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Title: Symmetric diophantine approximation over function fields
Issue Date: 2015-12-03

Chapter 1

Preliminaries

In this chapter we collect some results related to discriminants, resultants, valuations, heights and twisted heights.

Unless otherwise stated, throughout this dissertation, k will be an algebraically closed field of characteristic 0 and $K = k(t)$ the rational function field in the variable t. By a function field, we always mean a finite extension of K.

1.1 Discriminants and resultants

Let L be an arbitrary field. Let

$$F(X,Y) = a_0X^n + a_1X^{n-1}Y + \cdots + a_nY^n \in L[X,Y]$$

be a binary form of degree $n \geqslant 2$.

We have a factorization $F(X,Y) = \prod_{i=1}^{n}(\alpha_iX + \beta_iY)$ over an algebraic closure $\overline{L}$ of L. As usual, we define the discriminant of F to be

$$D(F) := \prod_{i<j}(\alpha_i\beta_j - \alpha_j\beta_i)^2.$$

This is a homogeneous polynomial of degree $2n-2$ in $\mathbb{Z}[a_0,\ldots,a_n]$. In particular, for a linear form, we define its discriminant to be 1.

It is easy to show that for $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}\,(2, \overline{L})$ and $\lambda \in \overline{L}^*$, we have

$$\begin{aligned} D(\lambda F) &= \lambda^{2n-2} D(F), \\ D(F_U) &= (\det U)^{n(n-1)} D(F), \end{aligned}$$

where $F_U(X, Y) = F(aX + bY, cX + dY)$.

Let $F(X, Y) = a_0 X^m + a_1 X^{m-1} Y + \cdots + a_m Y^m$ and $G(X, Y) = b_0 X^n + b_1 X^{n-1} Y + \cdots + b_n Y^n$ be two binary forms with coefficients in L. The resultant $R(F, G)$ of F, G is defined by the determinant

$$R(F,G) := \begin{vmatrix} a_0 & a_1 & \cdots & \cdots & a_m & & & \\ & a_0 & a_1 & \cdots & \cdots & a_m & & \\ & & \ddots & & & & \ddots & \\ & & & a_0 & a_1 & \cdots & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & & & & \\ & b_0 & b_1 & \cdots & b_s & & & \\ & & \ddots & & & \ddots & & \\ & & & \ddots & & & \ddots & \\ & & & & b_0 & b_1 & \cdots & b_n \end{vmatrix}, \tag{1.1.1}$$

where the first n rows consist of coefficients of F and the last m rows of coefficients of G.

Over the algebraic closure $\overline{L}$ of L, suppose that we have factorizations

$$F(X, Y) = \prod_{i=1}^{m} (\alpha_i X + \beta_i Y), G(X, Y) = \prod_{j=1}^{n} (\gamma_j X + \delta_j Y).$$

Then

$$R(F, G) = \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i \delta_j - \beta_i \gamma_j). \tag{1.1.2}$$

Hence $R(F, G) = 0$ holds exactly when F, G have a common factor.

The resultant has the following properties:

$$R(\lambda F, \mu G) = \lambda^n \mu^m R(F, G),$$

$$R(F_1 F_2, G) = R(F_1, G) R(F_2, G),$$

$$R(G,F) = (-1)^{mn} R(F,G),$$

$$R(F, G+HF) = R(F,G),$$

where $\lambda, \mu \in L$, F, G, F_1, F_2 are binary forms and H is a binary form of degree $n-m$ if $n \geqslant m$.

For an invertible matrix $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define

$$F_U(X,Y) := F(aX+bY, cX+dY).$$

Then $R(F_U, G_U) = (\det U)^{mn} R(F,G)$.

