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Title: Sterile neutrinos in the early Universe

Issue Date: 2015-12-09

Chapter 5

Chiral Magnetic Effect from parity-violating interactions

5.1 Chern-Simons term as a result of particle interactions

As we have seen before, the presence of chiral asymmetry in a medium of relativistic particles can have drastic consequences for the dynamics of primordial plasma. Namely, the medium can become *unstable* towards the spontaneous generation of long wavelength magnetic fields [202, 91, 92]. However, all the electrically charged particles are massive in the Standard Model and therefore the notion of *chirality* can be only approximate for them. Any asymmetry in numbers of left- and right-chiral particles, created in equilibrium will be quickly erased due to the chirality-flipping reactions, driven by the finite fermion mass m. As we have argued in Chap. 4, although in the regime $m \ll T$, the rate of chirality-flipping reactions Γ_f is strongly suppressed with respect to chirality-preserving reactions, this rate Γ_f is still extremely high, which means that the chirality gets quickly erased on the timescale of the lifetime of the Universe, $t \sim H^{-1}$, at temperatures below $\sim$ 80 TeV [95]. However, if sterile neutrinos inject large chiral asymmetry very fast, at $t \ll \Gamma_f^{-1}$, then the Chiral Magnetic Current is developed, which is proportional to this asymmetry. According to Sec. 1.3.5, it means that the parity-odd term (Chern-Simons term) is produced in the free energy of electromagnetic field (1.35).

Recently, however, it has been argued, that due to the parity-violating nature of weak interactions, the chiral asymmetry is produced effectively, in the states with large lepton asymmetry [98]. Below we discuss this mechanism in more detail.

In our discussion of the Chiral Magnetic Effect above (case *(I)* from Chap. 4), we have considered the particles as non-interacting entities. However, for the case of *dense* medium, each individual particle becomes dressed by presence of the background of all the other particles, so that its properties are modified as compared to vacuum (see similar discussion in Sec. 1.3.1 about neutrinos in the early Universe). If we consider an electron, for definiteness, and take into account the dressing due to his dominant interaction, the electromagnetic interaction, we find that the left and right electrons are dressed in the same way. Indeed, electromagnetic interactions themselves obey P-symmetry, the state of plasma is also P-symmetric in absence of chiral asymmetry, so the properties of the two types of particles, which are related to each other by the transformation under parity, are identical.

On the other hand, electrons participate as well in weak interactions, which violate parity. In analogy to neutrinos in the dense medium, electrons in medium are described by the effective Dirac equation

$$(i\partial_\mu \gamma^\mu - \Sigma_A - m)e(x) = 0, \tag{5.1}$$

where Σ_A is the medium self-energy correction of electron (similar to Eq. (1.12) for neutrinos) [51, 230, 98]

$$\Sigma_A \sim G_F L \gamma^0 \gamma^5, \tag{5.2}$$

where L is the density of lepton number.[1]

In order to understand Eq. (5.2) better, one can note that on the one hand, the expression (5.2) violates the combined symmetry CP (which corresponds to subsequent application of charge C and P transformations). On the other hand, according to the SM, weak interactions of leptons preserve CP, the state $L = 0$ is symmetric under CP, therefore, in absence of lepton asymmetry, Σ_A vanishes.[2]

The modified Dirac equation (5.1) with axial self-energy (5.2) has been already analyzed in Chap. 4 (it was called case *(II)*, and the parameter b_0 therein is $b_0 \sim G_F L$). And we have seen, that left and right electrons with given momentum are described by *different* energies in medium, so that their occupation numbers differ even in the state of thermal equilibrium, when the chirality-flipping processes have lead to $\mu_L - \mu_R \to 0$. As a result, we have found that indeed, the *effective* chiral asymmetry is developed. Our conclusion was that in this setup, the Chern-Simons term may be induced,

$$\Delta \mathcal{L}_{\text{CS}} \sim G_F L \int d^3x \boldsymbol{A} \cdot \boldsymbol{B}. \tag{5.3}$$

[1] Here we write only the parity-odd part of the self-energy induced by weak interaction. But there exists parity-even part, which renormalizes left and right particles in the same way (similarly to the case of electrodynamics considered above), and is not relevant for our further discussion.

