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Chapter 4

Sterile neutrinos between baryogenesis and nucleosynthesis

4.1 Leptogenesis and chiral magnetic effect

Let us consider first the production of lepton asymmetry by sterile neutrinos at $T \gtrsim 100$ GeV, when sphalerons still operate, so that the lepton asymmetry is transformed partially into the baryon asymmetry. Then, we deal with "restored" (or "unbroken") electroweak phase, where the ground state of the plasma is invariant under the electroweak $SU_L(2) \times U_Y(1)$ group, and massive gauge bosons $W^{\pm}$ and Z become massless, like photon.[1]

From the form of the Yukawa interaction (1.3), we see that sterile neutrinos interact directly only with left particles. Therefore, during the leptogenesis, lepton asymmetry is produced first in the sector of left particles. Afterwards, in reactions with the gauge bosons, the number of left- and right-chiral particles is conserved, since the interaction of fermions with gauge bosons

$$\Delta\mathcal{L}_{\text{gauge}} = \bar{L}\gamma^{\mu}(g_W V_{\mu} + g_L B_{\mu})L + g_R \bar{R}\gamma^{\mu} B_{\mu} R \tag{4.1}$$

does not mix different chiralities. Here L and R are the left-handed fermion doublets and right handed singlets, respectively, g_W is the weak coupling constant, g_{L} and g_{R} are the hypercharge couplings of left- and right-chiral particles, which are different, $g_{\text{L}} \neq g_{\text{R}}$. V_{μ} is the $SU_L(2)$ gauge field, B_{μ} is the $U_Y(1)$ field. (Since $SU_L(2)$ transformations are characterized by three parameters, V_{μ} actually includes three independent fields.) At lower temperatures, when the electroweak

[1] Actually, all the SM particles become massless in this phase. Only sterile neutrinos remain massive.

symmetry is broken (in the "broken" phase), the interaction $\Delta\mathcal{L}_{\text{gauge}}$ becomes

$$\Delta\mathcal{L}_{\text{gauge}} = \frac{g_W}{\sqrt{2}} W_\mu \bar{e}_{\text{L}}\gamma^\mu \nu_{\text{L}} + Z_\mu (g_\nu \bar{\nu}_{\text{L}}\gamma^\mu \nu_{\text{L}} + g_{eL}\bar{e}_{\text{L}}\gamma^\mu e_{\text{L}} + g_{eR}\bar{e}_{\text{R}}\gamma^\mu e_{\text{R}}) - eA_\mu(\bar{e}_{\text{L}}\gamma^\mu e_{\text{L}} + \bar{e}_{\text{R}}\gamma^\mu e_{\text{R}}), \tag{4.2}$$

where W_μ, Z_μ and A_μ are the fields of W-, Z-bosons and photon, respectively. (These fields are linear combinations of the fields V_μ and B_μ above.) This interaction does not mix chiralities, as well as in the unbroken phase. However, since sterile neutrinos interact with both W and Z, and Z-bosons couple to both types of chiralities, the leptogenesis in the broken phase produces *both* left and right particles. But it is important that due to parity violation, the numbers of these particles are different even in the broken phase (chiral asymmetry is generated).

According to what was said above, one may expect that once we produce chiral asymmetry, it remains conserved. However, as we have pointed out in Sec. 1.3.5, there exist scattering processes, which change chirality (the chirality-flipping processes). Indeed, in the SM, there exists Yukawa interaction

$$\Delta\mathcal{L}_{\text{Yukawa}} = y_f \bar{L}HR \tag{4.3}$$

which becomes

$$\Delta\mathcal{L}_{\text{Yukawa}} = m\bar{e}_{\text{L}}e_{\text{R}} + y_f\bar{e}_{\text{L}}e_{\text{R}}h, \tag{4.4}$$

after the spontaneous breaking of electroweak symmetry, and the first term on the right-hand side is the Dirac mass term. (This generation of Dirac mass is similar to case of neutrinos, see Sec. 1.2.1). Here y_f is the Yukawa constant, which is proportional to the fermion mass m, h is the field of the Higgs boson. Presence of the Yukawa interaction in the unbroken phase indicates that left fermion can transform into right fermion by emitting (or absorbing) the Higgs particle. However, for the lightest massive electrically charged fermion (electron), the Yukawa constant is relatively small, so the chirality-flipping processes with right-chiral fermions have rates $\Gamma_f \sim y_f^2 T$, which are much smaller than the rate of (chirality-conserving) interactions of fermions with gauge bosons, $\Gamma_{\text{SM}} \sim g_W^2 T$.

In the broken phase, there are two mechanisms of chirality flip. First mechanism is similar to the restored phase, where the Higgs particle can be emitted/absorbed by a left-chiral electron, and the electron will become right-chiral. However, this mechanism requires high enough density of the Higgs bosons, which is exponentially suppressed for small temperatures, since the Higgs boson becomes massive and quite heavy in the broken phase (recall that the Higgs boson mass in vacuum is $m_h \approx 125\,\text{GeV}$). The second mechanism is related to appearence of fermion mass: massive electrons do not have a definite chirality. The rate of the chirality-flipping processes in this case is $\Gamma_f \sim e^2m^2/T$.

Using the estimates for the chirality-flipping rates given above, one concludes $\Gamma_f \gg H(T)$ ($H(T)$ is the Hubble rate) at $T < 80\,\text{TeV}$ [95] in both broken and unbroken phases, so during the leptogenesis, chirality-flipping processes are kept in thermal equilibrium. However, if sterile neutrinos inject significant chiral asymmetry very fast, on the timescale smaller than Γ_f^{-1}, then before the chirality-flipping processes start to operate, the Chiral Magnetic Effect becomes important (see Sec. 1.3.8).[2] As we have noticed in Sec. 1.3.5, magnetic fields start to grow exponentially with time (system becomes unstable against creation of magnetic fields), so apart from the equations describing non-equilibrium dynamics of fermions, one has to consider Maxwell equations. According to Eq. (1.48), although only the magnetic fields with large coherence scale start to grow, this scale is finite, and as a result, plasma becomes inhomogeneous. It means that the *chiral asymmetry*, which is coupled to magnetic fields, *becomes inhomogeneous* as well. Another feature is that the scale of inhomogeneities is much smaller than the cosmological horizon $1/H(T)$, so the magnetic fields are *sub-horizon*.

Electric currents flowing in presence of magnetic fields lead to the Lorentz force, which acts on plasma. As a result, macroscopic motions are excited, and one expects that the system is described by some kind of magnetohydrodynamics. And we know, that such hydrodynamical effect as turbulence, appears, and it should be consistently taken into account in the analysis of the νMSM model. However, before one carries out such a study, all of its individual ingridients must be understood well. In particular, one may notice, that in the derivation of the Chiral Magnetic Current in Sec. 1.3.5, fermion mass m was neglected. How does the picture change if we take into account m? Answering this question turns out to be non-trivial and it will be the main subject of this Chapter.

4.2 Chiral Magnetic Effect and non-zero fermion mass

The Chiral Magnetic Current

$$\boldsymbol{j} = -\frac{e^2}{4\pi^2}(\mu_{\rm L} - \mu_{\rm R})\boldsymbol{B} \tag{4.5}$$

was discovered some 35 years ago in [201], although the importance of this work has not been immediately recognized. The paper [201] had zero citations for the first 18 years, while in the last 5 years it accumulated 100+ citations. The result (4.5) has been independently rediscovered by a number of authors [202, 203, 204, 93, 94, 205, 206, 207], for a recent review and historical introduction, see [208].