1.2 Valuations on function fields

Recall $K = k(t)$. Denote by M_K the collection of normalized discrete valuations on K that are trivial on k. This set is described as follows. For $f \in k[t] \backslash \{0\}$, define $\nu_p(f)(p \in k \cup \{\infty\})$ by $f = (t-p)^{\nu_p(f)} g$ where $g \in k[t]$ and $g(p) \neq 0$ if $p \in k$; further, define $\nu_\infty(f) = -\deg f$. We extend this to $k(t)$ by setting $\nu_p(0) := \infty$ and $\nu_p(\frac{f}{g}) = \nu_p(f) - \nu_p(g)$ for $f, g \in k[t], g \neq 0$. Then $M_K = \{\nu_p : p \in k \cup \{\infty\}\}$. In this thesis we often work with absolute values. We define the absolute value $|\cdot|_\nu$ by $e^{-\nu(\cdot)}$ for $\nu \in M_K$. These absolute values satisfy the product formula

$$\prod_{\nu \in M_K} |x|_\nu = 1$$

for every $x \in K^*$. All valuations of K are non-archimedean, so for a binary form $F \in K[X,Y]$ we have

$$|D(F)|_\nu \leqslant \max_{0 \leqslant j \leqslant n} (|a_j|_\nu^{2n-2}) \tag{1.2.1}$$

for every $\nu \in M_K$. Let S be a finite set of valuations of K, containing the 'infinite valuation' ν_∞. Define the ring of S-integers and group of S-units by

$$\begin{aligned} \mathcal{O}_S &= \{x \in K : |x|_\nu \leqslant 1 \text{ for } \nu \notin S\}, \\ \mathcal{O}_S^\times &= \{x \in K : |x|_\nu = 1 \text{ for } \nu \notin S\}. \end{aligned}$$

We define the S-norm of $x \in K$ by

$$|x|_S = \prod_{\nu \in S} |x|_\nu.$$

It is clear that $|x|_S \geqslant 1$ for $x \in \mathcal{O}_S \backslash \{0\}$ and $|x|_S = 1$ for $x \in \mathcal{O}_S^\times$.

Remark 1.2.1. *Let K be a purely transcendental extension of k of transcendence degree 1. Choose t such that $K = k(t)$. The 'infinite valuation' ν_∞ is the one with $\nu_\infty(t) < 0$. The choice of the infinite valuation depends on the choice of a transcendental element t generating K. In what follows, we make a distinction between the infinite valuation ν_∞ and the other valuations on K. But we should mention that in our arguments we could as well have chosen any other valuation to play the role of the infinite valuation.*

Recall that k is an algebraically closed field of characteristic 0, and $K = k(t)$. Let L be a finite extension of K. We say a valuation ω is normalized if $\omega(L^*) = \mathbb{Z}$. Denote by M_L the normalized valuations on L that are trivial on k. For valuations $\nu \in M_K$, $\omega \in M_L$, we say that ω lies above ν, and denote it by $\omega|\nu$, if the restriction of ω to K is a positive multiple of ν. Then for every $\nu \in M_K$, we have finitely many valuations $\omega \in M_L$ above ν. For every $\omega \in M_L$, we define the corresponding absolute value $|x|_\omega := e^{-\omega(x)}$. Then we have $\omega(x) = e(\omega|\nu)\nu(x)$ for $\omega|\nu, x \in K$, where $e(\omega|\nu)$ is called the ramification index. Let L_ω denote the completion of L at ω. In our case, k is algebraically closed with $\operatorname{char} k = 0$ and the residue field of ν is k, hence the residue degree is 1, implying that $e(\omega|\nu) = [L_\omega : K_\nu]$. Thus our chosen absolute value is a prolongation of $|\cdot|_\nu^{[L_\omega:K_\nu]}$, rather than $|\cdot|_\nu$, to L, hence by Proposition 1.2.7 of [4], we have the relation $|x|_\omega = |N_{L_\omega/K_\nu}(x)|_\nu$ for every $x \in L$. By assumption, K has characteristic 0, so the extension L/K is separable. Hence

$$N_{L/K}(x) = \prod_{\omega|\nu} N_{L_\omega/K_\nu}(x) \text{ for } x \in L,$$

so we have

$$\prod_{\omega|\nu} |x|_\omega = |N_{L/K}(x)|_\nu \text{ for } x \in L, \nu \in M_K$$

and

$$\prod_{\omega \in M_L} |x|_\omega = 1 \text{ for } x \in L^*.$$

Similarly, we define the T-norm of $x \in L$ by

$$|x|_T = \prod_{\omega \in L} |x|_\omega.$$

We recall some facts about Dedekind domains. For a non-zero fractional ideal $\mathfrak{a}$ of a Dedekind domain A and a prime ideal $\wp$ of A, we denote by $\upsilon_\wp(\mathfrak{a})$ the exponent of $\wp$ in the prime ideal factorization of $\mathfrak{a}$.