[2] CP violation happens in weak interactions of *quarks*, which leads to oscillations of K^0 mesons into $\overline{K^0}$, and similar oscillations of B^0 mesons into $\overline{B^0}$. However, according to the experimental data, the relative magnitude of this violation is very small.

At the same time, in the discussion of the case *(II)* in Chap. 4 we have implicitly assumed that the only way the medium renormalizes the properties of particles is through appearence of the self-energy (5.2). Another assumption was that this self-energy does not depend on particle momentum, which is true as long as 4-fermion Fermi interaction is taken to be local. Below, we take into account the medium effects *systematically*, in the framework of a theory with two Abelian gauge fields ($U(1)\times U(1)$ theory). One gauge field ("vector" gauge field) is massless and couples the same way to left- and right-chiral particles, the other field ("chiral" gauge field) is heavy, and couples asymmetrically to left and right chiralities. This model is a simplified version of the Standard Model, which captures its essential features: presence of massless "electromagnetic" field which does not distinguish electric charges of different fermion chiralities, and the intrinsic parity-violation, induced by coupling of fermions to W and Z bosons. At the same time, the $U(1)\times U(1)$ model is simpler, since it does not involve Yang-Mills interactions of the electroweak bosons, and has only two gauge bosons instead of four. In the $U(1)\times U(1)$ theory, the parity-violating 4-fermion coupling appears as a result of exchange of the heavy chiral field, and is therefore *non-local*.

In Sec. 5.2 we describe the $U(1)\times U(1)$ in more detail, and classify the medium contributions to the parity-odd part of the polarization tensor. The two different classes are studied in Sections 5.3 and 5.4, respectively. We conclude that the sum of diagrams inside each of the classes vanishes separately, so that no Chern-Simons term is induced in the state of thermal equilibrium ($\mu_{\rm L}=\mu_{\rm R}$), even in presence of non-zero lepton asymmetry.

5.2 Theory with $U(1)_{\textbf{vector}}\times U(1)_{\textbf{axial}}$ gauge group

We consider a model based on $U(1)\times U(1)$ gauge symmetry, where one of the gauge fields is massless (we will call it γ or "photon") and has vector-like couplings e_f with fermions, the other gauge field is massive (we will call it Z-boson) and has different couplings with left and right fermions, g_{Lf} and g_{Rf}, respectively. Difference in couplings provides explicit violation of P-symmetry at the level of particle interactions, and the Lagrangian is[3]

$$\mathcal{L}=\sum_f\bar\psi\left[i\gamma^\mu\left(\partial_\mu+ieA_\mu-i(g_LP_L+g_RP_R)Z_\mu\right)-m\right]\psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}Z_{\mu\nu}Z^{\mu\nu}+\frac{1}{2}M_Z^2Z_\mu Z^\mu, \tag{5.4}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, $Z_{\mu\nu}=\partial_\mu Z_\nu-\partial_\nu Z_\mu$ are the strength tensors of the massless and massive gauge fields, respectively. For simplicity of notation, we have

[3]More general form includes the mixing term $Z_{\mu\nu}F^{\mu\nu}$, which leads to appearence of the $Z\gamma$ vertex in perturbation theory. However, we choose the value of this vertex to be zero for momentum $q=0$, which enters the vertex. But a non-zero vertex contribution is nevertheless generated in the effective action for the other momenta, due to loop corrections (for example, by creation of a virtual fermion-antifermion pair).

dropped the flavour index f in fermionic fields, couplings and masses. Projectors $P_L = (1-\gamma^5)/2$, $P_R = (1+\gamma^5)/2$ extract states with definite chiralities. In order for this theory to be self-consistent, we choose the gauge charges such that all the gauge anomalies cancel.

Note that the theory (5.4) is not unitary by itself. For example, the tree-level process $f_L^- Z \to f_R^- Z$ violates unitarity of the S-matrix at high energies, provided that the coupling are indeed chiral, $g_{Lf} \neq g_{Rf}$. We overcome this difficulty by introducing an (Abelian) Higgs field, which provides finite mass to Z-boson after the spontaneous symmetry breaking. The resulting model is renormalizable. On the other hand, the additional neutral scalar particle h, which appears after the symmetry breaking, is not relevant to what is discussed below in our work, therefore we omit this degree of freedom. For convenience, we choose also a non-unitary gauge, $\xi = 1$. As a result, intermediate Goldstone bosons appear in our analysis, however, they do not contribute to the parity-odd part of the polarization tensor.