Our everyday intuition considers an electric current as a non-equilibrium process, that requires energy to be pumped into a system and whose flow generates

[2] However, as we have noticed in Sec. 1.3.3, above the temperature of the sphaleron freeze-out, lepton asymmetry in the νMSM is of order of baryon asymmetry, therefore the chiral asymmetry is small and is not expected to give rise to significant Chiral Magnetic Effect. When the Universe cools down below $T \simeq 100$ GeV, the sterile neutrinos generate much larger lepton asymmetry, until they come into equilibrium at $T = T_+$ (see Sec. 1.3.3).

entropy. However, as we saw in the derivation in Sec. 1.3.6, the chiral magnetic current flows in the system in the state of thermal equilibrium and is *dissipationless* – it does not generate entropy [206, 209] (unlike for example the Ohmic current, $\boldsymbol{j} = \sigma \boldsymbol{E}$), and the system with this current flowing has time-reversal symmetry [209]. The presence of the current (4.5) in the Maxwell equations has important consequences in the heavy ion collisions [210, 211], in the early Universe [212, 213], in the astrophysical systems [214, 215, 216, 217] and magnetohydrodynamics of relativistic plasmas [218, 219, 220], Weyl semi-metals [221, 222], see [208, 223] for review.

The quantum-mechanical derivation of the chiral magnetic current, which was presented in Sec. 1.3.6 (following the work [90]), is applicable only to the case of *static and homogeneous* magnetic field (although the strength of this field formally can be arbitrarily large). In order to find the current for the magnetic field with arbitrary time and spatial dependence, one may use the field-theoretical methods instead, and look for the parity-odd part of the effective action of the gauge field (1.35). The field-theoretical approach was applied in the same paper [90], and the result for the electric current was the same as in the quantum-mechanical method, in the limit of static and homogeneous magnetic field.

As it was noted in [202], the parity-odd polarization operator (1.36), which enters the effective action of the gauge field (Fig. 4.1a) reduces to the triangular graph of Fig. 4.1b with $\Delta\mu = \mu_{\rm L} - \mu_{\rm R}$.

The original derivation of the Chiral Magnetic Current [90] was obtained for *massless* fermions. By turning on non-zero fermion mass m, we see that sending $m \to 0$ while keeping the wave-number $|\boldsymbol{q}|$ finite means that we are interested in the *short-ranged* gauge fields, i.e. that we are considering a gradient expansion of the parity-odd polarization tensor $\Pi_2(\boldsymbol{q})$ in powers of $|\boldsymbol{q}|/m \gg 1$. While this expansion is possible, it is difficult to imagine physically relevant situations when on the one hand $m \ll |\boldsymbol{q}|$, and on the other hand, we are still in the infrared region $|\boldsymbol{q}| \ll e^2(\mu_{\rm L} - \mu_{\rm R})$ (recall (1.49)), where the Chiral Magnetic Effect dominates the dynamics of the magnetic field.

In this Chapter we are exploring the question: does the expression (4.5) hold for the electrodynamics of long-range magnetic fields (i.e. for $|\boldsymbol{q}| \ll m$) or, alternatively, can the long-wavelength Chern-Simons term appear in the effective action for the electromagnetic fields in the electron-positron plasmas.

In order to produce the chiral magnetic effect, one has to create chiral asymmetry in the plasma. There are two distinct ways to do this in the system of charged *massive* particles:

(I) One could consider a plasma with different populations of left- and right-helical states.

(II) One can introduce a parity-odd part to the dispersion relation of the charged particles.

As we have noted above, for the case *(I)* the chirality flipping processes due to

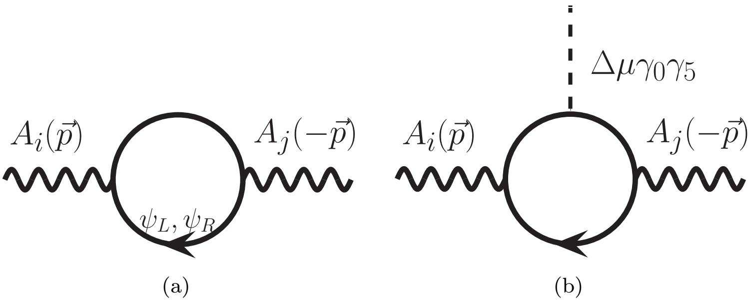

Figure 4.1: Parity-odd part of the polarization operator of the gauge fields in plasma (a) is related to the triangular anomaly with 1 axial and two vector (or 1 vector and two axial) vertices (b).

the finite fermion mass drive the chiral imbalance ($\mu_\mathrm{L} - \mu_\mathrm{R}$) to 0. Thus the chiral asymmetry can exist only for the *finite time* and the whole system should be treated as a *non-equilibrium* one (see e.g. [213, 218]) and one cannot use methods of equilibrium quantum field theory, like the imaginary-time (Matsubara) technique. Indeed, these methods operate with chemical potentials, that are defined for *conserved* quantum numbers. We discuss the case *(I)* in the Section 4.3.

In the case *(II)* chiral imbalance is due to dispersion relations of the fermions modified by an axial self-energy, $\Sigma_A = b_0\gamma^0\gamma^5$. Indeed, in absence of fermion mass, the effect of self-energy is very similar to the effect of chemical potentials $\mu_\mathrm{L} = b_0$, $\mu_\mathrm{R} = -b_0$. The key difference is that the self-energy actually modifies the energy spectrum of individual fermions, while chemical potentials define the initial state of the system. Therefore, equilibrium description can be used in the case *(II)*. When masses of the fermions are taken into account, the similarity between the two cases gets destroyed, even if the temperature of the system is high. The modification of dispersion relation can originate from the medium effects (e.g. as a result of weak interactions in the hot dense plasma with non-zero lepton or baryon number (see [178, 98]) or have "fundamental" origin, like in CPT-violating theories considered in [224].

The model with axial self-energy at finite temperatures is studied in Sec. 4.4, and we argue that the current similar to (4.5) *may not* appear actually, although a naive calculation indicates the opposite.

In the νMSM , both scenarios *(I)* and *(II)* are expected to be realized, since leptogenesis populates plasma with different numbers of left and right leptons, and presence of lepton asymmetry itself produces parity-violating self-energy of fermions ($b_0 \neq 0$) [51, 98] (see below and Chap. 5).

4.3 Asymmetric population of left/right helical states

We start by considering the non-equilibrium case *(I)* – massive electrically charged fermions – and show explicitly that Eq. (4.5) remains true in the leading $(m/T)^2$ order, if there exists asymmetry between the number of relativistic particles with left and right helicities. The resulting polarization operator is given by the expression (4.10) below. We will discuss the properties and origin of this result that will allow us to analyse a more intricate case *(II)* below (Section 4.4).

In case of free massless fermions, the state of definite chirality would also have definite *helicity* – projection of the spin of the particle on the direction of its momentum, $\boldsymbol{p}$. However, unlike the chirality operator, γ^5, the helicity operator,

$$h \equiv -\frac{\boldsymbol{p}\cdot\boldsymbol{\gamma}}{|\boldsymbol{p}|}\gamma^0\gamma^5 \ , \tag{4.6}$$

also commutes with the *massive* Dirac Hamiltonian:

$$\mathcal{H}_{\text{Dirac}} = \gamma^0(\boldsymbol{\gamma}\cdot\boldsymbol{p} + m) \tag{4.7}$$

Therefore, in presence of mass, states with definite energy are not characterized by definite chirality, but can be characterized by definite helicity instead.

Although during the leptogenesis, fermions are produced with definite chirality, afterwards, due to oscillations between different chiralities, and due to the scattering processes like $e^-\gamma \to e^-\gamma$ and $e^-e^+ \to \gamma\gamma$, these states will transform into the eigenstates of the massive Dirac Hamiltonian: the states with definite helicity.

Below, we consider two different approaches to find the chiral magnetic current, which give however the same answer. The first approach is described in Sec. 4.3.1, and there we study (quasi-)equilibrium distribution of fermions, whose energy levels are modified in presence of constant homogeneous magnetic field. The result is non-perturbative in magnetic field (i.e. it can be applicable to strong magnetic fields). In the second approach, which is described in Sec. 4.4.2, we consider (time-dependent and inhomogeneous) electromagnetic field as a perturbation, and find the linear response of the system to this perturbation.