Lemma 1.2.2. *There is a bijection between the non-zero prime ideals of A and the discrete valuations of F that are non-negative on A, given by $\mathfrak{p} \mapsto \nu_\mathfrak{p}$ such that $\nu_\mathfrak{p}(a)$ is the exponent of $\mathfrak{p}$ in the unique prime ideal factorization of the ideal generated by a.*

Proof. See [1]. □

Lemma 1.2.3. *Let A be a Dedekind domain with fraction field K_1. Let L be a finite separable extension of K_1, and B the integral closure of A in L. Assume that L/K_1 is tamely ramified. Denote by $D_{B/A}$ the discriminant ideal and $\mathfrak{D}_{B/A}$ the different ideal of B over A. Let $\mathfrak{p}$ be a prime ideal of A, let $\wp_1, \ldots, \wp_r$ be the prime ideals of B above $\mathfrak{p}$, and ν the valuation corresponding to $\mathfrak{p}$, and ω_i corresponding to $\wp_i$ for $i = 1, \ldots, r$. Then*

$$N_{L/K_1}(\mathfrak{D}_{B/A}) = D_{B/A}.$$

Further

$$\nu(D_{B/A}) = \sum_{i=1}^{r} \Big(e(\omega_i|\nu) - 1 \Big).$$

Proof. For the first part, see Proposition 6, §3, Chapter III of [22].

Since the extension L/K_1 is tamely ramified with residue degree $f(\omega_i|\nu) = 1$, we get by Proposition 13, §6, Chapter III of [22],

$$\omega_i(\mathfrak{D}_{B/A}) = e(\omega_i|\nu) - 1 \text{ for } i = 1, \ldots, r,$$

hence

$$\nu(D_{B/A}) = \nu\Big(N_{L/K_1}(\mathfrak{D}_{B/A})\Big) = \sum_{i=1}^{r} \Big(e(\omega_i|\nu) - 1\Big),$$

which gives the claim. $\square$

Later we will apply this lemma frequently to the case $K_1 = k(t), A = k[t]$ and $K_1 = K_\nu$, the completion of K at ν and $A = R_\nu := \{x \in K_\nu : \nu(x) \geqslant 0\}$ for $\nu \in M_K$.

1.3 Polynomials and heights

Recall $K = k(t)$. For $\nu \in M_K$, denote by K_ν the completion of K at the valuation ν. Then ν has a unique extension to K_ν. Define

$$R_\nu = \{x \in K_\nu : \nu(x) \geqslant 0\}$$

to be the local ring of K_ν. Then its group of units is

$$R_\nu^\times = \{x \in K_\nu : \nu(x) = 0\}.$$

For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in K_\nu^n$, define

$$\begin{aligned} \nu(\mathbf{x}) &= \min_{1 \leqslant i \leqslant n} \nu(x_i), \\ \|\mathbf{x}\|_\nu &= e^{-\nu(\mathbf{x})} = \max_{1 \leqslant i \leqslant n} |x_i|_\nu, \end{aligned}$$

and for $\mathbf{x} \in K^n$, define the homogeneous height and S-height

$$H_K(\mathbf{x}) = \prod_{\nu \in M_K} \|\mathbf{x}\|_\nu,$$

$$H_S(\mathbf{x}) = \prod_{\nu \in S} \|\mathbf{x}\|_\nu.$$

Clearly, the product is well-defined and $H_K(\mathbf{x}) \geqslant 1$ for every $\mathbf{x} \neq \mathbf{0}$ because of the product formula. Also, $H_K(\lambda\mathbf{x}) = H_K(\mathbf{x})$.