We consider fermionic masses m_f, which are much smaller than the temperature T, and temperature by itself is much smaller than the mass of Z-boson, $m_f \ll T \ll M_Z$ (we will call the fermions "leptons", and the lightest one we will call "electron"). According to the logic given above, we are interested in the situation where the difference of chemical potentials for different chiralities has relaxed to zero, so that left and right fermions share common chemical potential μ (which can be, nevertheless, different for different flavours f).

What is then the expected order of magnitude of the parity-odd polarization tensor? On the one hand, Π_2 is expected to involve at least two electromagnetic vertices, $ff\gamma$, since we have two electromagnetic fields in the effective action (1.35). On the other hand, violation of parity appears due to the exchange of an intermediate Z-boson, which means, that at least two vertices ffZ are present as well. As a result $\Pi_2 \propto e^2 g^2$, where g is either g_L or g_R. There are several classes of contributions, in this order of perturbation theory.

One class is given by two-loop *vacuum* diagrams with one or two fermion loops and one intermediate Z-boson. However, this class does not give parity-odd contributions to the polarization tensor (in the considered $\mathcal{O}(q)$ approximation), since any of such terms would contradict the Lorentz invariance. Therefore, the relevant contribution may be expected to appear only from presence of medium. It is worth noting at this point, that the density of real Z-bosons is suppressed by the Boltzmann factor $\exp(-M_Z/T)$, which is negligibly small, according to our assumption $T \ll M_Z$. On the other hand, real fermions in plasma are relativistic, $m \ll T$, and do not experience such a dramatic Boltzmann suppression. As a conclusion, we will consider only virtual Z-bosons, while fermions can be either virtual, or real.

The class with one initial and one final real fermion can be interpreted as Compton scattering process, in analogy with quantum electrodynamics. The typical diagrams are given in Figs. 5.1, 5.2, and each of them involves one (vacuum) loop integration, the same as one has in absence of medium. Therefore, this class

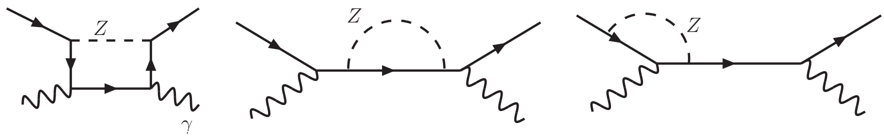

Figure 5.1: The first subclass of parity-violating vacuum 1-loop diagrams of the $f\gamma \to f\gamma$ scattering. The wavy line corresponds to photon, the line with arrow corresponds to the fermion f, the dashed line corresponds to the massive boson. Here one must include also diagrams with permutations of the $ff\gamma$ and ffZ vertices.

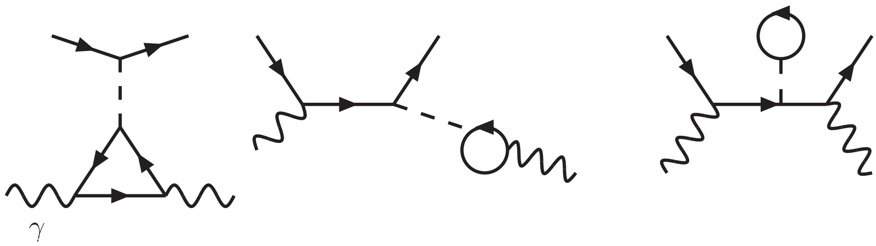

Figure 5.2: The second subclass of vacuum 1-loop diagrams of the $f\gamma \to f\gamma$ scattering in the $U(1) \times U(1)$ model.

will be referred to as "vacuum 1-loop corrections", and is discussed in detail in Sec. 5.3. The conclusion is that the sum of all these corrections is zero in the first order in photon momentum, owing to the electromagnetic gauge invariance, the analyticity of each term in the sum with respect to photon momentum, and the cancellation of chiral anomalies.