4.3.1 Plasma in homogeneous magnetic field

First, we extend the approach of Section 1.3.6, which was used for calculation of the Chiral Magnetic Current for the massless fermions in the *static and homogeneous magnetic field*, by considering non-zero mass m. Instead of repeating the derivation, we point out the essential differences between cases of massive and massless fermions. Presence of mass leads to mixing of left and right chiralities even in absence of magnetic field, hence there will be no separate left and right Landau levels. In particular, the zeroth Landau levels ($n = 0$) of left and right chirality will combine into one, and the momentum p^z along the magnetic field will not be bounded by (1.32) anymore, instead it will run in the full range $-\infty < p^z < \infty$. At the same time, all the particles at this level will remain polarized opposite to the direction of the magnetic field, therefore the sign of p^z will describe the fermion *helicity*: if $p^z > 0$, then the spin is opposite to momentum, the fermion is left-helical; if $p^z < 0$, the fermion is right-helical. The dispersion relation (1.29) is replaced by

$$\varepsilon_n(p^z) = \sqrt{(p^z)^2 + 2|eB|n + m^2}, \tag{4.8}$$

In the massive case, the equilibrium Fermi distributions (1.27) correspond to left- and right-*helical* particles, respectively (in order to distinguish the massive and massless cases, in the massive case we replace μ_{L} by $\mu_\downarrow$, and μ_{R} by $\mu_\uparrow$). $\mu_\downarrow \neq \mu_\uparrow$ implies that in this state, left and right helicities are populated asymmetrically. As we have discussed above, this state is *quasi*-equilibrium.

Using the statistical formula (1.33), we find once again that the contributions to the electric current from the $n \neq 0$ Landau levels are cancelled, and the remaining contribution

$$\boldsymbol{j} = -\frac{e^2}{4\pi^2}(\mu_\downarrow - \mu_\uparrow)\boldsymbol{B}\left(1 + \mathcal{O}\left(\frac{m^2}{T^2}\right)\right) \tag{4.9}$$

comes from the zeroth levels. This expression coincides with Eq. (4.5) for massless particles, after the identification $\mu_{\mathrm{L}} \leftrightarrow \mu_\downarrow$, $\mu_{\mathrm{R}} \leftrightarrow \mu_\uparrow$, up to the corrections suppressed by fermion mass. Therefore, the Chern-Simons coefficient is

$$\Pi_{\mathrm{CS}} = \frac{e^2}{4\pi^2}(\mu_\downarrow - \mu_\uparrow)\left(1 + \mathcal{O}\left(\frac{m^2}{T^2}\right)\right). \tag{4.10}$$

We note once again, that the described quantum-mechanical method with Landau levels is applicable only to the static and homogeneous magnetic field. Below, we consider an alternative approach, which can be extended to time-dependent and inhomogeneous fields.

4.3.2 Plasma in inhomogeneous magnetic field

Let us turn off the magnetic field for a moment. The density matrix, which describes the state with asymmetry of left and right helicities can be written as a product of electrons' and positrons' density matrices

$$\varrho = \varrho_{\mathrm{el}} \otimes \varrho_{\mathrm{pos}} \tag{4.11}$$

where

$$\begin{aligned}\varrho_{\mathrm{el}} = \prod_{\boldsymbol{p}} \quad & \left[\Big(1 - n_{\mathrm{F}}(E_{\boldsymbol{p}} - \mu_\downarrow)\Big)\left|0_\downarrow\right\rangle\left\langle 0_\downarrow\right| + n_{\mathrm{F}}(E_{\boldsymbol{p}} - \mu_\downarrow)\left|e^-_\downarrow\right\rangle\left\langle e^-_\downarrow\right|\right] \\ \otimes \quad & \left[\Big(1 - n_{\mathrm{F}}(E_{\boldsymbol{p}} - \mu_\uparrow)\Big)\left|0_\uparrow\right\rangle\left\langle 0_\uparrow\right| + n_{\mathrm{F}}(E_{\boldsymbol{p}} - \mu_\uparrow)\left|e^-_\uparrow\right\rangle\left\langle e^-_\uparrow\right|\right].\end{aligned} \tag{4.12}$$

Here $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ is the energy of the states $|e^-\rangle$ with *definite helicity* (4.6):[3]

$$h\left|e^-_\downarrow\right\rangle = -\left|e^-_\downarrow\right\rangle \quad ; \quad h\left|e^-_\uparrow\right\rangle = +\left|e^-_\uparrow\right\rangle \tag{4.13}$$

The positron density matrix $\varrho_{\rm pos}$ has a similar form, but $\mu_\downarrow, \mu_\uparrow$ are replaced by $-\mu_\uparrow, -\mu_\downarrow$ respectively. Here the function $n_{\rm F}(x)$ is the Fermi-Dirac distribution, so that

$$n_{\rm F}(E_{\boldsymbol{p}} - \mu_\downarrow) = \frac{1}{\exp\left(\frac{\sqrt{\boldsymbol{p}^2+m^2}-\mu_\downarrow}{T}\right) + 1} \tag{4.14}$$

Clearly the distribution (4.14) looks like a chiral distribution function under the identification $\mu_{\rm L} \leftrightarrow \mu_\downarrow$ and $\mu_{\rm R} \leftrightarrow \mu_\uparrow$. Since the helicity operator commutes with the Hamiltonian (4.7), the state (4.11) is an equilibrium state in the absence of particle interactions. Indeed, neither spin, nor momentum of a fermion can change if no interactions are present in the Hamiltonian.

Let us now perturb the system by turning on electromagnetic field:

$$\mathcal{H} = \mathcal{H}_{\rm Dirac} + \Delta\mathcal{H}_A\,, \tag{4.15}$$

where

$$\Delta\mathcal{H}_A = e\int d^3\boldsymbol{x}\, A_\mu(t,\boldsymbol{x})\hat{\jmath}^\mu(\boldsymbol{x}), \quad \hat{\jmath}^\mu(\boldsymbol{x}) =: \bar\psi(\boldsymbol{x})\gamma^\mu\psi(\boldsymbol{x}) : . \tag{4.16}$$

We use here the normal-ordered current, which is denoted by $:\ldots:$.

It can be demonstrated, that in the first order in perturbation $\Delta\mathcal{H}_A$ the external gauge potential $A_j(\boldsymbol{q})$ causes the current

$$\langle \boldsymbol{j}^i\rangle_{\mu_\downarrow,\mu_\uparrow} = i\int \frac{d^3\boldsymbol{r}}{(2\pi)^3}\Big[n_{\rm F}(e_\downarrow)\mathcal{M}^{ij}(e_\downarrow) + n_{\rm F}(e_\uparrow)\mathcal{M}^{ij}(e_\uparrow) + (e\leftrightarrow e^+)\Big]A_j(\boldsymbol{q}) \tag{4.17}$$

where $\mathcal{M}_{\mu\nu}(e^-_\downarrow)$ is the matrix element of the Compton scattering of negative-helicity electrons, $e^-_\downarrow\gamma \to e^-_\downarrow\gamma$ (see Fig. 4.2, right panel)[4]

$$\mathcal{M}^{ij}(e_\downarrow) = +ie^2\bar u_\downarrow(r)[\gamma^i S(r+q)\gamma^j + \gamma^j S(r-q)\gamma^i]u_\downarrow(r), \tag{4.18}$$

Here $u_\downarrow(r)$ are the Dirac spinors, $(\not r - m)u_\downarrow(r) = 0$, which characterize the electron state with negative helicity, $h(\boldsymbol{r})u_\downarrow(\boldsymbol{r}) = -u_\downarrow(\boldsymbol{r})$; $S(p) = i(\not p - m)^{-1}$ is the Dirac propagator. *It should be stressed* that the expression (4.17) is exact in

[3]If we think about the momentum of electron pointing *up* then the symbols ↑ and ↓ denote the states with positive (negative) projection of spins on momentum.