For a polynomial $P \in K[X_1, \ldots, X_n]$ or $P \in K_\nu[X_1, \ldots, X_n]$ we define $|P|_\nu$ to be the maximum of the $|\cdot|_\nu$-values of its coefficients.

Lemma 1.3.1 (Gauss' lemma)**.** *Let K be a field, $|\cdot|_\nu$ a non-archimedean absolute value on K, and $P = \prod_{i=1}^{t} P_i$ with $P_i \in K[X_1, \ldots, X_n]$ for $i = 1, \ldots, t$. Then*

$$|P|_\nu = \prod_{i=1}^{t} |P_i|_\nu.$$

Proof. See [14]. □

As a direct consequence, we have

Corollary 1.3.2. *Let $F = \prod_{i=1}^{n} (\alpha_i X + \beta_i Y)$ with $\alpha_i, \beta_i \in K$ for $i = 1, \ldots, n$. Then $|F|_\nu = \prod_{i=1}^{n} \max(|\alpha_i|_\nu, |\beta_i|_\nu)$ for every $\nu \in M_K$.*

For L a finite extension of K and a polynomial $P \in L[X_1, \ldots, X_m]$, we define

$$N_{L/K}(P) = \prod_{i=1}^{[L:K]} \sigma_i(P),$$

where $\sigma_1, \ldots, \sigma_{[L:K]}$ are the K-embeddings of L into $\overline{K}$, and $\sigma_i(P)$ is obtained by the action of σ_i on the coefficients of P.

1.4 Galois theory of valuations

In this section, we give a brief sketch of some aspects of Galois theory of valuations that will be needed later.

Lemma 1.4.1. *Let K be a field with a non-trivial absolute value $|\cdot|_\nu$, and L a finite Galois extension of K with Galois group $G = \mathrm{Gal}(L/K)$. Then for every two absolute values $|\cdot|_\omega, |\cdot|_{\omega'}$ on L prolonging $|\cdot|_\nu$, there is $\sigma \in G$ such that $|x|_\omega = |\sigma(x)|_{\omega'}$ for $x \in L$.*

Proof. See Corollary 1.3.5 of [4]. □

For $\nu \in M_K$ and L a Galois extension of K, denote by $\mathcal{A}(\nu)$ the set of normalized valuations of L above ν. Fix $\omega_1 \in \mathcal{A}(\nu)$. The completion L_{ω_1} of L at ω_1 is a Galois extension of K_ν. We may view L as a subfield of L_{ω_1}. As mentioned before, the absolute values on L defined above satisfy the relation $|x|_{\omega_1} = |N_{L_{\omega_1}/K_\nu}(x)|_\nu$ for $x \in L_{\omega_1}$. Without loss of generality, we may assume $K \subset K_\nu \subset L_{\omega_1} \subset \overline{K_\nu}$ and $K \subset L \subset L_{\omega_1} \subset \overline{K_\nu}$. Let $\mathcal{E}(\omega_1|\nu)$ be the set $\{\sigma \in G : \omega_1 \circ \sigma = \omega_1\}$ equipped with composition. This is by definition the decomposition group of ω_1 over ν. By, for instance, §9, Chapter II of [18], we have an isomorphism

$$\begin{array}{rcl} \mathrm{Gal}(L_{\omega_1}/K_\nu) & \xrightarrow{\sim} & \mathcal{E}(\omega_1|\nu), \\ \sigma & \longmapsto & \sigma|_L. \end{array}$$

Thus we may view $\mathrm{Gal}(L_{\omega_1}/K_\nu)$ as a subgroup of G. Further, let

$$\mathcal{E}(\omega|\nu) = \{\sigma \in G : \omega = \omega_1 \circ \sigma\} \text{ for } \omega \in \mathcal{A}(\nu). \tag{1.4.1}$$