There is another class, with *two* initial fermions, and two outgoing fermions, and some of its representatives are given in Fig. 5.3. The diagrams inside this class, however are singular in the limit of the vanishing photon momentum q, and require special resummation. Unlike the diagrams from Fig. 5.1, the resulting resummed diagrams are non-analytic in q, and therefore the argument of gauge invariance does not imply that they are $\mathcal{O}(q^2)$, so it is possible that $\Pi_2 \neq 0$. However, we show that the total contribution of diagrams from Fig. 5.3 gives $\Pi_2 = 0$.

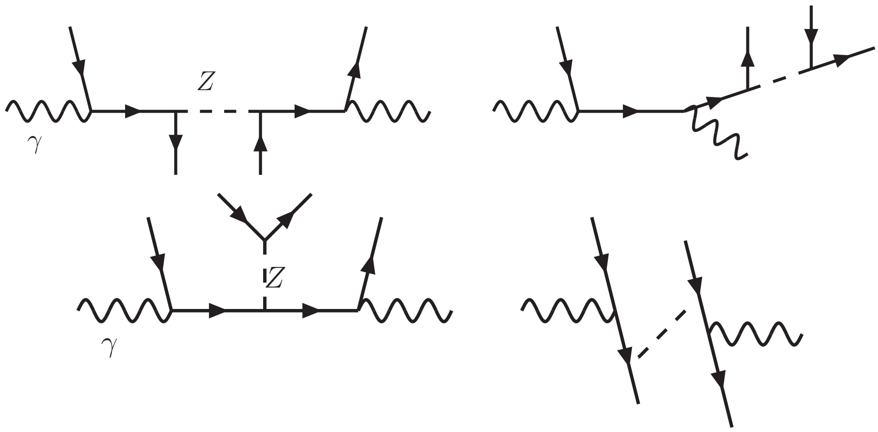

Figure 5.3: Some of the processes of the parity-violating $ff\gamma \to ff\gamma$ scattering.

5.3 1-loop vacuum corrections to Compton scattering

In this Section, we describe the contributions to Π_2 from class of diagrams with one initial real fermion, and one final real fermion. It is argued that separately these contributions are non-vanishing, but their total sum is zero.

Some of the considered diagrams are depicted in Figs. 5.1 and 5.2. (One also has to include the charged-conjugated processes, $\bar{f}\gamma \to \bar{f}\gamma$.) For a given diagram, the partial contribution to polarization tensor is

$$\Pi^{ij} = + \int \frac{d^3 r}{(2\pi)^3} n_F(\epsilon_r - \mu) \sum_s i\mathcal{M}^{ij}, \tag{5.5}$$

where $n_F(x) = (\exp(\beta x) + 1)^{-1}$ is the Fermi distribution, $\epsilon_r = \sqrt{\boldsymbol{r}^2 + m^2}$ is the fermion energy, μ is the chemical potential, s is one of two possible polarization states of fermion, $\mathcal{M}^{ij}$ is the quantum-mechanical scattering amplitude. More precisely, $\mathcal{M}^{ij}$ is the amputated amplitude, which is derived from the actual amplitude $\mathcal{M}$ by removing the polarization vectors ϵ of photons, so that $\mathcal{M} = \mathcal{M}^{\mu\nu}\epsilon_\mu\epsilon_\nu$. Everywhere in what follows, when amplitude is mentioned, we actually mean this kind of amputated amplitude.

The diagrams in Fig. 5.1 do not split into disconnected parts after cutting the Z-boson line. It is natural to call the first diagram the "box" diagram, the second - as the fermion propagator renormalization (or as vacuum self-energy), the third - as the vertex renormalization. The propagator renormalization can be expressed in terms of the vacuum self-energy $\Sigma_{\text{vac}}(p)$, while the vertex renormalization - in terms of the vacuum vertex correction $\Gamma^\mu(p, p')$. For example, the second and the third diagrams from Fig. 5.1 are, respectively,

$$\mathcal{M}^{\mu\nu}_{\Sigma} = +ie^2\bar{u}(p')\gamma^\nu S(p+q)\Sigma_{\rm vac}(p+q)S(p+q)\gamma^\mu u(p), \tag{5.6}$$
$$\mathcal{M}^{\mu\nu}_{\rm vert} = -e\ \bar{u}(p')\gamma^\nu S(p+q)\Gamma^\mu(p+q,p)u(p). \tag{5.7}$$

Here we have considered more general kinematic situation of scattering, when the photon momentum may change, $q \neq q'$, while for evaluation of Π_2^{ij} we need only $q = q'$. Using the explicit expressions for $\Sigma_{\rm vac}$ and Γ^μ, one finds that each of the amplitudes is *analytic* in photon momenta q, q'.

