{1 \leq i, j \leq n}$, then $\|A_\nu\|_\nu = \max_{i,j} |a_{ij}|_\nu$. Given a ring $R$ we denote by $R^n$ the module of $n$-dimensional column vectors with entries in $R$.

**Lemma 1.5.1.** Let $\nu \in M_K$. For $A_\nu \in \text{GL}_n(R_\nu)$ and $x \in K_\nu^n$, we have $\nu(A_\nu x) = \nu(x)$.

**Proof.** Let $A_\nu = (a_{ij}), x = (x_1, \ldots, x_n) \in K^n$. 

As \( \min_{i,j} \nu(a_{ij}) \geq 0 \), we have

\[
\nu(A_\nu x) \geq \min_{1 \leq i \leq n} \nu(a_{i1}x_1 + \cdots + a_{in}x_n) \\
\geq \min_{1 \leq i,j \leq n} \nu(a_{ij}) \\
\geq \min_{1 \leq j \leq n} \nu(x_j) + \min_{i,j} \nu(a_{ij}) \\
\geq \nu(x).
\]

Since \( A_\nu^{-1} \in \text{GL}_n(R_\nu) \), we have similarly for \( A_\nu \in \text{GL}_n(R_\nu), x \in K^n \) that \( \nu(x) = \nu(A_\nu^{-1}A_\nu x) \geq \nu(A_\nu x) \). This completes the proof. \( \square \)

For \( A \in \text{GL}_n(A_S), x \in K^n \) define the divisor

\[
\text{div}_A(x) := \sum_{\nu \in S} \nu(A_\nu x)\nu + \sum_{\nu \not\in S} \nu(x)\nu
\]

and its degree

\[
\text{deg}(\text{div}_A(x)) = \sum_{\nu \in S} \nu(A_\nu x) + \sum_{\nu \not\in S} \nu(x).
\]

Also define the corresponding twisted additive height

\[
h_A(x) := -\text{deg}(\text{div}_A(x)) = -\sum_{\nu \in S} \nu(A_\nu x) - \sum_{\nu \not\in S} \nu(x).
\]

The sum is well-defined by the fact that for every \( x \in K^\times \), we have \( \nu(x) = 0 \) for almost all \( \nu \in M_K \). Define the twisted multiplicative height for \( x \in K^n \) by:

\[
H_A(x) := \exp(h_A(x)) = \prod_{\nu \in S} \|A_\nu x\|_\nu \prod_{\nu \not\in S} \|x\|_\nu.
\]

It is projective in the sense that, by the product formula, \( H_A(\lambda x) = H_A(x) \) for \( x \in K^n, \lambda \in K^\times \).

Lastly, we define for \( A \in \text{GL}_n(A_S) \)

\[
\text{div}(A) := \text{div}(K^n) := \sum_{\nu \in S} \nu(\det(A_\nu))\nu,
\]

and

\[
h_A(K^n) := -\text{deg}(\text{div}(A)),
\]

\[
H_A(K^n) := \exp(h_A(K^n)) = \prod_{\nu \in S} |\det A_\nu|_\nu = |\det(A)|.
\]
Lemma 1.5.2. Let \( A \in \text{GL}_n(A_S) \). Then there exist positive constants \( c_1, c_2 \) depending on \( A \) such that \( c_2 H_K(x) \leq H_A(x) \leq c_1 H_K(x) \) for all \( x \in K^n \). In particular, for \( x \neq 0 \), we have \( H_A(x) \geq c_2 \).

Proof. Let \( c_1 = \prod_{\nu \in S} \| A_{\nu} \|_\nu \) and \( c_2 = \prod_{\nu \in S} \| A_{\nu}^{-1} \|_\nu^{-1} \).

Clearly, we have \( \| A_{\nu} x \|_\nu \leq \| A_{\nu} \|_\nu \| x \|_\nu \) because for all \( \nu \in S \), the valuation is non-archimedean. Similarly we have \( \| x \|_\nu = \| A_{\nu}^{-1} A_{\nu} x \|_\nu \leq \| A_{\nu}^{-1} \|_\nu \| A_{\nu} x \|_\nu \), hence \( \| A_{\nu}^{-1} \|_\nu^{-1} \| x \|_\nu \leq \| A_{\nu} x \|_\nu \leq \| A_{\nu} \|_\nu \| x \|_\nu \) for \( \nu \in S \). By taking the product over all \( \nu \in M_K \) we get \( c_2 H_K(x) \leq H_A(x) \leq c_1 H_K(x) \).