Since G acts transitively on $\mathcal{A}(\nu)$ (see §9, Chapter II, [18]), the sets $\mathcal{E}(\omega|\nu)$ form a partition of G, and in fact they are the right cosets of $\mathrm{Gal}(L_{\omega_1}/K_\nu)$ in G, so have the same cardinality:

$$[L_\omega : K_\nu] = [L_{\omega'} : K_\nu] \text{ for } \omega, \omega' \text{ above } \nu. \tag{1.4.2}$$

It is now reasonable to put $g_\nu := \#\mathcal{E}(\omega|\nu) = [L_{\omega_1} : K_\nu]$. If we still denote by $|\cdot|_\nu$ the prolongation of $|\cdot|_\nu$ from K to $\overline{K_\nu}$, and hence on L_{ω_1}, then $|x|_\nu = |N_{L_{\omega_1}/K_\nu}(x)|_\nu^{1/[L_{\omega_1}:K_\nu]}$ for $x \in L_{\omega_1}$. It follows that for $x \in L, \omega \in \mathcal{A}(\nu), \sigma \in \mathcal{E}(\omega|\nu)$, we have

$$|x|_\omega = |\sigma(x)|_{\omega_1} = |\sigma(x)|_\nu^{g_\nu}. \tag{1.4.3}$$

Notice that $\sigma \in \mathrm{Gal}(L/K)$, hence we may extend $\sigma \in \mathcal{E}(\omega|\nu)$ to a K_ν-isomorphism from L_ω to L_{ω_1}, by sending $\alpha = \lim\limits_{n\to\infty} \alpha_n$ to $\sigma(\alpha) = \lim\limits_{n\to\infty} \sigma(\alpha_n)$ where $\alpha \in L_\omega$ and $\alpha_n \in L$. Moreover, for every $x \in L_\omega$, we also have $|x|_\omega = |\sigma(x)|_{\omega_1} = |\sigma(x)|_\nu^{g_\nu}$.

1.5 Twisted heights

Let S be a finite set of valuations of K. We define the ring of S-adeles

$$\mathbb{A}_S := \prod_{\nu\in S} K_\nu = \{(x_\nu)|x_\nu \in K_\nu \text{ for every } \nu \in S\}$$

with componentwise addition and multiplication.

Further, let

$$\mathrm{GL}_n(\mathbb{A}_S) \quad = \quad \{(A_\nu)|A_\nu \in \mathrm{GL}_n(K_\nu) \text{ for every } \nu \in S\},$$

where $\mathrm{GL}_n(R_\nu)$ is the subgroup of $\mathrm{GL}_n(K_\nu)$ of $n \times n$ matrices whose entries are in R_ν and whose determinant is in $R_\nu^\times$.

For $A = (A_\nu) \in \mathrm{GL}_n(\mathbb{A}_S)$, define

$$|\det(A)|_S := \prod_{\nu\in S} |\det(A_\nu)|_\nu.$$

Also, we define the ν-norm of A_ν as follows: if $A_\nu = (a_{ij})_{1\leqslant i,j\leqslant n}$, then $\|A_\nu\|_\nu = \max\limits_{i,j} |a_{ij}|_\nu$. Given a ring R we denote by R^n the module of n-dimensional column vectors with entries in R.

Lemma 1.5.1. *Let $\nu \in M_K$. For $A_\nu \in GL_n(R_\nu)$ and $\mathbf{x} \in K_\nu^n$, we have $\nu(A_\nu\mathbf{x}) = \nu(\mathbf{x})$.*

Proof. Let $A_\nu = (a_{ij}), \mathbf{x} = (x_1, \ldots, x_n) \in K^n$.

As $\min_{i,j} \nu(a_{ij}) \geqslant 0$, we have

$$\begin{aligned}
\nu(A_\nu \mathbf{x}) &\geqslant \min_{1 \leqslant i \leqslant n} \nu(a_{i1}x_1 + \cdots + a_{in}x_n) \\
&\geqslant \min_{1 \leqslant i,j \leqslant n} \nu(a_{ij}x_j) \\
&\geqslant \min_{1 \leqslant j \leqslant n} \nu(x_i) + \min_{i,j} \nu(a_{ij}) \\
&\geqslant \nu(\mathbf{x}).
\end{aligned}$$