On the other hand, the Ward identities hold

$$(p-p')_\mu\Gamma^\mu(p,p') = e[\Sigma_{\rm vac}(p') - \Sigma_{\rm vac}(p)], \tag{5.8}$$

$$q_\mu{\rm Box}^{\mu\nu}(p,p';q,q') = e[\Gamma^\nu(p-q',p) - \Gamma^\nu(p',p'+q')], \tag{5.9}$$

where ${\rm Box}^{\mu\nu}$ is the sum of box amplitudes, in which not only the polarization vectors of photon are removed, but the spinors $u(p)$, $u(p')$ of the external fermions are absent as well. (We have checked these identities explicitly.) As a result, the total amplitude of the box, self-energy and vertex-renormalization channels satisfies the transversality property[4]

$$q_\mu\mathcal{M}^{\mu\nu}(q,q') = q'_\nu\mathcal{M}^{\mu\nu}(q,q') = 0 \tag{5.10}$$

These two properties imply, that longitudinal photons are neither emitted, nor absorbed. Note, however, that each of the amplitudes separately, for example $\mathcal{M}_\Sigma$, does not satisfy the transversality property. Together with the property of analyticity, we find as a corollary, that the total amplitude is at least second-order in photon momentum, $\mathcal{M}^{\mu\nu} = \mathcal{O}(q^\alpha q^\beta)$, for the forward regime of Compton scattering, when one puts $q = q'$. Therefore, these channels do not produce $\mathcal{O}(q)$ term in the polarization tensor (5.5).

At this point, one may ask, if the similar agrument is applicable to the result Vilenkin et al. Indeed, the expression, which is derived therein, can be obtained by plugging the *tree-level* Compton scattering amplitude in (5.5). However, although this tree-level amplitude is gauge-invariant, it is explicitly *non-analytic* in photon momentum (actually, the amplitude is singular at vanishing momentum q). Therefore, the property of gauge invariance does not forbid the presence of $\mathcal{O}(q)$ term in polarization tensor, in that model.

Contrary to the diagrams in Fig. 5.1, each of the diagrams in Fig. 5.2 *does* split into two disconnected parts after cutting the Z-boson line. The first diagram in that Figure involves the vacuum triangle diagram, more precisely the sum of

[4] In the on-shell renormalization, which we use, there are two distinct box diagrams, two self-energy diagrams, and four vertex-renormalization diagrams for the scattering $f\gamma \to f\gamma$. The same number of diagrams appears for the charge-conjugated process $\bar{f}\gamma \to \bar{f}\gamma$.

the triangle diagrams with the different possible fermion species, which run in the loop. As it was mentioned above, we consider the model, where the gauge charges are chosen in a way to prevent chiral gauge anomalies, therefore the sum of such triangles vanishes identically.

The second diagram in Fig. 5.2 is not analytic. This diagram comprises the 1-loop correction $\Pi^{\alpha\beta}_{Z\gamma}$ to the $Z\gamma$ kinetic mixing, and this correction does not involve the parity-odd part, while the parity-even part of this mixing is proportional to $q^\alpha q^\beta - q^2 g^{\alpha\beta}$. On the other hand, if remove this vacuum 1-loop bubble from the diagram and make thermal averaging of the remaining expression, the resulting parity-odd part will become first order in photon momentum q. (This situation repeats in Sec. 5.4.2, where it is described in more detail.) If one restores the removed piece, the resulting polarization tensor will be at least cubic in q, and hence it does not contribute to Π_2.

The third diagram in Fig. 5.2 can be thought of as diagram with self-energy insertion. However, contrary to the diagram with self-energy, which was considered before, the new diagram vanishes. The reason is that the tadpole contribution to self-energy is momentum-independent and therefore vanishes after renormalization.

5.4 $ff\gamma \to ff\gamma$ diagrams

In this Section, we consider the contribution to Π_2 from diagrams with two initial and two outgoing real fermions. Some of the diagrams from this class are drawn in Fig. 5.3. We conclude, that the total contribution from this class vanishes.