[4]The scattering is in the forward regime here, i.e. the helicities and momenta of incoming and outgoing particles are equal.

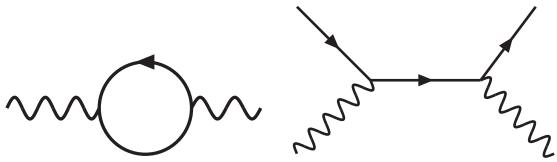

Figure 4.2: **Left**: The 1-loop contribution to the polarization tensor of electromagnetic field. **Right**: The Compton scattering amplitude at tree level, which leads to the 1-loop contribution after statistical averaging (the other amplitude should be taken into account as well, which results from the exchange of photon lines). In both digrams, wavy lines correspond to photons, solid lines correspond to charged fermions.

the first order in e^2. In Eq. (4.17) the helicity-flipping parts of the Compton scattering amplitude does not appear. Computing the expression (4.17) leads to the following current, proportional to the difference of $\mu_\downarrow - \mu_\uparrow$:

$$\langle \boldsymbol{j} \rangle_{\mu_\downarrow,\mu_\uparrow} = -\frac{e^2}{4\pi}(\mu_\downarrow - \mu_\uparrow)\boldsymbol{B}\left(1 + \mathcal{O}\left(\frac{m^2}{T^2}\right) + \mathcal{O}\left(\frac{\boldsymbol{q}^2}{T^2}\right)\right). \tag{4.19}$$

This answer is identical to Eq. (4.9).

4.4 Axial self-energy of the fermions

The previous Section discusses the situation when the parity-even system (electrons coupled to electromagnetic field) is placed into the parity-odd thermal state. Here we consider the case *(II)*. Namely, we modify the dispersion relation for left- and right-chiral particles so that the system has fundamental violation of parity. The system is put in a state of thermal equilibrium at finite temperature T (and no chemical potentials). Namely, the Lagrangian of the system is

$$\mathcal{L}_A = \bar\psi(i\partial\!\!\!/ - m - \Sigma_A)\psi, \tag{4.20}$$
$$\Sigma_A = b_0\gamma^0\gamma^5. \tag{4.21}$$

Such a system can appear if one considers fermions propagating in the hot plasma with non-zero lepton (baryon) number. In that case the self-energy part of the fermions acquires a parity odd self-energy Σ_A due to the one-loop weak corrections (see e.g. [178, 98]). A system with Lagrangian (4.20) at zero temperatures/densities has been also considered in [224, 225] (see also [226]), where the parameter b_0 was treated as a fundamental CPT-violating term, and the phenomenological consequences of its presence in the Lagrangian were discussed. In this Section we will not concentrate on the origin of b_0 and will perform the analysis of the system (4.20)–(4.21). The difference between "fundamental" and "plasma-induced" parity-odd self-energy (4.21) will be discussed in the Section 4.4.3 below.

The corresponding Hamiltonian of the system is given by

$$\mathcal{H}_A = \gamma^0(\boldsymbol{\gamma}\cdot\boldsymbol{p} + m) + b_0\gamma^5 \tag{4.22}$$

The dispersion relation of the fermions with the Lagrangian (4.20) is given by (see Appendix 4.A for details):

$$E_{\boldsymbol{p},\pm} = \sqrt{(|\boldsymbol{p}| \pm b_0)^2 + m^2} \quad \text{(particles)} \tag{4.23}$$

(and for anti-particles, the minus sign appears in front of the square root). It turns out that the operator of helicity (4.6) commutes with the Hamiltonian (4.22) and therefore *each particle with the energy* $E_{\boldsymbol{p},\pm}$ *has definite helicity:*[5]

$$h\,|E_{\boldsymbol{p},\pm}\rangle = \pm\,|E_{\boldsymbol{p},\pm}\rangle\,, \qquad E_{\boldsymbol{p},\pm} > 0 \tag{4.24}$$

The state of thermal equilibrium is described by $n_{\text{F}}(E_{\boldsymbol{p},\pm})$. In the limit $m \ll |\boldsymbol{p}|$ and $|\boldsymbol{p}| \gg b_0$, the Fermi-Dirac distribution of fermions (4.23) reduces to

$$n_{\text{F}}(E_{\boldsymbol{p},\pm}) \approx \frac{1}{\exp\left(\frac{|\boldsymbol{p}|\pm b_0}{T}\right) + 1}, \tag{4.25}$$

which looks like a Fermi-Dirac distribution for massless particles with chiral chemical potentials $\mu_{\text{L}} = b_0$ and $\mu_{\text{R}} = -b_0$. Clearly, one arrives to the same conclusion when putting $m = 0$ in the Hamiltonian (4.20) and considering the term (4.21) as the axial chemical potential term. Therefore, we expect that the model (4.20) at finite temperature emulates the chiral imbalance.

4.4.1 Homogeneous magnetic field

We have calculated the chiral magnetic current using the quantum-mechanical approach of [90], for massless particles with chiral asymmetry in Sec. 1.3.6, and for the asymmetrically populated state of massive fermions in Sec. 4.3. Here we want to extend this approach to the case of massive fermions with axial self-energy.

Lowest Landau level

In presence of electromagnetic field A_μ, the Dirac equation corresponding to the Lagrangian (4.20)–(4.21) is given by

$$(i\partial\!\!\!/ - m - eA\!\!\!/ - b_0\gamma_0\gamma^5)\psi = 0 \tag{4.26}$$

We will solve this equation for the static and homogeneous magnetic field, aligned in the z-direction, $\boldsymbol{B} = (0, 0, B)$ (we choose $eB < 0$, as before), and we choose

[5] Note the difference with the Dirac fermions, where a wave-function with a definite momentum $\boldsymbol{r}$ has fixed energy $\sqrt{\boldsymbol{r}^2 + m^2}$, but can have *both* helicities.

the Landau gauge, $A^\mu = (0, 0, xB, 0)$. Let us choose the ansatz $\psi = \exp(ip^z z + ip^y y)\chi F(x)$, where χ is a 4-component spinor, which does not depend on coordinates, and $F(x)$ is a scalar function of x only. The Dirac equation becomes

$$(\omega\gamma^0 - p^z\gamma^z - b_0\gamma_0\gamma^5 - m)F\chi + (-i\partial_x\gamma^x + p^y\gamma^y - exB\gamma^y)F\chi = 0 \tag{4.27}$$

To find the solution of the Eq. (4.27) we take the function $F(x)$ to be

$$F(x) = \exp\Big[-\frac{|eB|}{2}\left(x - \frac{p^y}{eB}\right)^2\Big] \tag{4.28}$$

so that

$$(-i\partial_x\gamma^x + p^y\gamma^y - exB\gamma^y)F\chi = 0 \tag{4.29}$$

where a spinor χ has a form $\chi = (0, \xi_1, 0, \xi_2)$. The solution of Eq. (4.27) is then reduces to

$$\begin{pmatrix} -m & \omega + p^z - b_0 \\ \omega - p^z + b_0 & -m \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0. \tag{4.30}$$

As a result, we find the dispersion relation for the lowest Landau level

$$\omega = \pm\sqrt{(p_z - b_0)^2 + m^2} \tag{4.31}$$

as well as the normalized via $\chi^\dagger\chi = 1$ solution:

$$\xi_1 = \sqrt{\frac{\omega + p_z - b_0}{2\omega}}, \quad \xi_2 = \frac{m}{\sqrt{2\omega(\omega + p_z - b_0)}}, \tag{4.32}$$

In all of these expressions, the limit $b_0 \to 0$ corresponds to the case of usual Dirac fermions (with no axial self-energy).