Since $A_\nu^{-1} \in \mathrm{GL}_n(R_\nu)$, we have similarly for $A_\nu \in \mathrm{GL}_n(R_\nu), \mathbf{x} \in K^n$ that $\nu(\mathbf{x}) = \nu(A_\nu^{-1} A_\nu \mathbf{x}) \geqslant \nu(A_\nu \mathbf{x})$. This completes the proof. □

For $A \in \mathrm{GL}_n(\mathbb{A}_S), \mathbf{x} \in K^n$ define the divisor

$$\mathrm{div}_A(\mathbf{x}) := \sum_{\nu \in S} \nu(A_\nu \mathbf{x})\nu + \sum_{\nu \notin S} \nu(\mathbf{x})\nu$$

and its degree

$$\deg(\mathrm{div}_A(\mathbf{x})) = \sum_{\nu \in S} \nu(A_\nu \mathbf{x}) + \sum_{\nu \notin S} \nu(\mathbf{x}).$$

Also define the corresponding twisted additive height

$$h_A(\mathbf{x}) := -\deg(\mathrm{div}_A(\mathbf{x})) = -\sum_{\nu \in S} \nu(A_\nu \mathbf{x}) - \sum_{\nu \notin S} \nu(\mathbf{x}).$$

The sum is well-defined by the fact that for every $\mathbf{x} \in K^*$, we have $\nu(\mathbf{x}) = 0$ for almost all $\nu \in M_K$. Define the twisted multiplicative height for $\mathbf{x} \in K^n$ by:

$$H_A(\mathbf{x}) := \exp(h_A(\mathbf{x})) = \prod_{\nu \in S} \|A_\nu \mathbf{x}\|_\nu \prod_{\nu \notin S} \|\mathbf{x}\|_\nu.$$

It is projective in the sense that, by the product formula, $H_A(\lambda \mathbf{x}) = H_A(\mathbf{x})$ for $\mathbf{x} \in K^n, \lambda \in K^\times$.

Lastly, we define for $A \in \mathrm{GL}_n(\mathbb{A}_S)$

$$\mathrm{div}(A) := \mathrm{div}_A(K^n) := \sum_{\nu \in S} \nu(\det(A_\nu))\nu,$$

and

$$\begin{aligned}
h_A(K^n) &:= -\deg(\mathrm{div}(A)), \\
H_A(K^n) &:= \exp(h_A(K^n)) = \prod_{\nu \in S} |\det A_\nu|_\nu = |\det(A)|.
\end{aligned}$$

Lemma 1.5.2. *Let $A \in GL_n(\mathbb{A}_S)$. Then there exist positive constants c_1, c_2 depending on A such that $c_2 H_K(\mathbf{x}) \leqslant H_A(\mathbf{x}) \leqslant c_1 H_K(\mathbf{x})$ for all $\mathbf{x} \in K^n$. In particular, for $\mathbf{x} \neq \mathbf{0}$, we have $H_A(\mathbf{x}) \geqslant c_2$.*

Proof. Let $c_1 = \prod_{\nu \in S} \|A_\nu\|_\nu$ and $c_2 = \prod_{\nu \in S} \|A_\nu^{-1}\|_\nu^{-1}$.

Clearly, we have $\|A_\nu \mathbf{x}\|_\nu \leqslant \|A_\nu\|_\nu \|\mathbf{x}\|_\nu$ because for all $\nu \in S$, the valuation is non-archimedean. Similarly we have $\|\mathbf{x}\|_\nu = \|A_\nu^{-1} A_\nu \mathbf{x}\|_\nu \leqslant \|A_\nu^{-1}\|_\nu \|A_\nu \mathbf{x}\|_\nu$, hence $\|A_\nu^{-1}\|_\nu^{-1} \|\mathbf{x}\|_\nu \leqslant \|A_\nu \mathbf{x}\|_\nu \leqslant \|A_\nu\|_\nu \|\mathbf{x}\|_\nu$ for $\nu \in S$. By taking the product over all $\nu \in M_K$ we get $c_2 H_K(\mathbf{x}) \leqslant H_A(\mathbf{x}) \leqslant c_1 H_K(\mathbf{x})$. □