In analogy with Sec. 5.3, the contributions to polarization tensor from Fig. 5.3 can be expressed in terms of *tree-level* scattering amplitudes $\mathcal{M}^{ij}$. But, contrary to the previously considered case, now the *double* thermal averaging of the amplitude should be performed,[5]

$$\Pi^{ij} = + \int \frac{d^3 r}{(2\pi)^3} n_F(\epsilon_r) \int \frac{d^3 p}{(2\pi)^3} n_F(\epsilon_p) \sum_{s,s'} i\mathcal{M}^{ij}. \tag{5.11}$$

Here $\boldsymbol{r}, s$ are the common momentum and polarization of one pair of incoming and outgoing fermions, and $\boldsymbol{p}, s'$ are the common momentum and polarization of the remaining pair. Therefore, in order to find the partial contribution to the polarization tensor, one has to specify which initial fermions are paired with which outgoing fermions. For each given diagram, it is possible to do in two different ways. For the top-left diagram in Fig. 5.3, one of the choices leads to the factorization of the amplitude in two matrix elements with *independent* momenta

[5]Since we consider processes like $e^- e^- \gamma \to e^- e^- \gamma$, where identical fermions are present in the initial and final states, one has to be careful with the definition of amplitude $\mathcal{M}^{ij}$, since it may acquire additional sign, which may be not taken into account in naive application of Feynman rules.

$$\mathcal{M}_1^{ij} = +\frac{e^2}{M_Z^2}\ \left[\bar{u}(r)\gamma^i S(r+q)\gamma_\alpha \hat{g} u(r)\right]\left[\bar{u}(p)\gamma^\alpha \hat{g} S(p+q)\gamma^j u(p)\right], \qquad (5.12)$$

while the other choice does not seem to admit such a factorization[6]

$$\mathcal{M}_2^{ij} = -\frac{e^2}{M_Z^2}\ \left[\bar{u}(p)\gamma^i S(p+q)\gamma_\alpha \hat{g} u(r)\right]\ \left[\bar{u}(r)\gamma^\alpha \hat{g} S(p+q)\gamma^j u(p)\right], \qquad (5.13)$$

at the first sight. (Note that only the leading-order term in momentum was kept in the propagator of Z-boson.) However, it is possible to do a factorization for the latter amplitude as well. Indeed, since we are actually interested in the parity-violating part of the amplitude, the mixed terms, which are proportional to $g_L g_R$, and which come from the chiral coupling $\hat{g} = g_L P_L + g_R P_R$, will not contribute. This can be understood from the observation that if one performs the parity transformation, P_L changes into P_R and vice versa, so that the sum of the mixed terms remains unchanged. Only the terms, which involve g_L^2 or g_R^2, are relevant. For them one may apply the identity

$$(\gamma^\alpha P_L)_{\lambda\rho}(\gamma_\alpha P_L)_{\lambda'\rho'} = -(\gamma^\alpha P_L)_{\lambda\rho'}(\gamma_\alpha P_L)_{\lambda'\rho}, \qquad (5.14)$$

which is commonly used in derivation of the Fierz identities, and a similar identity, where one replaces the left chiral projectors P_L with the right chiral projectors P_R. As a result, the parity-odd part of the amplitude becomes factorized

$$\begin{aligned}\mathcal{M}_2^{ij} = &+\frac{g_L^2 e^2}{M_Z^2}\left[\bar{u}(r)\gamma^\alpha P_L u(r)\right]\ \left[\bar{u}(p)\gamma^i S(p+q)\gamma_\alpha P_L S(p+q)\gamma^j u(p)\right] + \\ &+ (g_L, P_L \to g_R, P_R) \quad \text{(Parity-odd part)}\end{aligned} \qquad (5.15)$$

Graphically, the application of Eq. (5.14) is equivalent to a repairing of the four fermion lines, which are attached to the Z-boson line. The same factorization is applicable to the other diagrams from Fig. 5.3. As a result, all these diagrams become splitted into two subclasses. In the first one, one of the two factors involves both electromagnetic vertices (that is, it involves both γ^i and γ^j), and this subclass is described in Sec. 5.4.1. In the second subclass, each of the factors involves only one electromagnetic vertex, and this subclass is described in Sec. 5.4.2.