Expectation value of the electric current

We now find the quantum mechanical expectation value of the operator of the electric current $j^z = e\hat{\bar\psi}\gamma^z\hat\psi$ in thermal equilibrium:[6]

$$\langle j^z\rangle_{\text{thermal}} = e\times\frac{|eB|}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dp^z}{2\pi}n_{\text{F}}(\omega)\bar\chi\gamma^z\chi + \text{anti-particles}, \qquad \omega = \sqrt{(p^z - b_0)^2 + m^2} > 0 \tag{4.33}$$

[6]Below we consider only the contribution from the lowest Landau level, $n = 0$. The contributions of the $n \neq 0$ levels cancel each other, in the same manner as it happens in Sec. 4.3.1.

The prefactor $|eB|/2\pi$ is the transversal density of states, which is the number of localized Landau orbits per unit area in the $x0y$ plane, exactly as in the standard Landau levels' picture (cf. Sec. (1.3.6)). Since we have chosen $eB < 0$, $|eB| = -eB$. Observing that

$$\bar{\chi}\gamma^z\chi = \frac{p^z - b_0}{\sqrt{(p^z - b_0)^2 + m^2}} \tag{4.34}$$

we find that the expectation value of the current vanishes,

$$\langle j^z \rangle_{\text{thermal}} = 0 \tag{4.35}$$

Indeed, after the shift $p^z - b_0 \to p^{z'}$ the integral (4.33) becomes an integral of an odd function of $p^{z'}$ in the symmetric limits.

Let us now take into account the contribution of the filled Dirac sea, which is present even at zero temperature (in vacuum). Formally:

$$\langle j^z \rangle_{\text{vac}} = e \times \left(-\frac{eB}{2\pi}\right) \int\limits_{-\infty}^{\infty} \frac{dp^z}{2\pi} \frac{p^z - b_0}{\sqrt{(p^z - b_0)^2 + m^2}}, \tag{4.36}$$

which involves integral over all states with negative energy, $\omega < 0$ (the filled Dirac sea). The expression (4.36) is divergent at $p^z \to \pm\infty$. To make it finite, we notice that we are actually interested in the *change* of $\langle j^z \rangle$ that results from turning on the b_0 from $b_0 = 0$ at $t = -\infty$ to $b_0 \neq 0$. The resulting variation of the matrix element is

$$\left.\frac{\delta(\bar{\chi}\gamma^z\chi)}{\delta b_0}\right|_{\omega<0} = \frac{m^2}{|\omega|^3} \tag{4.37}$$

This variation is finite and therefore leads to

$$\frac{\delta \langle j^z \rangle_{\text{vac}}}{\delta b_0} = -\frac{e^2}{2\pi^2} B \tag{4.38}$$

which is the current that one could obtain from variation of the Chern-Simons term (1.37) with respect to A_z. Noting that in the massless case, $\mu_{\text{L}} = b_0$ and $\mu_{\text{R}} = -b_0$, we find that after the integration over b_0, the current $\langle j^z \rangle_{\text{vac}}$ coincides with the expression for the massless fermions, Eq. 4.5.

4.4.2 Inhomogeneous magnetic field

In this Section, we apply the imaginary-time (Matsubara) formalism to consider the theory (4.20)– (4.21) at finite temperature T. We compute the parity-odd part of the polarization tensor $\Pi_2^{ij}(i\omega_n = 0, \boldsymbol{q})$ at zero Matsubara frequency and check whether $\Pi_{\text{CS}} \neq 0$. The relevant diagram is the left diagram in Fig. 4.2. The polarization tensor is given by the formula

$$\Pi^{\mu\nu}(q) = ie^2 T \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \sum_{p^0 = i\omega_n} \operatorname{Tr}\left[\gamma^\mu S_A(p)\gamma^\nu S_A(p-q)\right], \tag{4.39}$$

where $i\omega_n = i\pi(2n+1)T$ are fermionic imaginary frequencies (n is an integer),

$$S_A(p) = \frac{i}{\not{p} - m - \Sigma_A} = -i\frac{\left(\omega_n^2 + \boldsymbol{p}^{\,2} + m^2 + b_0^2 + 2b_0\gamma_0\gamma^5(\boldsymbol{\gamma}\cdot\boldsymbol{p})\right)\left(\not{p} + m - b_0\gamma_0\gamma^5\right)}{\left(\omega_n^2 + \boldsymbol{p}^{\,2} + m^2 + b_0^2\right)^2 - 4b_0^2\boldsymbol{p}^{\,2}} \tag{4.40}$$

is the fermion propagator, which explicitly takes into account the axial self-energy (4.21) and $\not{p} = i\omega_n\gamma_0 - \boldsymbol{\gamma}\cdot\boldsymbol{p}$ so that $p^2 = -(\omega_n^2 + \boldsymbol{p}^2)$. In the second equality (4.40) we identified explicitly the poles in the propagator. One can easily see that the poles of the denominator of (4.40) for $\omega_n = iE$ are precisely in the positions (4.23). For $m = 0$ the expression (4.40) splits into

$$S_A\Big|_{m=0} = iP_{\text{L}}\frac{1}{(i\omega_n + b_0)\gamma^0 - \boldsymbol{\gamma}\cdot\boldsymbol{p}} + iP_{\text{R}}\frac{1}{(i\omega_n - b_0)\gamma^0 - \boldsymbol{\gamma}\cdot\boldsymbol{p}} \tag{4.41}$$

— sum of two propagators of chiral fermions with chiral chemical potentials $\mu_{\text{L}} = b_0$ and $\mu_{\text{R}} = -b_0$.

In order to evaluate the expression, it is important to specify the order of integration over 3-momentum $\boldsymbol{p}$ and summation over frequencies p_0, as will become clear below.

We will be interested in the spatial, parity odd part of the expression (4.39) in the limit

$$m \ll T \quad ; \quad b_0 \ll T \quad ; \quad |\boldsymbol{q}| \ll T \tag{4.42}$$

and we keep *arbitrary* the ratios m/b_0 and $|\boldsymbol{q}|/m$.

Let us expand the expression (4.39) in powers of b_0/T. The zeroth-order term does not give parity-violating terms, since in absence of Σ_A there is no source of the violation of P-symmetry in (4.20). Therefore, we extract the first order in b_0 (see Fig. 4.3). Each of the terms is potentially ultraviolet-divergent (the naive counting of degree of divergence indicates *linear divergence*). The well-known ambiguity in defining linearly divergent integrals (see e.g. [227], Section on $\pi^0 \to \gamma\gamma$ decay) lead the authors of [226] to argue that the parity-odd contribution to the polarization operator (4.39) is zero in this case. However, the parity-odd part of the integral in (4.39) is always convergent (cf. the discussion in [224, 225], and the discussion below) if one takes into account that the integrand of the polarization tensor involves two fermion propagators, the expansion produces two different terms (Fig. 4.3). We do the computation of the sum of two triangular diagrams, keeping the original momentum routing in both terms, and do not make any shifts of the integration variable therein. The linear expansion of the result in $\boldsymbol{q}$ produces

$$\Pi_2^{ij} = 4e^2 b_0 q_k \epsilon^{ijk} T \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \sum_{p^0=i\omega_n} \frac{3p_0^2 + \boldsymbol{p}^2 - 3m^2}{(p^2-m^2)^3} \tag{4.43}$$

Computation of the Chern-Simons coefficient in Matsubara formalism

A number of important technical issues arises when one computes the integral (4.43) and we comment on them below. The integral in (4.43) has explicit splitting into one-dimensional and three-dimensional integrations (due to the breaking of the Lorentz invariance by the presence of plasma). Superficially this integral is logarithmically divergent (in the sum of two triangular graphs of Fig. 4.3, the linearly divergent terms are cancelled).