Consider a finite extension \( L \) of \( K \). Let \( S \) be a finite subset of \( M_K \) and let \( T \subset M_L \) be the set of valuations of \( L \) lying above those of \( S \). For \( x \in L \) put \( |x|_T := \prod_{\omega \in T} |x|_\omega \). Define the ring of \( T \)-integers and \( T \)-units

\[
\mathcal{O}_T := \{ x \in L : |x|_\omega \leq 1 \text{ for } \omega \notin T \},
\]

\[
\mathcal{O}_T^x := \{ x \in L : |x|_\omega = 1 \text{ for } \omega \notin T \}.
\]

Then \( \mathcal{O}_T \) is the integral closure of \( \mathcal{O}_S \) in \( L \). We have

\[
|x|_T = |N_{L/K}(x)|_S \text{ for } x \in L,
\]

and in particular,

\[
|x|_T = |x|_S^{[L:K]} \text{ for } x \in K.
\]

For \( \omega \in M_L \), denote by \( L_\omega \) the completion of \( L \) at \( \omega \). Then there is a unique extension of \( \omega \) to \( L_\omega \). For \( x = (x_1, \ldots, x_n)^T \in L_\omega^n \), we define

\[
\omega(x) = \min_{1 \leq i \leq n} \omega(x_i),
\]

\[
||x||_\omega = \max_{1 \leq i \leq n} |x_i|_\omega = \max_{1 \leq i \leq n} e^{-\omega(x_i)}.
\]

Similarly as before, we define \( \text{div}_A(x), \text{div}(A) \) for \( x \in L^n, A \in \text{GL}_n(A_T) \) by replacing \( K, S \) with \( L, T \) respectively. That is,

\[
\text{div}_A(x) := \sum_{\omega \in M_L} \omega(A_{\omega} x) \omega,
\]
\[ \text{div}(A) := \sum_{\omega \in M_L} \omega(\det(A)) \omega. \]

Define
\[
\begin{align*}
    h_A(x) &:= -\deg(\text{div}(x)) / [L : K], \\
    h_A(L^n) &:= -\deg(\text{div}(A)) / [L : K],
\end{align*}
\]
and
\[
\begin{align*}
    H_A(x) &:= \exp(h_A(x)) = \left( \prod_{\omega \in M_L} \|A_\omega x\|_\omega \right)^{1/[L : K]}, \\
    H_A(K^n) &:= \exp(h_A(K^n)) = \left( \prod_{\omega \in M_L} |\det A_\omega|_\omega \right)^{1/[L : K]} = |\det(A)|^{1/[L : K]}.
\end{align*}
\]
The height $H_A$ on $L^n$ is compatible with the one on $K^n$: $H_A(L^n) = H_A(K^n)$.

We recall Thunder’s analogue of Minkowski’s convex body theorem for function fields.

**Lemma 1.5.3.** Let $L$ be a finite extension of $K$ of degree $m$, and $H_A$ be the twisted height on $L^n$ corresponding to $A \in GL_n(\mathbb{A}_S)$. Then there is a basis $a_1, \ldots, a_n$ of $L^n$ satisfying
\[
\prod_{i=1}^n H_A(a_i) \leq H_A(L^n)e^{n(g_L+m-1)/m}.
\]
where $g_L$ is the genus of $L$.

**Proof.** See Theorem 1 of [24].

**Lemma 1.5.4.** For every basis $\{x_1, \ldots, x_n\}$ of $L^n$, we have
\[
\prod_{i=1}^n H_A(x_i) \geq H_A(L^n).
\]
In particular, there is a basis $\{a_1, \ldots, a_n\}$ of $K^n$ such that
\[
\prod_{i=1}^n H_A(a_i) = H_A(K^n).
\]

**Proof.** See Lemma 5 of [24] for the inequality. The equality is a combination with Lemma 1.5.3.