It is worth noting that if we consider scattering with change of photon momentum, $q \neq q'$, the transversality property, $q_\mu \mathcal{M}_{\mu\nu}(q,q') = q'_\nu \mathcal{M}_{\mu\nu}(q,q') = 0$, holds for sum of the amplitudes of both subclasses. However, $\mathcal{M}_{\mu\nu}$ is not analytic in q, q', since at $q \to 0$, $q' \to 0$ all the intermediate fermion momenta become on-shell, therefore the denominators of the corresponding propagators become zero. As a result, the gauge invariance does not imply the absence of Π_2.

[6] In this amplitude, the additional sign "-" appears, which is related to the issue of identical fermions, mentioned above.

5.4.1 Self-energy diagrams

Some of the amplitudes from the first subclass of the $ee\gamma \to ee\gamma$ scattering are ill-defined. Indeed, if one considers the top-right diagram from Fig. 5.3, after the application of the Fierz-like identity (5.14), one receives the term

$$\mathcal{M}_3^{ij} = +\frac{e^2 g_L^2}{M_Z^2}[\bar{u}(r)\gamma^\alpha P_L u(r)][\bar{u}(p)\gamma_\alpha P_L S(p)\gamma^j S(p-q)\gamma^i u(p)] - (g_L, P_L \to g_R, P_R), \tag{5.16}$$

which involves the fermion propagator $S(p)$ at on-shell momentum, $p^2 = m^2$. It makes the whole amplitude singular, and requires more careful treatment, which is provided below.

Before we proceed, it is convenient to perform one out of the two thermal averagings of the amplitude in (5.11). Namely, we average over possible 3-momenta of the factor, which does not involve the electromagnetic vertices. As a result, this factor can be replaced by the expression

$$i\Sigma_{\text{med}} = b_\mu \gamma^\mu \gamma^5, \tag{5.17}$$

This expression is the medium contribution to the parity-piolating part of the self-energy of fermion. The spatial components of the vector b_μ vanish in the rest frame of plasma, while the temporal component, $b_0 \propto (g_L^2 - g_R^2)\Delta n_e / M_Z^2$, is finite and involves asymmetry in numbers of electrons and positrons, $\Delta n_e = \mu T^2/3$.

Note, that there is an infinite class of the diagrams, which also are ill-defined. In analogy with the discussion above, they all can be effectively reduced to diagrams with insertion of more self-energy corrections Σ_{med} in the fermionic lines (both internal and external). However, explicit resummation of these diagrams is possible, and is equivalent to the replacement of the "vacuum" Dirac propagators with $S_A(p) = i/(\not{p} - m - i\Sigma_{\text{med}})$, Dirac wavefunctions u, v by the eigenfunctions $\chi_\pm$ of the modified Hamiltonian, and the modification of the dispersion relation, which enters the remaining Fermi distribution in (5.11).The net result is described by

$$\Pi_2^{ij} = +\int \frac{d^3 r}{(2\pi)^3} \sum_s n_F(E_{rs} - \mu)\ i\mathcal{M}_{\text{eff}}^{ij} \tag{5.18}$$

where

$$\mathcal{M}_{\text{eff}}^{ij} = +ie^2 \left[\bar{\chi}_+(r)\gamma^i S_A(r+q)\gamma^j \chi_+(r) + \bar{\chi}_+(r)\gamma^j S_A(r-q)\gamma^i \chi_+(r)\right] \tag{5.19}$$

is the effective amplitude of process $f\gamma \to f\gamma$. One has also to include in (5.18) the contribution of the charge-conjugated process, $\bar{f}\gamma \to \bar{f}\gamma$. Note the similarity of Eq. (5.18) with Eq. (5.5).

In order to compute (5.18), one can proceed with the straightforward quantum-mechanical approach, but it seems to be more convenient to make a connection with the imaginary-time (Matsubara) formalism. One can check that if one performs first the summation over the imaginary frequencies $i\omega_n = i\pi T(2n+1)$ (n is integer) in the expression

$$ie^2 \int \frac{d^3p}{(2\pi)^3} T \sum_{p^0 = i\omega_n + \mu} \mathrm{Tr}\,[\gamma^\mu S_A(p)\gamma^\nu S_A(p-q)], \tag{5.20}$$

and subtracts in the resulting integrand the term, which is the limit of this integrand at $T = 0$, $\mu = 0$, then the result coincides with (5.18). This expression was considered in Chap. 4, and we saw that it vanishes.