The sum over frequencies ω_n can be done explicitly and the resulting integral over $d^3\boldsymbol{p}$ is convergent and *non-vanishing*. If one starts however with the integration over momentum $\boldsymbol{p}$, one can easily see that

$$\int \frac{4\pi \boldsymbol{p}^2 d\boldsymbol{p}}{(2\pi)^3} \frac{-3\omega_n^2 + \boldsymbol{p}^2 - 3m^2}{(-\omega_n^2 - \boldsymbol{p}^2 - m^2)^3} = 0 \quad \forall\, \omega_n, m \tag{4.44}$$

Although both answers are convergent, we understand that the reason for it is the manipulation with the order of summation and integration for superficially divergent integrals. This is a rare example, when every regularization gives convergent answer, but the answers are different (see the discussion in [224, 225]).

How should we choose the correct regularization prescription? Fortunately, we know the answer in the massless case. Repeating the computations of the Section 4.4.2 for the propagator (4.41) we find that the integral (4.44) would also be zero in a purely massless case. Using the summation before the integration, on the other hand recovers the known result (4.5) (with the identification $b_0 = (\mu_{\rm L} - \mu_{\rm R})/2$.

Finally, performing first the summation over the Matsubara frequencies:

$$T \sum_n \frac{-3\omega_n^2 + \boldsymbol{p}^2 - 3m^2}{(-\omega_n^2 - \boldsymbol{p}^2 - m^2)^3} \tag{4.45}$$

and then the integration over $d^3\boldsymbol{p}$ we arrive to the following expression for the polarization operator in the system with axial self-energy (4.20)(4.21)

$$\Pi_2^{ij} = -i\frac{e^2}{2\pi^2} b_0 \epsilon^{ijk} q^k \quad (|\boldsymbol{q}| \ll m, \quad m \ll T). \tag{4.46}$$

This expression, again, looks like Eq. (4.5) under the identification $b_0 \leftrightarrow (\mu_{\rm L} - \mu_{\rm R})/2$ and it is valid for the finite fermion mass (with the relative corrections due to the mass of the order $(m/T)^2$).

We discuss the result (4.46) in the next Section.

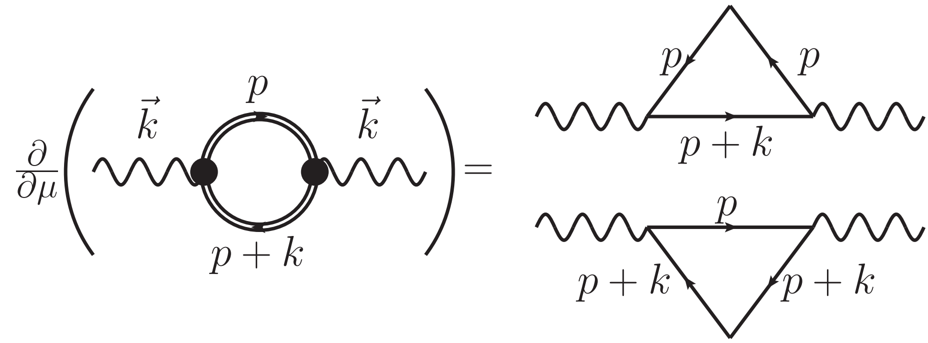

Figure 4.3: Differentiating Π_2 with respect to μ and putting $\mu = 0$ gives the following prescription of the loop momentum routing in triangular graphs as shown schematically on the Figure. Here the double line is the full propagator (4.40), while the propagators in the triangular graphs are free massive propagators without chemical potential. Please, note that this prescription for routing coincides with that of [225, 224].

4.4.3 Thermal and vacuum contributions to the parity-odd terms

The Matsubara formalism of the previous Section is technically simple, but makes the result (4.46) obscure. Why this expression is not suppressed by the mass and what is the difference with the previous example (Section 4.3.2) – we will clarify this in the current Section.

By definition, the thermal average $\langle \mathcal{O} \rangle_T$ is defined as a trace over the full system of states, *including* the vacuum state:

$$\langle \boldsymbol{j} \rangle_T = \langle 0| \boldsymbol{j} |0\rangle + \sum_{n, E_n \neq 0} \langle n| \boldsymbol{j} |n\rangle \, e^{-E_n/T} \tag{4.47}$$

where the system $\{|0\rangle, |n\rangle\}$ is a basis in the Hilbert space.

The Matsubara formalism computes the left hand side of the Eq. (4.47) for the current $\boldsymbol{j}$. The computation of the previous Section does not show what contributions are due to $|0\rangle$ or $\{|n\rangle\}$ states. The separation into the *vacuum part* $\langle 0| \mathcal{O} |0\rangle$ and *thermal contribution* (the sum over $E_n \neq 0$) in Eq. (4.47) can be done, using the following formal representation of the $\sum_n$ over Matsubara frequencies (see [228]):

$$T \sum_n f(\omega_n) = \frac{1}{2} \oint_{\mathcal{C}} \frac{d\omega}{2\pi i} f(\omega) \tan(\frac{\omega}{2T}) \tag{4.48}$$

where the contour $\mathcal{C}$ is shown as a solid line in Fig. 4.4. We can rewrite the r.h.s.

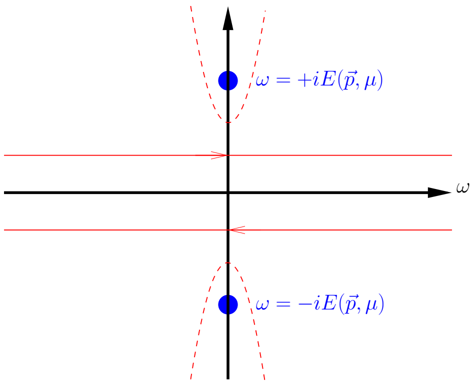

Figure 4.4: Choice of contours that allows to reduce the sum over Matsubara frequencies to the sum over filled states. The contour $\mathcal{C}$ in the integral (4.48) (red solid line) gets deformed into the dashed contour that feels only poles in the dispersion relation.

of (4.48) as a sum over the residues. The residues are computed in the "physical poles" – poles of the propagator (4.40).

Let us perform this computation in the simplest case of (4.45). The function $f(\omega)$ is in this case

$$f(\omega) = \frac{-3\omega^2 + \boldsymbol{p}^2 - 3m^2}{(-\omega^2 - \boldsymbol{p}^2 - m^2)^3} \tag{4.49}$$

The poles of this function are in $\omega = \pm\sqrt{\boldsymbol{p}^2 + m^2}$ and the residues of the expression (4.48) can be computed:

$$T\sum_n f(\omega_n) = \frac{3m^2}{4E_{\boldsymbol{p}}^5} + \frac{(\boldsymbol{p}^2 E_{\boldsymbol{p}}^2 - 3m^2T^2 - 3m^2 E_{\boldsymbol{p}} T)n_{\text{F}}(E_{\boldsymbol{p}})}{2E_{\boldsymbol{p}}^5 T^2} + (\dots)n_{\text{F}}^2(E_{\boldsymbol{p}}) + (\dots)n_{\text{F}}^3(E_{\boldsymbol{p}}) \tag{4.50}$$

The sum (4.50) contains several terms independent of temperature and a series of terms with the powers of Fermi-Dirac distribution functions, $n_{\text{F}}(E_{\boldsymbol{p}})$. This result is generic: any Matsubara sum splits into the "vacuum" part that does not depend on temperature and is equivalent to the vacuum QFT expression, integrated over ω *and* into sum over thermal states (weighted by the Fermi-Dirac distributions). In our case there is the only first term in (4.50), that is temperature-independent.[7] The integration of the r.h.s. of Eq. (4.50) over $d^3\boldsymbol{p}$ reveals that the integral of all terms, containing $n_{\text{F}}(E_{\boldsymbol{p}})$ is zero, while the integral

[7] If we did not expand the original expression in b_0/T, the thermal contributions would include Fermi-Dirac distributions evaluated for both $E_{\boldsymbol{p},\pm}$ given by (4.23).

over the first term is non-zero and leads to

$$\Pi_{\rm CS} = \frac{e^2 b_0}{2\pi^2} \qquad \text{(Fermions with axial self-energy, zero-temperature contribution)} \tag{4.51}$$

Thus, even at *zero temperature*, the parity-odd part of the polarization tensor, Π_2, or, equivalently, the current (4.5) with $\mu_{\rm L} - \mu_{\rm R} \leftrightarrow 2b_0$ exists in the system (4.20)–(4.21) for arbitrarily small $|\boldsymbol{q}|$ (i.e. in the limit $|\boldsymbol{q}| \ll m$).