Consider a finite extension L of K. Let S be a finite subset of M_K and let $T \subset M_L$ be the set of valuations of L lying above those of S. For $x \in L$ put $|x|_T := \prod_{\omega \in T} |x|_\omega$. Define the ring of T-integers and T-units

$$\mathcal{O}_T := \{x \in L : |x|_\omega \leqslant 1 \text{ for } \omega \notin T\},$$

$$\mathcal{O}_T^\times := \{x \in L : |x|_\omega = 1 \text{ for } \omega \notin T\}.$$

Then $\mathcal{O}_T$ is the integral closure of $\mathcal{O}_S$ in L. We have

$$|x|_T = |N_{L/K}(x)|_S \text{ for } x \in L, \tag{1.5.1}$$

and in particular,

$$|x|_T = |x|_S^{[L:K]} \text{ for } x \in K. \tag{1.5.2}$$

For $\omega \in M_L$, denote by L_ω the completion of L at ω. Then there is a unique extension of ω to L_ω. For $\mathbf{x} = (x_1, \dots, x_n)^{\mathrm{T}} \in L_\omega^n$, we define

$$\begin{aligned} \omega(\mathbf{x}) &= \min_{1 \leqslant i \leqslant n} \omega(x_i), \\ \|\mathbf{x}\|_\omega &= \max_{1 \leqslant i \leqslant n} |x_i|_\omega = \max_{1 \leqslant i \leqslant n} e^{-\omega(x_i)}. \end{aligned}$$

Similarly as before, we define $\mathrm{div}_A(\mathbf{x}), \mathrm{div}(A)$ for $\mathbf{x} \in L^n, A \in \mathrm{GL}_n(\mathbb{A}_T)$ by replacing K, S with L, T respectively. That is,

$$\mathrm{div}_A(\mathbf{x}) := \sum_{\omega \in M_L} \omega(A_\omega \mathbf{x})\omega,$$

$$\operatorname{div}(A) := \sum_{\omega \in M_L} \omega(\det(A_\omega))\omega.$$

Define

$$\begin{aligned} h_A(\mathbf{x}) &:= -\deg(\operatorname{div}_A(\mathbf{x}))/[L:K], \\ h_A(L^n) &:= -\deg(\operatorname{div}(A))/[L:K], \end{aligned}$$

and

$$\begin{aligned} H_A(\mathbf{x}) &:= \exp(h_A(\mathbf{x})) = \Big(\prod_{\omega \in M_L} \|A_\omega \mathbf{x}\|_\omega \Big)^{\frac{1}{[L:K]}}, \\ H_A(L^n) &:= \exp(h_A(K^n)) = \Big(\prod_{\omega \in M_L} |\det A_\omega|_\omega \Big)^{\frac{1}{[L:K]}} = |\det(A)|_L^{\frac{1}{[L:K]}}. \end{aligned}$$

The height H_A on L^n is compatible with the one on K^n: $H_A(L^n) = H_A(K^n)$.

We recall Thunder's analogue of Minkowski's convex body theorem for function fields.

Lemma 1.5.3. *Let L be a finite extension of K of degree m, and H_A be the twisted height on L^n corresponding to $A \in GL_n(\mathbb{A}_S)$. Then there is a basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of L^n satisfying*

$$\prod_{i=1}^{n} H_A(\mathbf{a}_i) \leqslant H_A(L^n) e^{n(g_L+m-1)/m}.$$

where g_L is the genus of L.

Proof. See Theorem 1 of [24]. □

Lemma 1.5.4. *For every basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of L^n, we have*

$$\prod_{i=1}^{n} H_A(\mathbf{x}_i) \geqslant H_A(L^n).$$

In particular, there is a basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of K^n such that

$$\prod_{i=1}^{n} H_A(\mathbf{a}_i) = H_A(K^n).$$

Proof. See Lemma 5 of [24] for the inequality. The equality is a combination with Lemma 1.5.3. □