5.4.2 $Z\gamma$ mixing diagrams

In this Section, we consider partial contribution to the polarization tensor, which comes from the subclass of amplitudes that can be written as a product of two factors, where each of the factors involves *one* electromagnetic vertex. The conclusion is that this contribution gives vanishing Π_2.

One example of the amplitude under consideration was given in Eq. (5.12). Note that each of the two factors therein is proportional to the amplitude of $f\gamma \to fZ$ scattering. The thermal averaging in (5.11) gives a product of $\Pi^{\mu\alpha}(Z\gamma)\Pi^{\alpha\nu}(Z\gamma)$, where

$$\Pi^{\alpha\beta}(Z\gamma) = -e \int \frac{d^3r}{(2\pi)^3} \sum_s n_{\mathrm{F}}(\epsilon_r - \mu)\bar{u}(r)\gamma^\alpha \hat{g} S(r+q)\gamma^\beta u(r) + (\text{cross terms}) \tag{5.21}$$

is the medium correction to the $Z\gamma$ mixing tensor, which comes from exchange of fermions (The sum is over possible electron helicities $s = \pm$). Application of the method of Sec. 5.4.1, which was based on connection with the imaginary-time technique, to calculation of this mixing tensor gives

$$\Pi^{\alpha\beta}(Z\gamma) \propto \mu\epsilon^{0\alpha\beta\gamma} q_\gamma \quad \text{(Parity-odd part)} \tag{5.22}$$

for the parity-odd part. The parity-even expression is basically the same as the thermal polarization tensor of *photon* in QED, only the prefactor e^2 in the QED expression must be replaced by eg. In the limit $q^0 \ll |\boldsymbol{q}|$ that we consider, the polarization tensor in QED is equal to $\Pi^{\alpha\beta}(QED) = e^2T^2\delta_0^\mu\delta_0^\nu/3 + \mathcal{O}(q_0^2/\boldsymbol{q}^2)$ (for a single fermion flavour that runs in the loop) [231]. Therefore, the expansion of $\Pi^{ij}(Z\gamma)$ in $\boldsymbol{q}$ starts from the linear term, and the contraction of two mixing tensors $\Pi(Z\gamma)$ does not comprise any $\mathcal{O}(q)$ terms, so that $\Pi_2 = 0$ for the considered subclass of diagrams.

5.5 Discussion

In this Chapter, we have studied the question whether Chiral Magnetic Effect can result from particle interactions, for plasma that is initially in the state of thermal equilibrium. We have analyzed a particular model with two gauge fields, where one of the fields is massless and plays a role of electromagnetic field, while the other is massive and mediates the parity-violating interaction of fermions. Our results demonstrate that the Chiral Magnetic Effect is *absent* in such a system. This situation is non-trivial, and becomes possible due to cancellation of several types of contributions to the Chiral Magnetic Current. The key ingridients here are gauge invariance and analyticity of scattering amplitudes.

This model with two gauge fields should be contrasted with the model with *local* Fermi interaction of four fermions, where no massive gauge field is present, and where the value of the Chiral Magnetic Current is ambiguous due to ultraviolet divergences as it was discussed in Chap. 4. On the other hand, the model with two gauge fields is renormalizable, therefore all the ultraviolet divergences can be unambiguously removed, and the physical observables like currents become well-defined.

Finally, we want to relate the analysis of this Chapter to the realistic case of the Standard Model, where instead of two gauge fields one deals with *four* fields (the electromagnetic field plus fields of the massive gauge bosons Z, $W^{\pm}$), and where the fermion flavours are not conserved in particle reactions, in general. Although in the case of the Standard Model, there are more different classes of contributions to the Chiral Magnetic Current, preliminary inspection shows that they cancel each other in the sum, similarly to how the cancellation happens in the $U(1) \times U(1)$ model. Therefore, we expect that the Chiral Magnetic Effect is absent in thermal equilibrium, and may appear only for out-of-equilibrium states.