What we have discovered here is actually a known result [224]. The photon polarization tensor in the model with axial self-energy was previously studied in Ref. [224] that considered the CPT-violating fermion self-energy at a fundamental level. There it was shown that indeed the Chern-Simons term (1.37) is generated in such an electrodynamics in the absence of medium. In the subsequent discussion [226, 225] authors argued that the resulting expression depends on the regularization prescription.

Pure massless case

It is instructive to repeat the previous computation in the model (4.20)–(4.21) but with $m = 0$. Performing the sum as in (4.50), we will find an expression

$$T\sum_n f_{m=0}(\omega_n) = \frac{1}{2pT^2}\Big(2n_{\rm F}^3(p) - 3n_{\rm F}^2(p) + n_{\rm F}(p)\Big) \tag{4.52}$$

Notice that *all terms* in the r.h.s. of Eq. (4.52) are temperature-dependent, and there is no analog of the first term from the sum (4.50). However, it *does not mean* that the vacuum contribution to $\Pi_{\rm CS}$ vanishes! Indeed, integrating the expression (4.52) over $d^3\boldsymbol{p}$ we get the same answer as (4.51): $\Pi_{\rm CS} = \frac{e^2 b_0}{2\pi^2}$, which is temperature-independent! The reason is that no matter how small T is, the dominant part of the integral comes from the region $p \lesssim T$, where $n_{\rm F} \sim 1$, and we deal with the integral $\sim \int_0^T p^2 dp/pT^2 \sim 1$, which does not vanish in the limit $T \to 0$. Therefore, the answer $\Pi_{\rm CS} = \frac{e^2 b_0}{2\pi^2}$ comes from vacuum, and does not receive finite-temperature corrections, like in the previous calculation with $m \neq 0$! The difference with the case $m \neq 0$ comes from the fact that the energies of massive fermions are bounded by m from below, therefore the Fermi distribution for any momentum p is suppressed at least as $\exp(-m/T)$ in the limit $T \ll m$. On the other hand, massless fermions do not receive such a suppression, and there exists range of low momenta ($p \lesssim T$) where the distribution function is unsuppressed.

$$\Pi_{\rm CS} = \frac{e^2 b_0}{2\pi^2}, \quad \text{(vacuum contribution)}, \tag{4.53}$$

$$\Pi_{\rm CS} = 0, \quad \text{(thermal contribution)} \tag{4.54}$$

Why this happens can be understood from another perspective. For massless fermions, one can find Π_{CS} without expansion in b_0, by recalling that in this case the fermionic propagator splits into the sum of left- and right-chiral propagators (4.41). Therefore, the polarization tensor splits into sum of two integrals, and the expansion in linear order in q together with the frequency summation will give

$$\Pi_{\text{CS}} \propto \int_0^\infty dp(n_{\text{F}}(p-b_0) - n_{\text{F}}(p+b_0)) \sim b_0. \tag{4.55}$$

Although the integrand is proportional to the Fermi distribution, it does not vanish in absence of medium. Indeed, the absence of medium corresponds to the limit $T \to 0$ while keeping b_0 fixed (since b_0 is a vacuum property of fermion, which remains non-zero after the removal of medium). In this limit, $n_{\text{F}}(p \pm b_0) \to \theta(-(p \pm b_0))$, where $\theta(x)$ is the Heaviside step function, which gives non-zero Π_{CS}. Once we add the medium back by turning on temperature, we see that the coefficient Π_{CS} does not change.

Short wave-length regime

Finally, let us analyse the same system in the limit $m \ll |\boldsymbol{q}| \ll T$. One can expect that this limit is similar to the case of massless fermions.

Note that the result (4.46) is applicable only in linear order in $\boldsymbol{q}/m$. As an additional step, we have evaluated the polarization tensor *without* expansion in external momentum. In the limit $|\boldsymbol{q}| \gg m$ the numerical evaluation of the parity-odd part of (4.39) gives

$$\Pi_{\text{CS}} = \mathcal{O}\left(\frac{m^2}{q}\right) \quad (m \ll |\boldsymbol{q}| \ll T, \quad \text{medium contribution}). \tag{4.56}$$

$$\Pi_{\text{CS}} = \frac{e^2}{2\pi^2} b_0 \quad (m \ll |\boldsymbol{q}| \ll T, \quad \text{vacuum contribution}), \tag{4.57}$$

This result coincides with the massless case, which is considered above, as it was expected. Note, that Eqs. (4.46) and (4.57) have identical form, although the relation between photon wavenumber and electron mass are different, and in both cases the dominant contribution comes from vacuum, while the medium corrections are suppressed.

4.5 Gauge invariance and Chern-Simons term

Medium corrections to the effective action of electromagnetic field are caused by the interactions of the probe electromagnetic field (photon) with plasma. In particular, we have seen that in the case *(I)*, the effective action is expressed through the forward scattering amplitude $\mathcal{M}^{\mu\nu}$ of photon by electron, Eq. (4.17).

Using the expansion (4.70) of the fermion propagator with axial self-energy (4.40) in terms of the eigenfunctions χ of the Hamiltonian (4.22), and evaluating the sum over frequencies according to Eq. (4.48), one can see that the thermal part of the polarization tensor (4.39) for the case *(II)* takes the form

$$\Pi^{\mu\nu}(q) = \int d^3 r \sum_{s=\pm} \left[n_{\text{F}}(E_{\boldsymbol{r}s}) \mathcal{M}_A^{\mu\nu} + \text{anti-particles} \right], \tag{4.58}$$

where

$$\mathcal{M}_A^{\mu\nu} = ie^2 \bar{\chi}_s(\boldsymbol{r}) [\gamma^\mu S_A(r+q)\gamma^\nu + \gamma^\nu S_A(r-q)\gamma^\mu] \bar{\chi}_s(\boldsymbol{r}) \tag{4.59}$$

is the forward scattering amplitude of fermions dressed with axial self-energy. The polarization tensor (4.58) is expressed via the scattering amplitude in the way that is very similar to the case *(I)* (Eq. (4.17)).

Although in both cases *(I)* and *(II)* only the *forward* amplitude is involved in the expression for the polarization tensor (or electric current), one can consider the scattering amplitude in a more general kinematic regime, when the initial photon with momentum q scatters into a state with a *different* momentum q'. As a consequence of the gauge invariance this amplitude is transversal:

$$q_\mu \mathcal{M}^{\mu\nu}(q, q') = 0, \quad q'_\nu \mathcal{M}^{\mu\nu}(q, q') = 0. \tag{4.60}$$

From these two independent relations, one may argue that the amplitude is at least second-order in photon momenta, $\mathcal{M}(q, q') \propto \mathcal{O}\big(q^2, (q')^2, (q \cdot q')\big)$ (the argument essentially repeats that of [229]) and therefore, after thermal averaging in Eq. (4.17) it is not expected to give $\mathcal{O}(q)$ term. However, the amplitude is explicitly *non-analytic* in q and q'. Indeed, in the case $q = q' = 0$ the expression (4.17) is actually singular, since the intermediate electrons become on-shell. Therefore, the transversality of the amplitude does not imply absence of Π_{CS}. The important relation between the analyticity, gauge invariance and presence of the Chern-Simons term will be actively used in Chapter 5.

4.6 Discussion

In this Chapter, we have considered the Chiral Magnetic Current for massive fermions, for two different scenarios, which are both realized in the νMSM . In the first scenario, the left- and right-helical fermions are populated asymmetrically in plasma, and the timescales that we consider are large enough for each sort of the helical particles to come into quasi-thermal equilibrium individually. At the same time, we choose this timescale to be small enough so that the disbalance in populations does not relax to zero due to chirality-flipping reactions. For this setup, we conclude that there exists electric current proportional to magnetic field in the limit $q \ll m$ of wavenumbers q that are much smaller than the fermion mass m. The current (4.19) has the same form as in the case of massless particles.

The result (4.19) is valid to the order $\mathcal{O}(e^2)$ and exact in T. Higher-order corrections $\mathcal{O}(e^{2+n})$ will lead in particular to the helicity-flipping contributions that will drive the asymmetry in populations to zero. These processes determine the timescale of helicity-flipping and thus the timescale on which Eq. (4.19) is valid. In the context of the νMSM , one has to compare the rate of helicity-flipping reactions with the rate of the development of instability against the growth of large-scale magnetic fields.

In the second scenario, we do not introduce chemical potentials, but modify instead the dispersion relation of Dirac fermions, by adding the axial self-energy. This modification breaks down the parity symmetry, and in the state of exact thermal equilibrium, the distribution functions of relativistic particles coincide with the Fermi distribution of massless fermions with chiral chemical potential. Although one would expect that the electric current is the same as in the scenario with asymmetric population, the relation between the two cases turns out to be more subtle. We employ the standard and straightforward method for systems in thermal equilibrium – the Matsubara (imaginary-time) formalism. Although in the long-wavelength limit $q \ll m$ this method gives the same Chiral Magnetic Current, as in the case of asymmetric population, we point out that the dominant contribution to the current is temperature-independent, while the medium correction to this contribution is suppressed. The temperature-independent part is present in vacuum, and evaluation by standard zero-temperature methods reveals that the contribution is ultraviolet divergent (with logarithmic degree of divergence). Thus, the current in second scenario depends on regularization, and becomes ambiguous. Matsubara formalism only provides a particular regularization prescription, which is no better than the other prescriptions.

As we have noticed before, the parity-violating correction to the Lagrangian (axial self-energy) appears effectively as a result of weak interactions in plasma. The mentioned ultraviolet ambiguity for the vacuum term is a feature of models with local 4-fermion interaction, but in a realistic renormalizable theory (like the Standard Model) there is an intermediate boson (W-boson or Z-boson), which makes the 4-fermion interaction non-local, and results in *well-defined* vacuum terms. In Chapter 5, we consider the fate of the Chern-Simons term induced by parity-violating interactions of a renormalizable theory.

Appendices

4.A Quantum mechanics of fermions with axial self-energy

Below we provide necessary details about the single-particle quantum mechanics of fermion with axial self-energy, central for the Sections 4.4 of this work. There results are fairly straightforward, however, to our knowledge they have not been worked out in necessary details in monographs and research papers. Therefore, we present this Appendix for completeness.

Starting from the Hamiltonian (4.22),

$$\mathcal{H}_A = \gamma^0 \boldsymbol{\gamma}(-i\boldsymbol{\nabla}) + m\gamma^0 + b_0\gamma^5 \; , \tag{4.61}$$

we search for the plane wave solutions with the 3-dimensional wavevector $\boldsymbol{r}$, pointing along z-axis, $\boldsymbol{p} = (0, 0, p)$, $p > 0$. Then the eigenstates of the Hamiltonian are

$$\chi_- \propto \begin{pmatrix} 0 \\ m \\ 0 \\ E_{\boldsymbol{p},-} - (p - b_0) \end{pmatrix} e^{ipz}, \quad \mathcal{H}\chi_- = E_{\boldsymbol{r},-}\chi_-, \tag{4.62}$$

$$\chi_+ \propto \begin{pmatrix} m \\ 0 \\ E_{\boldsymbol{p},+} + (p + b_0) \\ 0 \end{pmatrix} e^{ipz}, \quad \mathcal{H}\chi_+ = E_{\boldsymbol{p},+}\chi_+, \tag{4.63}$$

$$\psi_- \propto \begin{pmatrix} 0 \\ E_{\boldsymbol{p},-} - (p - b_0) \\ 0 \\ -m \end{pmatrix} e^{ipz}, \quad \mathcal{H}\psi_- = -E_{\boldsymbol{p},-}\psi_-, \tag{4.64}$$

$$\psi_+ \propto \begin{pmatrix} -m \\ 0 \\ E_{\boldsymbol{p},+} - (p + b_0) \\ 0 \end{pmatrix} e^{ipz}, \quad \mathcal{H}\psi_+ = -E_{\boldsymbol{p},+}\psi_+ \tag{4.65}$$

$$E_{\boldsymbol{p},-} = \sqrt{(p-b_0)^2+m^2}, \quad E_{\boldsymbol{p},+} = \sqrt{(p+b_0)^2+m^2} \tag{4.66}$$

We have two positive-energy branches of solutions (χ_+ and χ_-) and two negative-energy branches (ψ_+ and ψ_-), which are divided by the energy gap $2m$. In order to make physical sense of this model, let us fill all the negative-energy levels, and call this state vacuum (so the vacuum is actually the filled Dirac sea). Then the interpetation of the positive-energy branches remains unchanged: χ_- is an electron with energy $E_{p,-}$ and momentum $\boldsymbol{p}$, χ_+ is an electron with energy $E_{p,+}$ and momentum $\boldsymbol{p}$.

The values of helicities for these states can be recovered from the observation that the spin operator for electrons is the same as for usual Dirac particles,

$$\hat{\boldsymbol{s}} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{4.67}$$

so that the helicity operator has the form (4.6). As a result, χ_- corresponds to negative helicity, and χ_+ – to positive helicity. That these states can have definite energy and definite helicity at the same time, can be understood from the commutation property

$$[\hat{\mathcal{H}}_A(\boldsymbol{p}), \hat{h}(\boldsymbol{p})] = 0 \tag{4.68}$$

which is valid for non-zero b_0, as well as for the case of Dirac fermions (when $b_0 = 0$).

The state with an unoccupied negative-energy level in the filled Dirac sea is a hole, and corresponds to positive-energy and positive-charged particle (all of the other quantum numbers, like spin projection on z axis, should be flipped). The spinor of this state is the charge-conjugation of ψ_- (ψ_+), the energy is $E_{\boldsymbol{p},-}$ ($E_{\boldsymbol{p},+}$) the momentum is $-\boldsymbol{p}$, and the helicities is negative (positive).

Assuming $b_0 \ll m$, one can note that for $p \gg m$, left-chiral components of ψ_1 and ψ_4 dominate over their right-chiral components, and vice versa for ψ_2 and ψ_3. In this relativistic regime, we can expand the energy in powers of b_0/p, and find that

$$E_{\boldsymbol{p},-} \approx \sqrt{\boldsymbol{p}^2+m^2} - b_0, \quad E_{\boldsymbol{p},+} \approx \sqrt{\boldsymbol{p}^2+m^2} + b_0. \quad (p \gg m) \tag{4.69}$$

It means that the energies of left-helical electrons and positrons are shifted by b_0 downwards, while the energies of right-helical electrons and positrons are shifted by b_0 upwards, with respect to the case of pure Dirac fermions. however, for the mid- and non-relativistic particles, the expansion (4.69) is not valid.

Finally, we want to notice that the fermionic propagator (4.40) can be expanded in the wavefunctions as

$$S_A(\omega, \boldsymbol{p}) = \frac{i}{\not{p} - m - \Sigma_A} = i \sum_{s=\pm} \left(\frac{\chi_s(\boldsymbol{p})\bar{\chi}_s(\boldsymbol{p})}{\omega - E_{\boldsymbol{p},s}} + \frac{\psi_s^C(-\boldsymbol{p})\bar{\psi}_s^C(-\boldsymbol{p})}{\omega + E_{\boldsymbol{p},s}} \right), \tag{4.70}$$

where $\psi^C \equiv i\gamma^2\psi^*$ is the charge-conjugated spinor.